# Shedding light on exclusive $B$ decays

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KhodjamirianFest
Siegen, October 28, 2021

***Systematic treatment of electromagnetic corrections with the tools of factorization and effective theory***

MB, Bobeth, Szafron, 1708.09152, 1908.07011 [$B_s \to \mu^+\mu^-$]

MB, Böer, Toelstede, Vos, 2008.10615 [$B \to \pi K$, charmless]

MB, Böer, Finauri, Vos, 2107.03819 [$B \to D_{(s)}^{(*)+} L^-$, colour-allowed + semi=leptonic]

MB, Böer, Toelstede, Vos, 2108.05589 + in preparation [LCDAs of light and heavy mesons]

# Motivation: Theory

- Precision: Traditionally focus on hadronic uncertainties. Time to look at QED. QED effects violate isospin symmetry and can cause large "lepton-flavour violating" logarithms, $\log m_\ell$.

- Photons couple weakly to strongly interacting quarks $\rightarrow$ probe of hadronic physics, requires factorization theorems, which mostly don't exist yet.

- Photons have long-range interactions with the charged particles in the initial/final state $\rightarrow$ QED factorization is more complicated than QCD factorization.

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## Observables

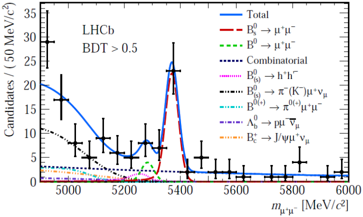
IR finite observable is

$$\Gamma_{\rm phys} = \sum_{n=0}^{\infty} \Gamma(B \to f + n\gamma, \sum_n E_{\gamma,n} < \Delta E)$$
$$\equiv \omega(\Delta E) \times \Gamma_{\rm non-rad.}(B \to f)$$

Signal window $|m_B - m_f| < \Delta \quad \Longrightarrow \quad \Delta E = \Delta$

Assume $\Delta \ll \Lambda_{QCD} \sim$ size of hadrons

Large $\ln \Delta E$.

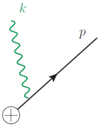



[LHCb, $B_s \to \mu^+\mu^-$, 1703.05748]

# Ultrasoft photons and the point-like approximation

Universal soft radiative amplitude

$$A^{i\to f+\gamma}(p_j,k) = A^{i\to f}(p_j) \times \sum_{j=\text{legs}} \frac{-eQ_j p_j^\mu}{\eta_j p_j\cdot k + i\epsilon}$$




The amplitude implies that the charged particles (B-meson, pion, lepton, ...) are treated as point-like. Exponentiates for the decay rate, but the virtual correction is UV divergent in the soft limit. Cut-off $\Lambda$.

$$\Gamma = \Gamma^{i\to f}_{\text{tree}} \times \left(\frac{2\Delta E}{\Lambda}\right)^{-\frac{\alpha}{\pi}\sum_{i,j} Q_i Q_j f(\beta_{ij})}$$

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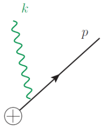


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**What is $\Lambda$?**

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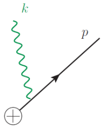


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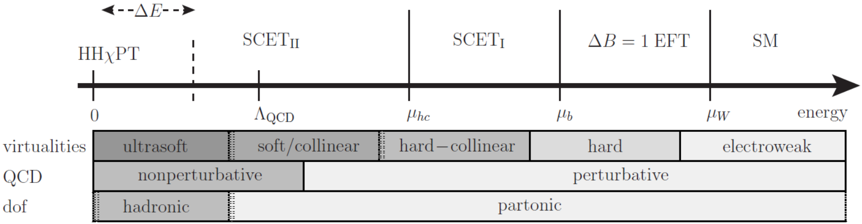
**What is $\Lambda$?**

- Present treatment of QED effects sets $\Lambda = m_B$ (e.g. using a theory of point-like mesons)
- Experimental analyses uses the PHOTOS Monte Carlo [Golonka, Was, 2005], which in addition neglects radiation from charged initial state particles.

However, the derivation implies that $\Lambda \ll \Lambda_{\text{QCD}} \sim$ size of the hadron (B-meson). Otherwise virtual corrections resolve the structure of the hadron and higher-multipole couplings are unsuppressed.

# Scales and Effective Field theories (EFTs)

Multiple scales: $m_W$, $m_b$, $\sqrt{m_b \Lambda_{\rm QCD}}$, $\Lambda_{\rm QCD}$, $m_\mu$, $\Delta E$



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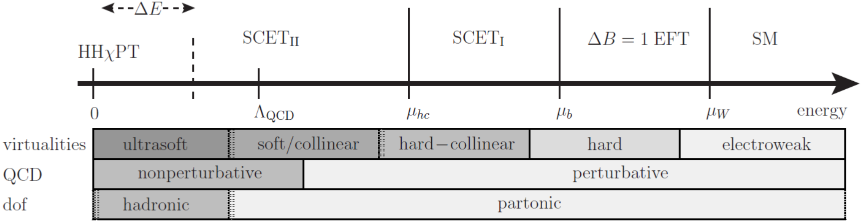
Short-distance QED at $\mu \gtrsim m_b$ can be included in the usual weak effective Lagrangian (extended Fermi theory) + renormalization group.



