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More on a new sphaleron in SU(3) Yang-Mills-Higgs theory

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The Standard Model beyond perturbation theory

Most predictions of the Standard Model (SM) are made using perturbation theory, but not exclusively:

Two non-perturbative examples are

- The **Instanton I** [Belavin, Polyakov, Schwatz & Tyupkin, 1975] [1] of QCD ($= SU(3)$ Yang-Mills theory)
- The **Sphaleron S** [Klinkhamer & Manton, 1984] [2] of the electroweak SM ($= SU(2) \times U(1)$ Yang-Mills-Higgs theory)

- **Instanton**: localized, finite-action, solution of field equations in **imaginary** time
- **Topological soliton**: static, **stable**, finite-energy, solution of field equations in **real** time
- **Sphaleron**: static, **unstable**, finite-energy, solution of field equations in **real** time

Topological objects in field theory

Example: (1+1)-dimensional scalar field with a double well potential

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right) - \frac{\lambda}{4} \left(\phi^2 - v^2\right)^2$$

the corresponding field equations are:

$$\partial_t^2 \phi - \partial_x^2 \phi + \lambda \left(\phi^2 - v^2\right) \phi = 0$$

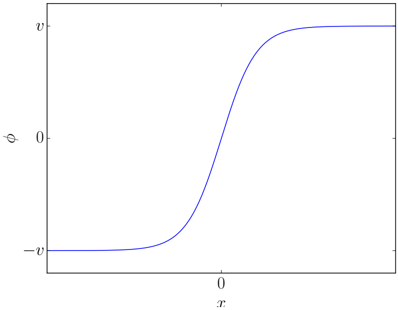
We are now looking for **static** solutions ($\partial_t \phi = 0$).

It is easy to find two static solutions: $\phi = \pm v$

Topological objects in field theory

A little harder is to find a non-trivial one, connecting both constant values:

$$\phi = v \tanh\left[v\sqrt{\frac{\lambda}{2}}\, x\right]$$



At $x = \pm\infty$ the field is fixed to $\phi = \pm v \;\Rightarrow$ Hence, the **kink** solution is stable.

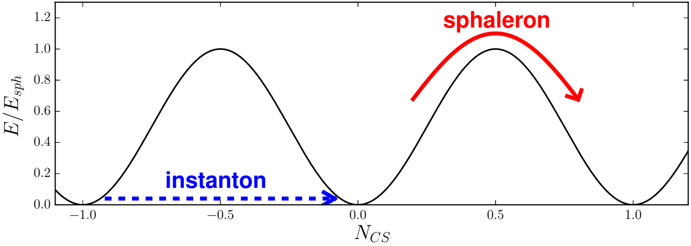
Sphalerons and Instantons

A jump forward to (3+1)-dimensional Yang-Mills-Higgs theory:

Here we have not 2 ($\phi = \pm v$), but infinitely many static vacuum configurations [all gauge transformations of $A_m(x_1, x_2, x_3) = 0$], labeled by integers N_{CS}:

$$A_m = -\frac{1}{g}\Omega\left(\partial_m \Omega^{-1}\right), \quad \Omega \in SU(N)$$

Going from one vacuum to another, one needs to go over (sphaleron) or tunnel through (instanton) an energy barrier:



The sphalerons S and $\widehat{S}$

The well-known sphaleron S exists in $SU(2) \times U(1)$ Yang-Mills-Higgs theory [2].

It can be embedded in $SU(3)$.

The $\widehat{S}$ is a completely different object, of distinct structure, which exists in $SU(3)$ Yang-Mills-Higgs theory, but not in $SU(2)$.

Finding the $\widehat{S}$ sphaleron configuration

We start from the $SU(3)$ Yang-Mills-Higgs theory Lagrangian with a single complex scalar triplet Φ:

$$\mathcal{L} = \frac{1}{2}\,\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^\dagger(D^\mu\Phi) - \lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2,$$

with the YM field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\,[A_\mu, A_\nu]$, the covariant derivative $D_\mu = \partial_\mu + gA_\mu$ and the gauge fields $A_\mu(x) = A^a_\mu(x)T^a$, with $su(3)$ generators $T^a = \lambda^a/(2i)$ and g the strong coupling constant.

We are looking for solutions of the corresponding static field equations:

$$[D_i, F_{ij}] = g\left(\Phi^\dagger\, T^a\,(D_j\Phi) - (D_j\Phi)^\dagger\, T^a\Phi\right) T^a,$$

$$D_iD_i\Phi = 2\lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)\Phi,$$

with appropriate boundary conditions.

