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Third order corrections to the semi-leptonic $b \to c$ and the muon decays

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Matteo Fael, Kay Schönwald, Matthias Steinhauser | May 20, 2021

TTP KARLSRUHE

[based on: Fael, Schönwald, Steinhauser (arxiv:2011:13654)]

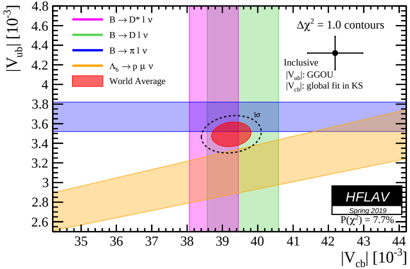
Institute for Theoretical Particle Physics

TRR 257 - Particle Physics Phenomenology after the Higgs Discovery

Outline

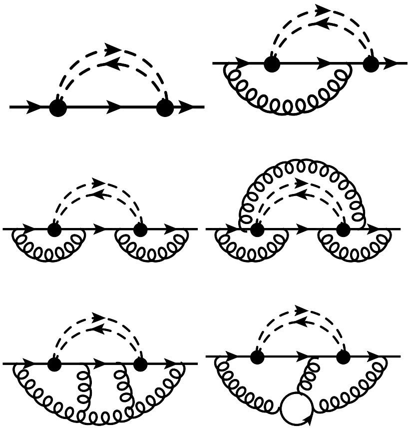
Motivation

- $b \to c\ell\nu$ is an important ingredient in the inclusive determination of $|V_{cb}|$:
  - Currently there is a tension between inclusive and exclusive determination of $|V_{cb}|$.
  - Errors are mostly theory dominated.
  - Precise measurements of the CKM matrix elements $|V_{ib}|$ are among main goals of Belle II and LHCb.
  - The semi-leptonic decay rate is an important ingredient in the global fit for the inclusive determination.
- $\mu \to e\nu\nu$ is the most precise way to determine G_F.



Method of Calculation

- We calculate the inclusive decay rate to third order via the optical theorem, i.e. we consider the imaginary part of 5-loop forward scattering diagrams.
- We consider massless leptons, i.e. we have two dimensionful scales, the bottom mass m_b and the charm mass m_c.
- Analytical dependence on charm and bottom mass seems out of reach:
  $\Rightarrow$ consider expansion in mass difference

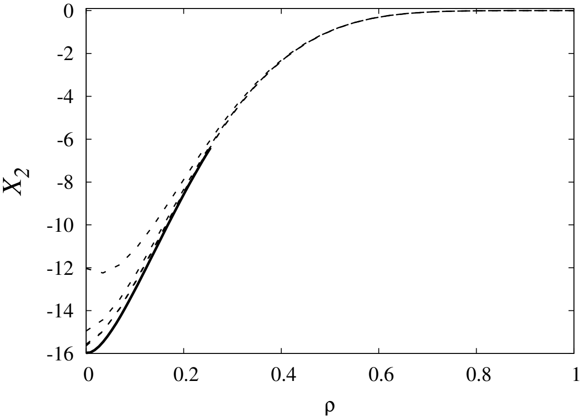


The Heavy-Daughter Expansion

- Perform the expansion in the limit $m_c \sim m_b$: $\delta = 1 - \rho = 1 - \frac{m_c}{m_b} \ll 1$
- Limit has been shown to converge well down to $m_c/m_b \to 0$ at 2-loop order.
  [Czarnecki, Dowling, Piclum (Phys. Rev. D 78 (2008))]

$$\Gamma(b \to c\ell\nu) = \Gamma_0 \left[X_0 + C_F \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n X_n \right]$$

with $\Gamma_0 = G_F^2 m_b^2 |V_{cb}|^2 / 192\pi^3$



Asymptotic Expansion

$$\delta = 1 - \frac{m_c}{m_b}$$

- We use the method of regions to perform the expansion. [Beneke, Smirnov (Nucl. Phys. B (1998))]
- Loop momenta can either scale hard $k_i \sim m_b$ or ultra-soft $k_i \sim \delta m_b$ (regions have been cross-checked with `Asy`). [Pak, Smirnov (Eur. Phys. J. C (2011))]
- The momentum of the electron-neutrino loop can be integrated trivially.
- The momentum of the lepton q has to scale ultra-soft, otherwise no imaginary part is generated. This reduces the number of regions to be considered.
- After the asymptotic expansion the integrals over the QCD loops (k_1, k_2, k_3) have a definitive scaling in $2p \cdot q + 2m_b^2\delta$:
  $\Rightarrow$ factorize out this dependence and perform the 1-loop tensor integral over q first.
- We are left with 3-loop integrals with integer powers in the end.