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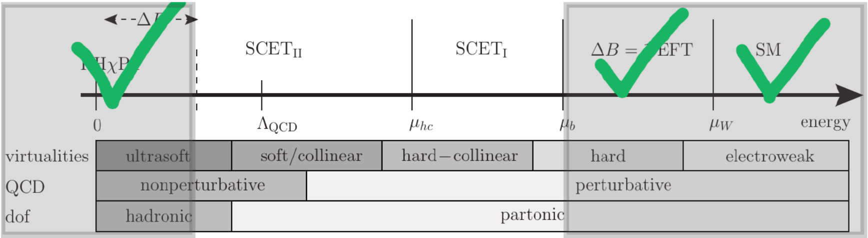
Far IR (ultrasoft scale) described by theory of point-like hadrons.



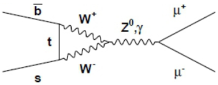

Goal: Theory for QED corrections between the scales $m_b$ and $\Lambda_{\rm QCD}$ (structure-dependent effects).

# $B_s \to \mu^+\mu^-$

1708.09152, 1908.07011, with C. Bobeth and R. Szafron

# Some facts about $B_s \to \mu^+\mu^-$

"Instantaneous", "non-radiative" branching fraction




$$\mathrm{Br}(B_s \to \mu^+\mu^-) = \frac{G_F^2\alpha^2}{64\pi^3} f_{B_s}^2 \tau_{B_s} m_{B_s}^3 |V_{tb}V_{ts}^*|^2 \sqrt{1 - \frac{4m_\mu^2}{m_{B_s}^2}}$$
$$\times \left\{ \left| \frac{2m_\mu}{m_{B_s}} (C_{10} - C_{10}') + (C_P - C_P') \right|^2 + \left(1 - \frac{4m_\mu^2}{m_{B_s}^2}\right) |C_S - C_S'|^2 \right\}$$

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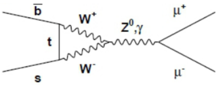


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- Long-distance QCD effects are very simple. Local annihilation. Only $\langle 0|\bar{q}\gamma^\mu\gamma_5 b|\bar{B}_q(p)\rangle = i f_{B_q} p^\mu$ Task for lattice QCD (1.5% [Aoki et al. 1607.00299], 0.5% [FNAL/MILC 1712.09262]).
- Only the operator $Q_{10}$ from the weak effective Lagrangian enters.
- No scalar lepton current $\bar{\ell}\ell$, only $\bar{\ell}\gamma_5\ell \Longrightarrow$

$$\mathcal{A}_{\Delta\Gamma}^\lambda = 1 \qquad C_\lambda = S_\lambda = 0$$

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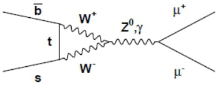


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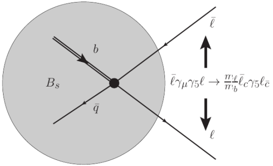
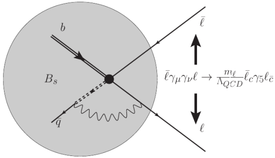
None of these are exactly true in the presence of electromagnetic corrections

# Interpretation of the $m_B/\Lambda$-enhanced correction




$$\langle 0|\bar{q}\gamma^\mu\gamma_5 b|\bar{B}_q(p)\rangle$$

Local annihilation and helicity flip.

$$\langle 0|\int d^4x\, T\{j_{\rm QED}(x), \mathcal{L}_{\Delta B=1}(0)\}|\bar{B}_q\rangle$$

Helicity-flip and annihilation delocalized by a hard-collinear distance

The virtual photon probes the $B$ meson structure. Annihilation/helicity-suppression is "smeared out" over light-like distance $1/\sqrt{m_B\Lambda}$ [$\to$ B-LCDA]. Still short-distance.

Logarithms are not the standard soft logarithms, but due to hard-collinear, collinear and soft regions, including final-state soft lepton exchange.

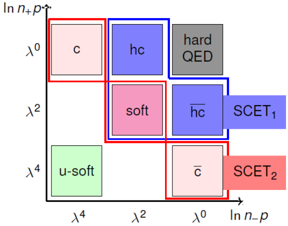
# All orders, EFT, summation of logarithms

Back-to-back energetic lepton pair
Collinear (lepton $n_+p_\ell$ large) and anti-collinear (anti-lepton $n_-p_{\bar\ell}$ large) modes

$$n_+^2 = n_-^2 = 0, \quad n_+ \cdot n_- = 2, \qquad p^\mu = n_+p\,\frac{n_-^\mu}{2} + n_-p\,\frac{n_+^\mu}{2} + p_\perp^\mu$$

$$p = (n_+p, p_\perp, n_-p), \qquad \lambda \sim \frac{\Lambda_{\rm QCD}}{m_b} \sim \frac{m_\mu}{m_b}$$

- Modes in the EFT classified by virtuality and rapidity
- Matching QCD+QED $\to$ SCET$_{\rm I}$ $\to$ SCET$_{\rm II}$

