OUTER AUTOMORPHISM ANOMALIES

Tom Melia, Kavli IPMU
Based on work with Brian Henning, Xiaochuan Lu, and Hitoshi Murayama
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CHARGE CONJUGATION, PARITY & ANOMALIES

All crucial to developing our understanding of gauge theories

All played an important historic role through studies of the strong and electromagnetic interactions

This talk is about their intersection, namely anomalies of symmetries like charge conjugation and parity, and their consequences

CHARGE CONJUGATION, PARITY & ANOMALIES

Main message: anomalies of charge conjugation and parity are useful

charge/parity — Gauge / Gauge

Check whether it is a symmetry of quantum theory or not

charge/parity — Global / Global

IR - IR must match

UV - IR spontaneous breaking or not

OUTLINE

- Outer Automorphisms
- Anomaly matching IR dual theories
- Spontaneous breaking
- Outlook

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CHARGE CONJUGATION

Consider SU(N) gauge theory

Charge conjugation interchanges a representation with its complex conjugate. It is an example of an outer automorphism of the gauge group.

Isomorphism of G onto itself that can not be written in the form $g \to hgh^{-1}$

For some fixed $h \in G$

PARITY

Similarly, parity is an outer automorphism that exchanges two inequivalent spinor reps of SO(2r)

e.g. for $SO(6)$ $\mathcal{P}_1 = (+,-,-,-,-,-)$

is the element of O(6)

Isomorphism of G onto itself that can not be written in the form $g \to hgh^{-1}$

For some fixed $h \in G$

CHARGE CONJUGATION

We can see it very explicitly. We require charge conjugation to be:

1. Linear
2. Unitary
3. C^2 = 1 (up to a phase)
4. Compatible with SU(N)

CHARGE CONJUGATION

Charge conjugation works on the direct sum of fundamental and anti-fundamental reps $\mathrm{N} \oplus \overline{\mathrm{N}}$.

(Action on other irreps specified as they are are tensor products of these)

1. **Linear**

$$\mathcal{C}\begin{pmatrix}\mathbf{N}\\ \overline{\mathbf{N}}\end{pmatrix} = \begin{pmatrix}0 & C_-\\ C_+ & 0\end{pmatrix}\begin{pmatrix}\mathbf{N}\\ \overline{\mathbf{N}}\end{pmatrix}$$

2. **Unitary**

$$\mathcal{C}^\dagger\mathcal{C} = \begin{pmatrix}0 & C_+^\dagger\\ C_-^\dagger & 0\end{pmatrix}\begin{pmatrix}0 & C_-\\ C_+ & 0\end{pmatrix} = 1$$

➡ $C_-^\dagger C_- = C_+^\dagger C_+ = 1$

3. **C^2 = 1 ***

$$\mathcal{C}^2 = \begin{pmatrix}0 & C_-\\ C_+ & 0\end{pmatrix}\begin{pmatrix}0 & C_-\\ C_+ & 0\end{pmatrix} = 1$$

➡ $C_-C_+ = C_+C_- = 1$

➡ $C_- = C_+^{-1} = C_+^\dagger$

*** up to phase**

write $C_+ = C$ and $C_- = C^\dagger$

CHARGE CONJUGATION

4. Compatible with SU(N)

$G = SU(n) \rtimes \mathcal{C}$ where SU(N) is normal subgroup: $g\,h\,g^{-1} \in SU(N)$

for $h \in SU(N)$

$$\mathcal{C}\,h\,\mathcal{C}^{-1} \in SU(N)$$

$$\mathcal{C} = \begin{pmatrix} 0 & C^* \\ C & 0 \end{pmatrix}$$

INVARIANT SUBGROUPS

C of SU(N)

Under unitary transform

$$C' = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} 0 & C^* \\ C & 0 \end{pmatrix} \begin{pmatrix} U^\dagger & 0 \\ 0 & U^T \end{pmatrix} = \begin{pmatrix} 0 & UC^*U^T \\ U^*CU^\dagger & 0 \end{pmatrix}$$

$$C' = U^*CU^\dagger$$

Using SU(N) transforms and using unitarity condition, we can write

$$C = e^{i\theta}\mathbf{1}$$

Invar. subgroup of SU(N) under C

$$C = e^{i\theta}\mathbf{1} = C' = U^*CU^\dagger = U^*e^{i\theta}U^\dagger \implies U^TU = 1$$

is SO(N)

P of SO(2r)

Invar. subgroup of SO(2r) under P

$$\mathcal{P} = (+, -, -, \ldots, -)$$

is SO(2r-1)