Asymptotic Expansion – Example

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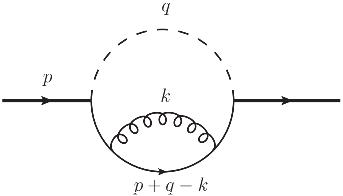
Look at the 1-loop integral (we already integrated out the electron-neutrino loop):



$$\sim \int \frac{\mathrm{d}q\mathrm{d}k}{[q^2]^\alpha[(p+q)^2-m_c^2]^2[k^2][(p+q+k)^2-m_c^2]}$$

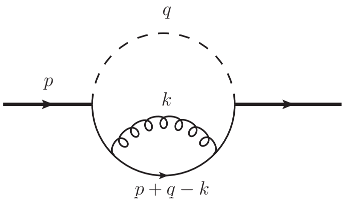
- case 1: q has to be ultra-soft, k is hard;

$$\to \int \frac{\mathrm{d}q}{[q^2]^\alpha[2p\cdot q+2m_b^2\delta]^2} \times \int \frac{\mathrm{d}k}{[k^2][(k+p)^2-m_b^2]}$$

  - We see an explicit factorization.

Asymptotic Expansion – Example

Look at the 1-loop integral (we already integrated out the electron-neutrino loop):



$$\sim \int \frac{\mathrm{d}q\mathrm{d}k}{[q^2]^\alpha[(p+q)^2 - m_c^2]^2[k^2][(p+q+k)^2 - m_c^2]}$$

- case 2: q and k are ultra-soft;

$$\to \int \frac{\mathrm{d}q\mathrm{d}k}{[q^2]^\alpha[2p\cdot q + 2m_b^2\delta]^2[k^2][2p\cdot k + \underbrace{2p\cdot q + 2m_b^2\delta}_{\text{fixed combination}}]}$$

$$= \int \frac{\mathrm{d}q}{[q^2]^\alpha[2p\cdot q + 2m_b^2\delta]^\beta} \times \int \frac{\mathrm{d}k}{[k^2][2p\cdot k + 1]}$$

- Integrals can be factorized through definite power counting in the asymptotic expansion.

Details on the Calculation

- We can always perform the integrations over the electron-neutrino loop and lepton momenta analytically via 1-loop tensor reduction. The remaining loop integration have the following scalings:

	scaling	n. regions
$\mathcal{O}(\alpha_s)$	h, u	2
$\mathcal{O}(\alpha_s^2)$	hh, hu, uu	4
$\mathcal{O}(\alpha_s^3)$	hhh, huu, hhu, uuu	8

- In case a single region with either hard or ultra-soft scaling remains we can also integrate it out analytically.
- The remaining two- or three-loop integrals have integer powers of the propagators and can be reduced to master integrals via IBP reduction.
- Since we expand up to $\mathcal{O}(\delta^{12})$ we have to reduce about 25M three-loop integrals with positive and negative indices up to 12. We used `FIRE` together with `LiteRed` for this task.
  [Smirnov,Chuharev (2020), Lee (2013)]

Details on the Calculation

Different kinds of master integrals appear in hard or ultra-soft regions:

- hard regions: up to three-loop on-shell master integrals.
  [Melnikov, van Ritbergen (Nucl. Phys. B (2000) ; Lee, Smirnov (JHEP (2011))]
- soft regions: three-loop ultra-soft master integrals with eikonal propagators
  - up to 2-loop integrals expressible in terms of Γ-functions
  - 3-loop integrals computed for the $m^{\mathrm{OS}} - m^{\mathrm{kin}}$ relation at $O(\alpha_s^3)$ [see talk by Matteo Fael].
    [Fael, KS, Steinhauser (2020); hep-ph/2011.11655]

Renormalization:

- For the renormalization of the decay width the wave function and mass renormalization constants with two massive quarks need to be known in the expansion $m_c \sim m_b$ up to $O(\alpha_s^3)$.
- Previously they were only known in the expansion $m_c \ll m_b$ and numerically for larger values of m_c. [Bekavaz, Grozin, Seidel, Steinhauser (JHEP (2007))]
- We computed them analytically and expanded around the equal mass limit to obtain the needed quantities. [see talk by Matthias Steinhauser]