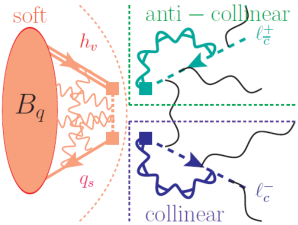


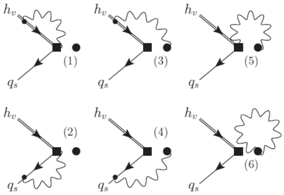

| mode | relative scaling | absolute scaling | virtuality $k^2$ |
|---|---|---|---|
| hard | $(1,1,1)$ | $(m_b, m_b, m_b)$ | $m_b^2$ |
| hard-collinear | $(1,\lambda,\lambda^2)$ | $(m_b, \sqrt{m_b\Lambda_{\rm QCD}}, \Lambda_{\rm QCD})$ | $m_b\Lambda_{\rm QCD}$ |
| anti-hard-collinear | $(\lambda^2,\lambda,1)$ | $(\Lambda_{\rm QCD}, \sqrt{m_b\Lambda_{\rm QCD}}, m_b)$ | $m_b\Lambda_{\rm QCD}$ |
| collinear | $(1,\lambda^2,\lambda^4)$ | $(m_b, m_\mu, m_\mu^2/m_b)$ | $m_\mu^2$ |
| anticollinear | $(\lambda^4,\lambda^2,1)$ | $(m_\mu^2/m_b, m_\mu, m_b)$ | $m_\mu^2$ |
| soft | $(\lambda^2,\lambda^2,\lambda^2)$ | $(\Lambda_{\rm QCD}, \Lambda_{\rm QCD}, \Lambda_{\rm QCD})$ | $\Lambda_{\rm QCD}^2$ |

# SCET$_{\text{II}}$ factorization and soft rearrangement

$$\widetilde{\mathcal{J}}^{B1}_{\mathcal{A}\chi}(v,t) = \bar{q}_s(vn_-)Y(vn_-,0)\frac{\not{n}_-}{2}P_L h_v(0)\left[Y_+^\dagger\, Y_-\right](0)\left[\bar{\ell}_c(0)(2\mathcal{A}_{c\perp}(tn_+)P_R)\ell_{\bar{c}}(0)\right] = \widehat{\mathcal{J}}_s \otimes \widehat{\mathcal{J}}_c \otimes \widehat{\mathcal{J}}_{\bar{c}}$$

- $s$, $c$, $\bar{c}$ do not interact in SCET$_{\text{II}}$. Sectors are factorized.
  Anomalous dimension should be separately well defined.
- But the anomalous dimension of the soft graphs is IR divergent.



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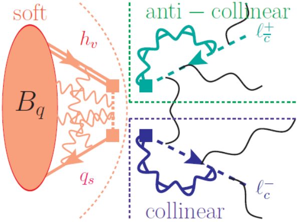
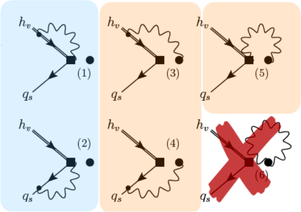



$$\langle 0|\left[Y_+^\dagger\, Y_-\right](0)|0\rangle \equiv R_+R_-$$

- Soft rearrangement $\quad \widehat{\mathcal{J}}_s\otimes\widehat{\mathcal{J}}_c\otimes\widehat{\mathcal{J}}_{\bar c} = \dfrac{\widehat{\mathcal{J}}_s}{R_+R_-}\otimes R_+\widehat{\mathcal{J}}_c\otimes R_-\widehat{\mathcal{J}}_{\bar c}$

  Soft matrix element defines a generalized B-LCDA

# Structure of the final result

## Amplitude [evolved to $\mu_c$]

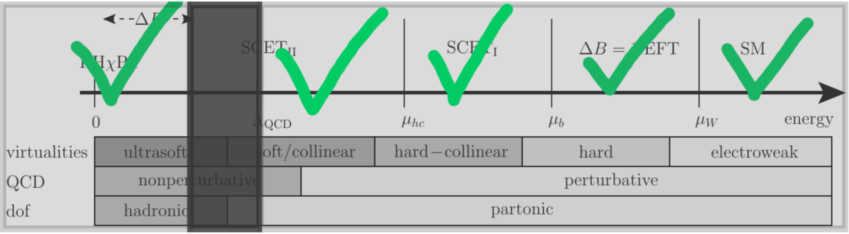
$$i\mathcal{A}_9 \;=\; e^{S_\ell(\mu_b,\,\mu_c)}\, T_+(\mu_c) \times \int_0^1 du\, e^{S_q(\mu_b,\,\mu_{hc})}\, 2H_9(u;\mu_b) \int_0^\infty d\omega\, U_s^{\rm QED}(\mu_{hc},\mu_s;\omega)\, m_{B_q} F_{B_q}(\mu_{hc})\, \phi_+(\omega;\mu_{hc})$$
$$\times \left[ J_m(u;\omega;\mu_{hc}) + \int_0^1 dw\, J_A(u;\omega,w;\mu_{hc}) \left( M_A(w;\mu_c) - \frac{Q_\ell \overline{w}}{\beta_{0,\rm em}} \ln \eta_{\rm em} \right) \right]$$
$$\equiv\; e^{S_\ell(\mu_b,\,\mu_c)} \times A_9\, [\bar{u}_c(1+\gamma_5)v_{\bar{c}}]$$

– defines the non-radiative amplitude $A_9$. QED+QCD Logs between $m_b$ and $\mu_c$ summed.