Boundary conditions

Obtain constraints on the boundary conditions at infinity by demanding **finiteness of energy**:

$$A_m(r \to \infty) = -\frac{1}{g} U(x) \left(\partial_m U(x)^{-1} \right), \quad U(x) \in SU(3)$$

$$\Phi(r \to \infty) = \frac{v}{\sqrt{2}} U(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and boundary conditions at the origin by demanding **regularity**:

$$A_m(r \to 0) = 0, \qquad \Phi(r \to 0) = 0$$

The key step is finding the right U(x).
Here, insights into the topology of the group space $SU(3)$ are crucial.

Symmetries and structure

The $\widehat{S}$ has **axial symmetry** around the z-axis and **reflection symmetry** on the equator:

This means that a change of the azimuthal angle ϕ and a parity transformation can be compensated by respective gauge transformations

The $\widehat{S}$ also has a residual **gauge symmetry**, which we need to fix.

A self-consistent Ansatz [3], based on these symmetries, made in the T,V,U-spin basis of $su(3)$[1], is remarkably structured:

A_r and A_θ require only one $su(2)$ subalgebra of $su(3)$ and A_ϕ precisely the other 5 remaining generators of $su(3)$.

[1] To be precise, a slight modification of this basis, since T_3, U_3 and V_3 are not independent

The Ansatz

The following Ansatz was obtained in [3] with fixed radial gauge ($A_r = 0$):

$$
\begin{aligned}
gA_r(r,\theta,\phi) =& 0 \\
gA_\phi(r,\theta,\phi) =& \alpha_1(r,\theta)\cos\theta\, T_\rho + \alpha_2(r,\theta) V_\rho - \alpha_3(r,\theta)\cos\theta\, U_\rho + \alpha_4(r,\theta)\frac{\lambda_3}{2i} + \alpha_5(r,\theta)\frac{\lambda_8}{2i}, \\
gA_\theta(r,\theta,\phi) =& \alpha_6(r,\theta)\ T_\phi + \alpha_7(r,\theta)\cos\theta\ V_\phi - \alpha_8(r,\theta) U_\phi, \\
\Phi(r,\theta,\phi) =& \frac{v}{\sqrt{2}}\left[\beta_1(r,\theta)\lambda_3 + \beta_2(r,\theta)\cos\theta\ 2iT_\rho + \beta_3(r,\theta)\ 2iV_\rho\right]\begin{pmatrix}1\\0\\0\end{pmatrix}
\end{aligned}
$$

This *Ansatz* solves most components of the field equations directly, leaving “only” 11 partial differential equations (PDEs) in r and θ to solve.

The boundary conditions of the 11 real functions α_i and β_j are obtained by matching with the boundary conditions of the fields.

Numerical solution

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There is no analytic solution to these equations. In fact, it is even hard to solve them numerically.

We use 2 different approaches:

- Solve the PDEs directly
- Minimize the energy functional (i.e. find an extremum of the action)

Both yield similar results, which solve the PDEs approximately[2].

The obtained numerical value for the energy is:

$$E_{\widehat{S}}\Big|_{\lambda/g^2=0} \approx 1.283 \left[\frac{4\pi v}{g}\right] \approx 0.844\, E_S,$$

with $E_S \equiv 1.52[4\pi v/g]$ the energy of the $SU(2)$ sphaleron S embedded in $SU(3)$.

[2] All calculations were performed on the ITP cluster.

$\widehat{S}$ energy density

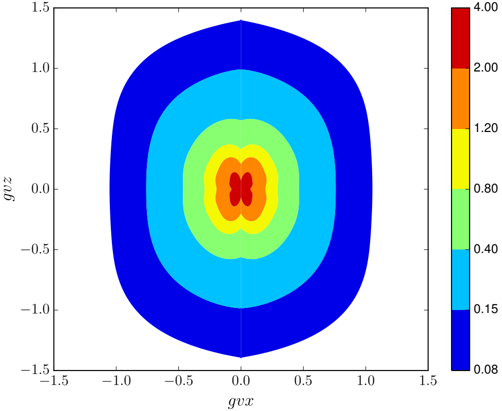

Fig.: Contour plot of the $\widehat{S}$ energy density for $\lambda/g^2 = 0$.

Conclusion and open questions

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It is remarkable, that the obtained energy for $\widehat{S}$ (with 8 gauge fields) is lower, than that of S (with 4 gauge fields).

The result for the energy $E_{\widehat{S}}$ shows, that related processes may not be very much suppressed.

Which physical effects does the $\widehat{S}$ have on the non-perturbative dynamics of QCD?

Can we see them in the quark-gluon plasma at heavy ion colliders (e.g. RHIC)?