Results

$$\Gamma(b \to c\ell\nu) = \Gamma_0 \left[X_0 + C_F \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n X_n \right], \qquad X_n = \sum_{j=5}^{\infty} \delta^j X_{n,j}$$

$$C_F X_3 = \delta^5 \left[\frac{533858}{1215} - \frac{20992 a_4}{81} + \frac{8744\pi^2\zeta_3}{135} - \frac{6176\zeta_5}{27} - \frac{16376\zeta_3}{135} - \frac{2624 l_2^4}{243} + \frac{5344\pi^2 l_2^2}{1215} + \frac{179552\pi^2 l_2}{405} - \frac{39776\pi^4}{6075} - \frac{1216402\pi^2}{3645} \right] + \mathcal{O}(\delta^6),$$

with $l_2 = \ln(2)$, $a_4 = \mathrm{Li}_4(1/2)$ and $\zeta_i = \sum_{j=1}^{\infty} 1/j^i$.

- We have calculated the expansion up to δ^{12} (for general color factors).
- A subset of color factors has been independently been computed up to δ^9. [Czakon, Czarnecki, Dowling (2021)]

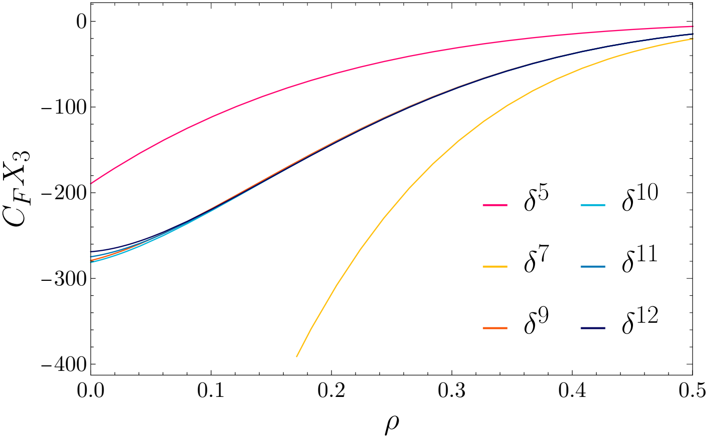
Convergence – Quark Decays

- We see a good convergence at the physical point of $\rho = m_c/m_b \approx 0.28$.
- We find:

$$X_3(\rho = 0.28) = -68.4 \pm 0.3$$

- We use the difference of the last two expansion terms to estimate the uncertainty.
- For $\rho \to 0$ we can extract values for $b \to u\ell\nu$:

$$X_3^u = -202 \pm 20$$



Convergence – Muon Decays

- Specifying the color factor to QED and setting $\rho = m_e/m_\mu \approx 0$ we get the 3-loop contributions to the muon decay.
- We find:

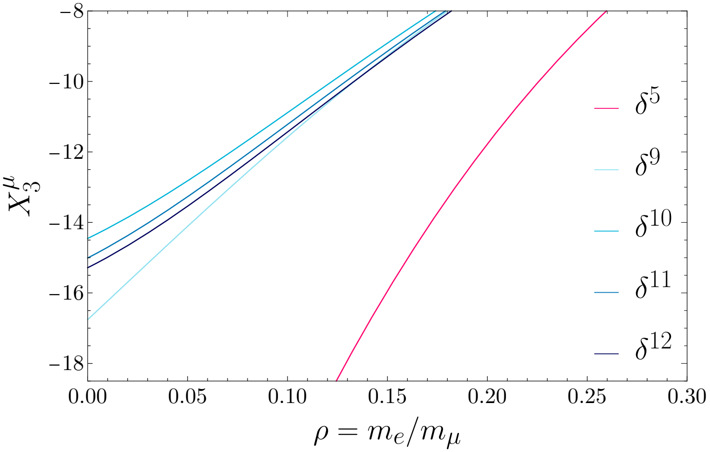
$$X_3^\mu = -15.3 \pm 2.3$$

- This leads to the shift:

$$\Delta\tau_\mu = (-9 \pm 1) \cdot 10^{-8}\,\mu s$$

- The current experimental value reads:

$$\tau_\mu = (2.1969811 \pm 0.0000022)\,\mu s$$



Different Renormalization Schemes

- The total decay rate of quarks expressed in terms of on-shell masses converges poorly:

$$\Gamma_{\rm sl} \sim 1 - 1.72\frac{\alpha_s(m_b)}{\pi} - 13.1\left(\frac{\alpha_s(m_b)}{\pi}\right)^2 - 163\left(\frac{\alpha_s(m_b)}{\pi}\right)^3$$