## Decay rate [including ultrasoft photon radiation]

$$\Gamma[B_q \to \mu^+\mu^-](\Delta E) \;=\; \underbrace{\frac{m_{B_q}}{8\pi}\beta_\mu \left( |A_{10}+A_9+A_7|^2 + \beta_\mu^2\, |A_9+A_7|^2 \right)}_{\text{non-radiative rate}} \times \underbrace{\left| e^{S_\ell(\mu_b,\,\mu_c)} \right|^2 \mathcal{S}(v_\ell, v_{\bar{\ell}}, \Delta E)}_{\text{ultrasoft radiation}}$$
$$=\; \Gamma^{(0)}[B_q \to \mu^+\mu^-] \left( \frac{2\Delta E}{m_{B_q}} \right)^{-\frac{2\alpha}{\pi}\left(1+\ln\frac{m_\mu^2}{m_{B_q}^2}\right)}$$

$$\mathcal{S}(v_\ell, v_{\bar{\ell}}, \Delta E) = \sum_{X_s} \left| \langle X_s | S_{v_\ell}^\dagger(0) S_{v_{\bar{\ell}}}(0) | 0 \rangle \right|^2 \theta(\Delta E - E_{X_s}) \qquad \text{Ultrasoft function}$$




Can sum leading logs, and calculate all QED effects between scale $m_b$ and a few times $\Lambda_{\mathrm{QCD}}$.

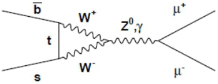
BUT: matching of SCET$_{\mathrm{II}}$ to the ultrasoft theory of point-like hadrons at a scale $\mu_c \sim \Lambda_{\mathrm{QCD}}$ must be done non-perturbatively.

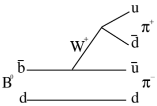
# Hadronic B two-body decays ($B \to \pi K$ ..., $D^+ L^-$, ...)

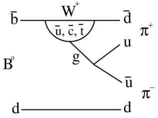
2008.10615 (charmless) and 2107.03819 (heavy-light + semi-leptonic), with P. Böer, G. Finauri, J. Toelstede and K. Vos

# Charmless decays, $B \to \pi^+\pi^-$ vs. $\mu^+\mu^-$

- Same kinematics, charges, composite pions instead of elementary leptons.
  QED effects similar, identical for ultrasoft photons.
- But QCD dynamics is very different.




$B \to \mu^+\mu^-$
$\langle 0|\bar{q}\gamma^\mu\gamma_5 b|\bar{B}\rangle$

$B \to \pi^+\pi^-$
$\langle \pi^+\pi^-|Q_i|\bar{B}\rangle$

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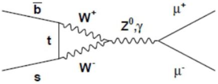


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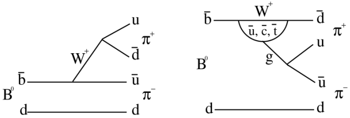



$B \to \pi^+\pi^-$
$\langle \pi^+\pi^-|Q_i|\bar{B}\rangle$

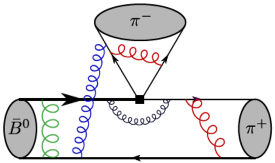
"QCD factorization" [BBNS, 1999-2001], later understood and formulated as a SCET$_{\rm II}$ problem:

$$\text{QCD} \;\overset{\text{remove h}}{\longrightarrow}\; \text{SCET}_{\rm I} \;\overset{\text{remove hc}}{\longrightarrow}\; \text{SCET}_{\rm II}(c,\bar{c},s)$$

$$\langle M_1 M_2|Q_i|\bar{B}\rangle = \underbrace{F^{BM_1}(0)}_{\text{form factor}} \int_0^1 du\, T_i^I(u)\phi_{M_2}(u)$$

$$+ \int_0^1 dz du\, H_i^{\rm II}(z,u) \int_0^\infty d\omega \int_0^1 dv\, J(\omega,u,v)\, \underbrace{\phi_B(\omega)\phi_{M_1}(v)\phi_{M_2}(u)}_{\text{LCDAs}}$$

# Including virtual QED effects into the factorization theorem




SCET$_\mathrm{I}$ operators

$$\mathcal{O}^\mathrm{I}(t) = [\bar{\chi}_{\bar{C}}(tn_-)\not{n}_-\gamma_5\chi_{\bar{C}}]\,[\bar{\chi}_C h_v]$$

$$\mathcal{O}^\mathrm{II}(t,s) = \underbrace{[\bar{\chi}_{\bar{C}}(tn_-)\not{n}_-\gamma_5\chi_{\bar{C}}]}_{M_2}\,\underbrace{[\bar{\chi}_C \not{\mathcal{A}}_{C,\perp}(sn_+)h_v]}_{B\to M_1}$$

*QCD factorization formula*

$$\langle M_1M_2|Q_i|\bar{B}\rangle = F^{BM_1}(0)\int_0^1 du\, T_i^{\mathrm{I,QCD}}(u)\, f_{M_2}\phi_{M_2}(u)$$
$$+\int_0^\infty d\omega\int_0^1 dudv\, T_i^{\mathrm{II,QCD}}(z,u)\, f_B\phi_B(\omega) f_{M_1}\phi_{M_1}(v) f_{M_2}\phi_{M_2}(u)$$

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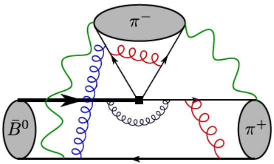



SCET$_{\mathrm{I}}$ operators

$$\mathcal{O}^{\mathrm{I}}(t) = [\bar{\chi}_{\bar{C}}(tn_-)\not{n}_-\gamma_5\chi_{\bar{C}}]\,[\bar{\chi}_C\,\mathbf{S}^{\dagger(Q_{M_2})}_{n_+}h_v]$$