Does the $\widehat{S}$ gauge field contribute significantly to the QCD glueball content?

References

[1] A.A. Belavin, A.M. Polyakov, A.S. Schwartz and Y.S. Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations*, Phys. Lett. B59 (1975) 85.

[2] F.R. Klinkhamer and N.S. Manton, *A saddle point solution in the Weinberg-Salam theory*, Phys. Rev. D30 (1984) 2212.

[3] F.R. Klinkhamer and C. Rupp, *A sphaleron for the non-Abelian anomaly*, Nucl. Phys. B709 (2005) 171 [arXiv:hep-th/0410195].

Backup

- $\widehat{S}$ map:

$$U(\theta,\phi) = \begin{pmatrix} \cos^2\theta & -\cos\theta\sin\theta\ e^{i\phi} & \sin\theta\ e^{-i\phi} \\ -\cos\theta\sin\theta\ e^{i\phi} & \sin^2\theta\ e^{2i\phi} & \cos\theta \\ -\sin\theta\ e^{-i\phi} & -\cos\theta & 0 \end{pmatrix}$$

- $su(3)$ basis:

$$T_\phi = \sin\phi\frac{\lambda_1}{2i} + \cos\phi\frac{\lambda_2}{2i} \qquad T_\rho = \cos\phi\frac{\lambda_1}{2i} - \sin\phi\frac{\lambda_2}{2i}$$

$$V_\phi = -\sin\phi\frac{\lambda_4}{2i} + \cos\phi\frac{\lambda_5}{2i} \qquad V_\rho = \cos\phi\frac{\lambda_4}{2i} + \sin\phi\frac{\lambda_5}{2i}$$

$$U_\phi = -\sin(2\phi)\frac{\lambda_6}{2i} + \cos(2\phi)\frac{\lambda_7}{2i} \qquad U_\rho = \cos(2\phi)\frac{\lambda_6}{2i} + \sin(2\phi)\frac{\lambda_7}{2i}$$

Backup

- Example PDE:

$$
\begin{aligned}
&r^2 \sin\theta \cos\theta \partial_r^2 \alpha_1 + \sin\theta\cos\theta\partial_\theta^2\alpha_1 - \frac{3}{2}\partial_\theta\alpha_1 + \frac{1}{2}\cos 2\theta\ \partial_\theta\alpha_1 - \sin\theta\cos\theta\alpha_1\alpha_6^2 \\
&- \frac{1}{4}\sin\theta\cos^3\theta\alpha_1\alpha_7^2 - \frac{1}{4}\sin\theta\cos\theta\alpha_1\alpha_8^2 - \frac{1}{4}g^2r^2v^2\sin\theta\cos\theta\alpha_1\beta_1^2 \\
&- \frac{1}{4}g^2r^2v^2\sin\theta\cos^3\theta\alpha_1\beta_2^2 - \sin\theta\partial_\theta\alpha_2\alpha_8 - \frac{3}{4}\sin\theta\cos\theta\alpha_2\alpha_6\alpha_7 - \frac{1}{2}\sin\theta\alpha_2\partial_\theta\alpha_8 \\
&+ \frac{1}{2}\cos\theta\alpha_2\alpha_8 - \frac{1}{4}g^2r^2v^2\sin\theta\cos\theta\alpha_2\beta_2\beta_3 - \sin\theta\cos^2\theta\partial_\theta\alpha_3\alpha_7 + \frac{3}{4}\sin\theta\cos\theta\alpha_3\alpha_6\alpha_8 \\
&- \frac{1}{2}\sin\theta\cos^2\theta\alpha_3\partial_\theta\alpha_7 + \cos\theta\alpha_3\alpha_7 - \frac{1}{2}\cos 2\theta\ \cos\theta\alpha_3\alpha_7 - \frac{1}{4}g^2r^2v^2\sin\theta\cos\theta\alpha_3\beta_1\beta_3 \\
&+ 2\sin\theta\partial_\theta\alpha_4\alpha_6 + \sin\theta\alpha_4\partial_\theta\alpha_6 - \cos\theta\alpha_4\alpha_6 - \frac{1}{2}\sqrt{3}\sin\theta\cos\theta\alpha_5\alpha_7\alpha_8 \\
&- \frac{g^2r^2v^2\sin\theta\cos\theta\alpha_5\beta_1\beta_2}{2\sqrt{3}} + \sin\theta\partial_\theta\alpha_6 - \cos\theta\alpha_6 + \frac{3}{2}\sin\theta\cos\theta\alpha_7\alpha_8 \\
&+ \frac{1}{2}g^2r^2v^2\sin\theta\cos\theta\beta_1\beta_2 = 0
\end{aligned}
$$