- Also the $\overline{\rm MS}$ scheme usually behaves poorly, since the scale has to be chosen rather low.
- Different threshold masses like the $\rm PS$ [Beneke (1998)] , $\rm 1S$ [Hoang, Ligeti, Manohar (1998)] or kinetic mass [Bigi, Shifman, Uraltsev, Vainshtein (1996)] have been proposed to improve the convergence.
- We see a much better behavior in the convergence for the schemes used for the global fits of inclusive quantities.
- E.g. for the kinetic mass:

$$m_b^{\rm kin}, m_c^{\rm kin}: \quad \Gamma(b \to c\ell\nu)/\Gamma_0 = 0.633\,(1 - 0.066 - 0.018 - 0.007) \quad \approx 0.575$$

$$m_b^{\rm kin}, \overline{m}_c(3\ \text{GeV}): \quad \Gamma(b \to c\ell\nu)/\Gamma_0 = 0.700\,(1 - 0.116 - 0.035 - 0.010) \quad \approx 0.587$$

$$m_b^{\rm kin}, \overline{m}_c(2\ \text{GeV}): \quad \Gamma(b \to c\ell\nu)/\Gamma_0 = 0.648\,(1 - 0.087 - 0.018 - 0.0003) \quad \approx 0.580$$

Different Renormalization Schemes

BLM and non-BLM part

$$\Gamma(b \to c\ell\nu) = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} X_0 \left[1 + C_F \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n Y_n\right]$$

$$Y_2 = \beta_0 Y_2^{\beta_0} + Y_2^{\mathrm{rem}}$$
$$Y_3 = \beta_0^2 Y_3^{\beta_0} + Y_3^{\mathrm{rem}}$$

	Y_1	Y_2^{rem}	$\beta_0 Y_2^{\beta_0}$	Y_3^{rem}	$\beta_0^2 Y_3^{\beta_0^2}$
$m_b^{\mathrm{OS}}, m_c^{\mathrm{OS}}$	-1.72	3.08	-16.17	48.8	-212.1
$m_b^{\mathrm{kin}}, m_c^{\mathrm{kin}}$	-0.94	0.33	-4.08	-5.4	-15.4
$m_b^{\mathrm{kin}}, \overline{m}_c(3\ \mathrm{GeV})$	-1.67	-3.39	-3.85	-97.7	69.1
$m_b^{\mathrm{kin}}, \overline{m}_c(2\ \mathrm{GeV})$	-1.25	-1.21	-2.43	-68.8	67.9
$\overline{m}_b(\overline{m}_b), \overline{m}_c(3\ \mathrm{GeV})$	3.07	-21.81	35.17	-56.7	119.4
$m_b^{\mathrm{PS}}, \overline{m}_c(2\ \mathrm{GeV})$	-0.47	-6.10	-2.31	-93.1	-7.19
$m_b^{\mathrm{1S}}, \overline{m}_c(2\ \mathrm{GeV})$	-3.59	-0.98	-19.39	-39.83	-80.22
m_b^{1S}, m_c via HQET	-1.38	0.73	-7.05	5.04	-38.09

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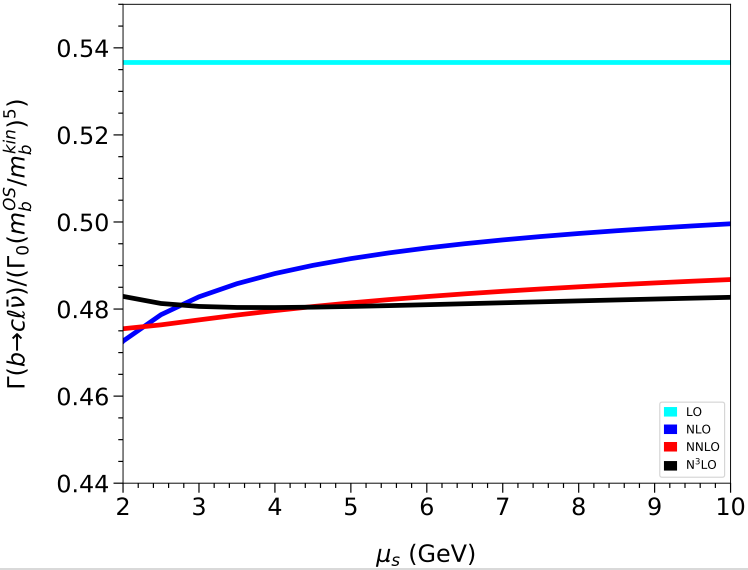
Different Renormalization Schemes – kinetic mass