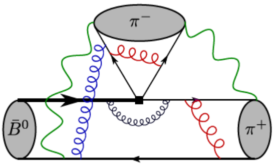
$$\mathcal{O}^{\mathrm{II}}(t,s) = [\bar{\chi}_{\bar{C}}(tn_-)\not{n}_-\gamma_5\chi_{\bar{C}}]\,[\bar{\chi}_C\not{\mathcal{A}}_{C,\perp}(sn_+)\,\mathbf{S}^{\dagger(Q_{M_2})}_{n_+}h_v]$$

$$S^{(q)}_{n_\pm} = \exp\left\{-iQ_q e\int_0^\infty ds\, n_\pm A_s(sn_\pm)\right\}$$

*QCD + QED factorization formula*

$$\langle M_1M_2|Q_i|\bar{B}\rangle_{|\mathrm{non-rad.}} = \mathcal{F}^{BM_1}_{Q_2}(0)\int_0^1 du\, T^{\mathrm{I,QCD+QED}}_{i,Q_2}(u)\, f_{M_2}\Phi_{M_2}(u)$$
$$+\int_{-\infty}^{\infty} d\omega\int_0^1 du dv\, T^{\mathrm{II,QCD+QED}}_{i,\otimes}(z,u)\, f_B\Phi_{B,\otimes}(\omega) f_{M_1}\Phi_{M_1}(v) f_{M_2}\Phi_{M_2}(u)$$

# Including virtual QED effects into the factorization theorem




SCET$_\mathrm{I}$ operators

$$\mathcal{O}^{\mathrm{I}}(t) = [\bar{\chi}_{\bar{C}}(tn_-)\not{n}_-\gamma_5\chi_{\bar{C}}]\,[\bar{\chi}_C\,\mathbf{S}^{\dagger(Q_{M_2})}_{n_+}h_v]$$

$$\mathcal{O}^{\mathrm{II}}(t,s) = [\bar{\chi}_{\bar{C}}(tn_-)\not{n}_-\gamma_5\chi_{\bar{C}}]\,[\bar{\chi}_C\not{A}_{C,\perp}(sn_+)\,\mathbf{S}^{\dagger(Q_{M_2})}_{n_+}h_v]$$

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- Formula retains its form, but the hadronic matrix elements are generalized. They become process-dependent through the directions and charges of the *other* particles.
- Computation of $\mathcal{O}(\alpha_{\mathrm{em}})$ corrections to the h and hc short-distance coefficient (all poles cancel).

# Example of a hard-scattering kernel

Same generalization for colour-allowed decays to $D^+L^-$.
Hard scattering kernel

$$
\begin{aligned}
H_-^{(1)} = &- Q_d^2 \left\{ \frac{L_b^2}{2} + L_b \left( \frac{5}{2} - 2\ln(u(1-z)) \right) + h\big(u(1-z)\big) + \frac{\pi^2}{12} + 7 \right\} \\
&- Q_u^2 \left\{ \frac{L_c^2}{2} + L_c \left( \frac{5}{2} + 2\pi i - 2\ln\left( \bar{u}\,\frac{1-z}{z} \right) \right) + h\left( \bar{u}\left( 1 - \frac{1}{z} \right) \right) + \frac{\pi^2}{12} + 7 \right\} \\
&+ Q_d Q_u \left\{ \frac{L_b^2}{2} + \frac{L_c^2}{2} - 6L_\nu + 2L_b \left( 2 - \ln(\bar{u}(1-z)) \right) \right. \\
&\left. - 2L_c \left( 1 - i\pi + \ln\left( u\,\frac{1-z}{z} \right) \right) + g\big(\bar{u}(1-z)\big) + g\left( u\left( 1 - \frac{1}{z} \right) \right) + \frac{\pi^2}{6} - 12 \right\} \\
&+ Q_d Q_u f(z)\,, \\
H_+^{(1)} = &- Q_d^2 \sqrt{z}\, w\big(u(1-z)\big) - Q_u^2 \frac{1}{\sqrt{z}}\, w\left( \bar{u}\left( 1 - \frac{1}{z} \right) \right) - Q_d Q_u \sqrt{z}\, \frac{\ln z}{1-z}\,,
\end{aligned}
$$

$$
z \equiv \frac{m_c^2}{m_b^2}\,, \qquad L_c \equiv \ln \frac{\mu^2}{m_c^2} = L_b - \ln z\,, \qquad L_\nu \equiv \ln \frac{\nu^2}{m_b^2}\,,
$$

$$
h(s) \equiv \ln^2 s - 2\ln s + \frac{s\ln s}{1-s} - 2\mathrm{Li}_2\left( \frac{s-1}{s} \right),
$$

$$
f(z) \equiv \left( 1 - \frac{1+z}{1-z}\ln z \right) L_b + \frac{\ln z}{1-z} \left( \frac{1}{2}(1+z)\ln z - 2 - z \right), \ldots
$$

# Enlightened LCDAs

2108.05589 (light mesons) + in preparation (heavy mesons),
with P. Böer, J. Toelstede and K. K. Vos

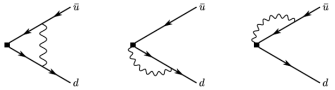
# LCDA of a charged pion in QCD×QED

$$\langle \pi^-(p)|R_c^{(Q_M)}\,(\bar{d}W^{(d)})(tn_+)\frac{\not{n}_+}{2}(1-\gamma_5)(W^{\dagger(u)}u)(0)|0\rangle = \frac{in_+p}{2}\int_0^1 du\, e^{iu(n_+p)t} f_M \Phi_M(u;\mu)$$

QED part of the Wilson lines does not combine to a finite-distance Wilson line.

Renormalization/evolution kernel for the (anti-)collinear operator well-defined after soft rearrangement

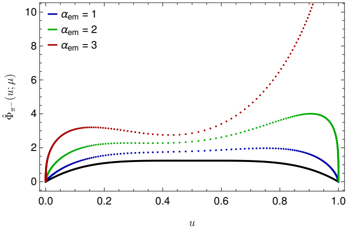
$$\Gamma(u,v;\mu) = -\frac{\alpha_{\rm em} Q_M}{\pi}\,\delta(u-v)\left(Q_M\left(\ln\frac{\mu}{2E}+\frac{3}{4}\right) - Q_d\ln u + Q_u \ln\bar{u}\right)$$
$$-\left(\frac{\alpha_s C_F}{\pi}+\frac{\alpha_{\rm em}}{\pi}Q_u Q_d\right)\left[\left(1+\frac{1}{v-u}\right)\frac{u}{v}\,\theta(v-u)+\left(1+\frac{1}{u-v}\right)\frac{1-u}{1-v}\,\theta(u-v)\right]_+$$




- The cusp anomalous dimension, endpoint logarithms $\ln u$, $\ln(1-u)$ and energy dependence are a remnant of the soft physics.
- Gegenbauer polynomials are no longer eigenfunctions, asymptotic behaviour $\Phi_\pi(u,\mu) \stackrel{\mu\to\infty}{\to} 6u(1-u)$ no longer holds. QED evolution is asymmetric and endpoint behaviour changes from linear.

# Endpoint behaviour and numerical QED effect

Endpoint behaviour $u \to 0$ governed by the an asymptotic kernel, which has the same properties as the kernel for the $B$-meson LCDA!
RGE kills itself at high-scales by developing too singular behaviour.



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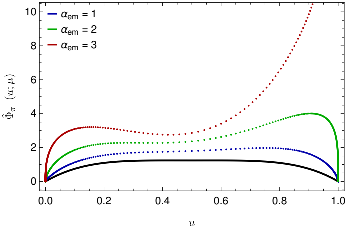



Not relevant for real value of $\alpha_{\rm em}$.
Expand in Gegenbauer polynomials, which, however, are not eigenfunctions and mix.

Example:

$$\left\langle \bar{u}^{-1} \right\rangle_{M^-}(\mu) = \int_0^1 \frac{du}{1-u} \Phi_{M^-}(u;\mu) = 3Z_\ell(\mu) \sum_{n=0}^{\infty} a_n^{M^-}(\mu)$$

$$\left\langle \bar{u}^{-1} \right\rangle_{\pi^-}(5.3\,\text{GeV}) = 0.9997\Big|^{\text{QED}}_{\text{point charge}} \Big(3.285^{+0.05}_{-0.05}\Big|_{\text{LL}} - 0.020\Big|_{\text{NLL}} + 0.017\Big|^{\text{QED}}_{\text{partonic}}\Big)$$

$$\left\langle \bar{u}^{-1} \right\rangle_{\pi^-}(80.4\,\text{GeV}) = 0.985\Big|^{\text{QED}}_{\text{point charge}} \Big(3.197^{+0.03}_{-0.03}\Big|_{\text{LL}} - 0.022\Big|_{\text{NLL}} + 0.042\Big|^{\text{QED}}_{\text{partonic}}\Big)$$

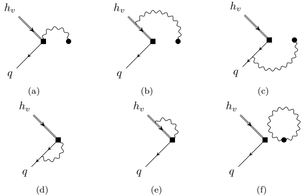
(Initial value: 3.42 at $\mu = 1$ GeV.

QED effects of similar size as NLL evolution for the inverse moments.

# B-LCDA alias soft function for $B \to M_1 M_2$ in QCD$\times$QED

$$\frac{1}{R_c^{(Q_{M_1})} R_{\bar{c}}^{(Q_{M_2})}} \langle 0 | \bar{q}_s^{(q)}(tn_-)[tn_-, 0]^{(q)} \not{n}_- \gamma_5 h_v(0) S_{n_+}^{\dagger (Q_{M_2})}(0) S_{n_-}^{\dagger (Q_{M_1})}(0) | \bar{B}_v \rangle$$
$$= iF_{\text{stat}}(\mu) \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \, \Phi_{B,\otimes}(\omega, \mu)$$

- Four different soft functions depending on final state charges $00, -0, 0-, +-$.
- Not really LCDAs, but soft functions: contain electromagnetic final-state rescattering.
- Different support in $\omega = n_- \cdot l$ than in QCD where ($\omega \in [0, \infty[$).




If $Q_{M_2} \neq 0$, the out-going meson with $n_- \cdot q = m_B \to \infty$ can *absorb* soft photons of arbitrary $n_- \cdot l$.