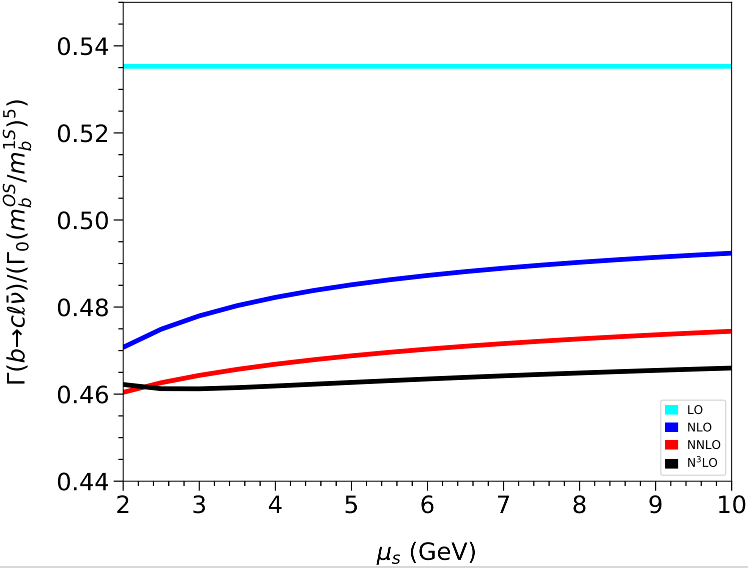
- m_b is expressed in the kinetic scheme.
- m_c is expressed in the $\overline{\mathrm{MS}}$ scheme at 2 GeV.



Different Renormalization Schemes – $1\mathrm{S}$ mass

- m_b is expressed in the $1\mathrm{S}$ scheme.
- m_c is expressed through meson masses via the HQET relation:

$$m_b^{\mathrm{OS}} - m_c^{\mathrm{OS}} = \overline{m}_B - \overline{m}_D + \dots$$

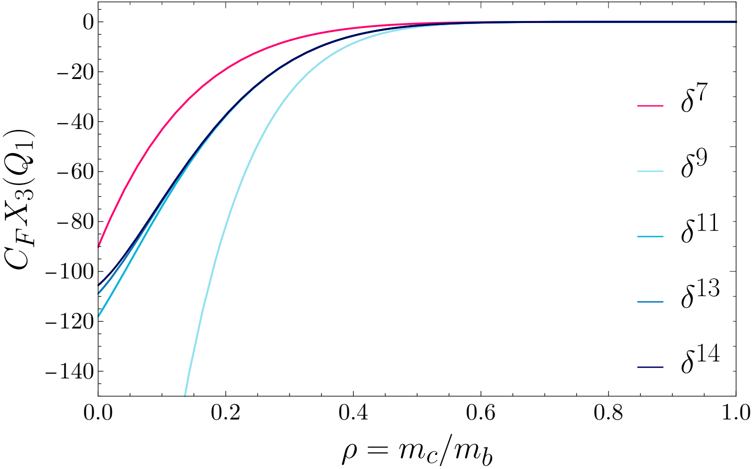


Moments of Differential Distributions

- The method can be used to calculate inclusive moments of differential distributions.
- For example we can calculate-q^2 moments:

$$Q_n = \frac{1}{\Gamma_0} \int dq^2 \left(\frac{q^2}{m_b^2}\right)^n \frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}$$

Preliminary



Conclusions and Outlook

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Conclusions

- We have computed the α_s^3 corrections to the width of $b \to c\ell\nu$.
- We performed an expansion in the limit $1 - m_c/m_b \ll 1$ and demonstrated its good convergence.
- The result is one of the few third order corrections involving two mass scales.
- The results are also relevant for $b \to u\ell\nu$ and the muon decay.

Outlook

- The method of calculation can be applied for the calculation of moments of the differential distributions.

Backup

Master Integrals in the ultrasoft region

- Denominator structures:
  - solid lines: $1/(2k_i \cdot p + 1)$, with $p^2 = 1$
  - double lines: $1/(2k_i \cdot p)$, with $p^2 = 1$
  - dotted lines: $1/k_i^2$
- Analytically calculated via direct integration, symbolic summation and differential equations using the packages `Sigma` [Schneider (2007-)] and `HarmonicSums` [Ablinger et al (2011-)] , verified with `PSLQ`.
- Only ζ values appear in the results.
- Already needed for the calculation of the relation between the OS and kinetic mass (although to lower order in ϵ).