$\Longrightarrow$ $B$-"LCDA" has support $\omega \in ] - \infty, \infty[$

# B-LCDA alias soft function for $B \to M_1 M_2$ in QCD×QED (II)

Anomalous dimension kernel for $B \to M_1^+ M_2^-$.
($>$) ($<$) means $\omega > 0$ ($\omega < 0$)

$$\begin{aligned}
\Gamma^{>}_{+-}(\omega,\omega',\mu) &= \frac{\alpha_s(\mu)C_F}{\pi}\left[\left(\ln\frac{\mu}{\omega - i0} - \frac{5}{4}\right)\delta(\omega-\omega') - F^{>}(\omega,\omega')\right] \\
&+ \frac{\alpha_{\rm em}(\mu)}{\pi}\left[\left((Q_d^2 - 2Q_dQ_{M_2})\ln\frac{\mu}{\omega - i0} + i\pi Q_u Q_{M_2} - \frac{5}{4}Q_d^2\right)\delta(\omega-\omega')\right. \\
&\left. -Q_d^2 F^{>}(\omega,\omega') + Q_d Q_{M_2} G^{>}(\omega,\omega')\right.
\end{aligned}$$

$$\begin{aligned}
\Gamma^{<}_{+-}(\omega,\omega',\mu) &= \frac{\alpha_s(\mu)C_F}{\pi}\left[\left(\ln\frac{\mu}{\omega - i0} - \frac{5}{4}\right)\delta(\omega-\omega') - G^{<}(\omega,\omega')\right] \\
&+ \frac{\alpha_{\rm em}(\mu)}{\pi}\left[\left((Q_d^2 - 2Q_dQ_{M_2})\ln\frac{\mu}{\omega - i0} + i\pi Q_u Q_{M_2} - \frac{5}{4}Q_d^2\right)\delta(\omega-\omega')\right. \\
&\left. -Q_d^2 G^{<}(\omega,\omega') + Q_d Q_{M_2} F^{<}(\omega,\omega')\right]
\end{aligned}$$

The distribution $F^{<}$ in the QED correction generates support for $\omega < 0$ even if the initial condition $\Phi_B(\omega')$ has support only for $\omega' > 0$.

$$F^{>}(\omega,\omega') = \omega\left[\frac{\theta(\omega'-\omega)}{\omega'(\omega'-\omega)}\right]_+ + \left[\frac{\theta(\omega-\omega')}{\omega-\omega'}\right]_{\oplus}$$

$$F^{<}(\omega,\omega') = \omega\left[\frac{\theta(\omega-\omega')}{\omega'(\omega-\omega')}\right]_+ + \left[\frac{\theta(\omega'-\omega)}{\omega'-\omega}\right]_{\ominus}$$

# Inverse moments relevant to leading-power factorization theorems

$$\frac{1}{\lambda_B(\mu)} = \int_{-\infty}^{\infty} \frac{d\omega}{\omega - i0}\, \phi_B(\omega,\mu) \qquad \frac{\sigma_n(\mu)}{\lambda_B(\mu)} = \int_{-\infty}^{\infty} \frac{d\omega}{\omega - i0} \ln^n\left(\frac{\mu_0}{\omega - i0}\right) \phi_B(\omega,\mu)$$

$$\begin{aligned}\frac{d}{d\ln\mu}\lambda_B^{-1}(\mu) &= \frac{\alpha_s C_F}{\pi}\left[-\sigma_1(\mu) + \ln\frac{\mu_0}{\mu} + \frac{5}{4}\right]\lambda_B^{-1}(\mu) \\ &\quad + \frac{\alpha_{\rm em}}{\pi}\left[(Q_d^2 - 2Q_dQ_{M_2})\left(-\sigma_1(\mu) + \ln\frac{\mu_0}{\mu}\right) + \frac{5}{4}Q_d^2 - i\pi Q_uQ_{M_2}\right]\lambda_B^{-1}(\mu)\end{aligned}$$

$$\begin{aligned}\frac{d\sigma_1}{d\ln\mu} &= \frac{\alpha_s C_F + \alpha_{\rm em}Q_u^2}{\pi}\left[-\sigma_2 + \sigma_1^2\right] - \frac{\alpha_{\rm em}2Q_dQ_{M_2}}{\pi}\left[-\sigma_2 + \sigma_1^2\right] \\ \frac{d\sigma_2}{d\ln\mu} &= \frac{\alpha_s C_F + \alpha_{\rm em}Q_u^2}{\pi}\left[-\sigma_3 + \sigma_1\sigma_2 + 4\zeta_3\right] - \frac{\alpha_{\rm em}2Q_dQ_{M_2}}{\pi}\left[-\sigma_3 + \sigma_1\sigma_2 + 2\zeta_3\right]\end{aligned}$$

- $i0$-prescription arises from the hard-collinear propagators in the jet function that supplies the $1/\omega$.
- Hierarchy of coupled equations as in QCD. Otherwise numerical solution of the evolution equation.

# Size of QED effects

2009.10615 ($\pi K$) and 2107.03819 (colour-allowed $DL$),
with P.Böer, G. Finauri, J. Toelstede and K.K. Vos

# Numerical estimate of QED effects for $\pi K$ and $D^+L^-$ final states

Up to now virtual corrections to the non-radiative amplitude.
Add (ultra)soft photon radiation.

- Electroweak scale to $m_B$: QED corrections to Wilson coefficients included
- $m_B$ to $\mu_c$: $\mathcal{O}(\alpha_{\rm em})$ corrections to short-distance kernels included.
  QED effects in form factors and LCDA not included.
- Ultrasoft photon radiation included (same formalism as for $\mu^+\mu^-$ with $m_\mu \to m_\pi, m_K$)

$$U(M_1M_2) = \left(\frac{2\Delta E}{m_B}\right)^{-\frac{\alpha_{\rm em}}{\pi}\left(Q_B^2 + Q_{M_1}^2\left[1+\ln\frac{m_{M_1}^2}{m_B^2}\right] + Q_{M_2}^2\left[1+\ln\frac{m_{M_2}^2}{m_B^2}\right]\right)} \qquad (M_1, M_2 \text{ light mesons})$$

$$U(\pi^+K^-) = 0.914$$
$$U(\pi^0K^-) = U(K^-\pi^0) = 0.976$$
$$U(\pi^-\bar{K}^0) = 0.954 \qquad [\text{for } \Delta E = 60\,\text{MeV}]$$
$$U(\bar{K}^0\pi^0) = 1$$

$$U(D^+K^-) = 0.960$$
$$U(D^+\pi^-) = 0.938$$

# Isospin-protected ratios / sum rules for the $\pi K$ final states

Consider ratios / sums where some QCD uncertainties drop out.

[MB, Neubert, 2003]

$$R_L = \frac{2\mathrm{Br}(\pi^0 K^0) + 2\mathrm{Br}(\pi^0 K^-)}{\mathrm{Br}(\pi^- K^0) + \mathrm{Br}(\pi^+ K^-)} = R_L^{\mathrm{QCD}} + \cos\gamma \mathrm{Re}\, \delta_{\mathrm{E}} + \delta_U$$

$$R_L^{\mathrm{QCD}} - 1 \approx (1 \pm 2)\% \qquad \delta_E \approx 0.1\% \qquad \delta_U = 5.8\%$$

QED correction larger than QCD and QCD uncertainty, but short-distance QED negligible.

[Gronau, Rosner, 2006]

$$\Delta(\pi K) \equiv A_{\mathrm{CP}}(\pi^+ K^-) + \frac{\Gamma(\pi^- \bar{K}^0)}{\Gamma(\pi^+ K^-)} A_{\mathrm{CP}}(\pi^- \bar{K}^0) - \frac{2\Gamma(\pi^0 K^-)}{\Gamma(\pi^+ K^-)} A_{\mathrm{CP}}(\pi^0 K^-) - \frac{2\Gamma(\pi^0 \bar{K}^0)}{\Gamma(\pi^+ K^-)} A_{\mathrm{CP}}(\pi^0 \bar{K}^0) \equiv \Delta(\pi K)^{\mathrm{QCD}} + \delta\Delta(\pi K)$$

$$\Delta(\pi K)^{\mathrm{QCD}} = (0.5 \pm 1.1)\% \qquad \delta_\Delta(\pi K) \approx -0.4\%$$

QED correction of similar size but small.

# Numerical estimate of QED effects for $D^{(*)+}L^-$ final states

$$R_L^{(0),(*)}(\Delta E) \equiv \frac{\Gamma(\bar{B}_d \to D^{(*)+}L^-)(\Delta E)}{d\Gamma^{(0)}(\bar{B}_d \to D^{(*)+}\mu^-\bar{\nu}_\ell)/dq^2 \,|_{q^2=m_L^2}}$$

$$R_L^{(*)}(\Delta E) \equiv \frac{\Gamma(\bar{B}_d \to D^{(*)+}L^-)(\Delta E)}{d\Gamma(\bar{B}_d \to D^{(*)+}\mu^-\bar{\nu}_\ell)(\Delta E)/dq^2 \,|_{q^2=m_L^2}}$$

- Short-distance QED effects $\approx -1\%$, ultrasoft up to $\approx -7\%$ for pions, depending on the semi-leptonic normalization.
- Not large enough to explain the apparent amplitude deficit of $-15\%$ [Bordone et al., 2020], but highlights the importance of proper treatment of ultrasoft radiation effects.

| $R_L^{(*)}$ | LO | QCD NNLO | $+\delta_{\rm QED}$ | $+\delta_{\rm U}$ ($\delta_{\rm U}^{(0)}$) |
|---|---|---|---|---|
| $R_\pi$ | $0.969 \pm 0.021$ | $1.078^{+0.045}_{-0.042}$ | $1.069^{+0.045}_{-0.041}$ | $1.074^{+0.046}_{-0.043}$ $(1.003^{+0.042}_{-0.039})$ |
| $R_\pi^*$ | $0.962 \pm 0.021$ | $1.069^{+0.045}_{-0.041}$ | $1.059^{+0.045}_{-0.041}$ | $1.065^{+0.047}_{-0.042}$ $(0.996^{+0.043}_{-0.039})$ |
| $R_K \cdot 10^2$ | $7.47 \pm 0.07$ | $8.28^{+0.27}_{-0.26}$ | $8.21^{+0.27}_{-0.26}$ | $8.44^{+0.29}_{-0.28}$ $(7.88^{+0.26}_{-0.25})$ |
| $R_K^* \cdot 10^2$ | $6.81 \pm 0.16$ | $7.54^{+0.31}_{-0.29}$ | $7.47^{+0.30}_{-0.29}$ | $7.68^{+0.32}_{-0.30}$ $(7.19^{+0.29}_{-0.28})$ |

Table 3: Theoretical predictions for $R_L^{(*)}$ expressed in GeV$^2$ at LO, NNLO QCD and subsequently adding $\delta_{\rm QED}$ given in (82) and the ultrasoft effects $\delta_{\rm U}$ (or in brackets $\delta_{\rm U}^{(0)}$). The last column presents our final results.