CHARGE CONJUGATION

We will need these C-invariant and P-invariant subgroups

SO(N) **SO(2r-1)**

as we will consider self-conjugate gauge field configurations

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \exp\left[i\int dx\,\bar{\psi}\,i\not{D}(A)\,\psi\right] \xrightarrow{C,P} \pm\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \exp\left[i\int dx\,\bar{\psi}\,i\not{D}(A)\,\psi\right]$$

Its enough to prove symmetry is anomalous, if there is a phase

One obtains non-trivial anomaly matching consistency checks

We haven't generalized to understand $\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \exp\left[i\int dx\,\bar{\psi}\,i\not{D}(A)\,\psi\right] \xrightarrow{C} \pm\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \exp\left[i\int dx\,\bar{\psi}\,i\not{D}(A^C)\,\psi\right]$

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C IN QCD-LIKE THEORIES

SU(N) gauge theory

Nf quark fields as left-handed Weyl fermions:

q_i — In fundamental rep $\mathbf{N}$

$\tilde{q}_i = (q_{iR})^c$ — In anti-fundamental rep $\overline{\mathbf{N}}$

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}G_{\mu\nu}G^{\mu\nu} + \begin{pmatrix} \bar{q}_i & \bar{\tilde{q}}_i \end{pmatrix} \begin{pmatrix} i\gamma^\mu(\partial_\mu - igA_\mu) & 0 \\ 0 & i\gamma^\mu(\partial_\mu + igA_\mu^T) \end{pmatrix} \begin{pmatrix} q_i \\ \tilde{q}_i \end{pmatrix}$$

Charge conjugation:

$$q \to C^\dagger \tilde{q}\,,$$

$$\tilde{q} \to Cq\,,$$

$$A_\mu \to -C^\dagger A_\mu^T C$$

C IN QCD-LIKE THEORIES

$$\text{Tr}\; T^a\, T^b = I[R] \cdot \frac{1}{2}\delta^{ab}$$

For an odd eigenstate of C $\psi \to -\psi$

$$Z[A] \to \exp\left(\frac{i\pi I[R]}{16\pi^2} \int d^4x\, F \star F\right)\, Z[A]$$

Need to first check symmetry is non-anomalous under the gauge group (and thus is an actual symmetry of the quantum theory)

C IN QCD-LIKE THEORIES

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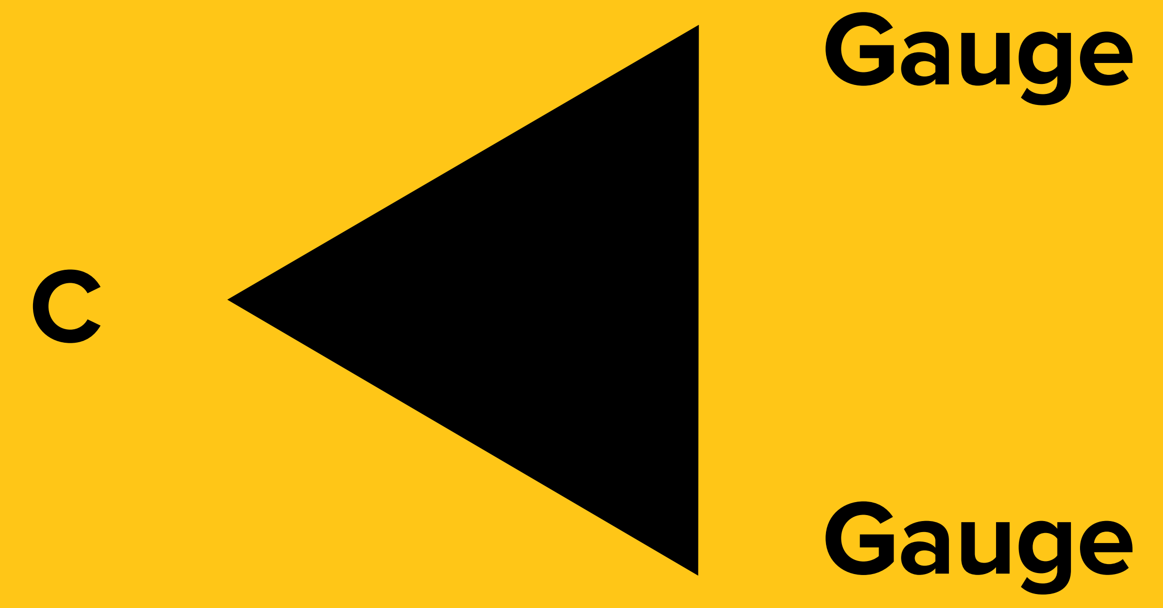

- We consider SO(N) instanton background
- q_i and $\tilde{q}_i$ decompose in fundamental of SO(N)
- $q_i - \tilde{q}_i$ is odd under C
- Dynkin index I[R] = 2

Not anomalous (under this background)

IR ANOMALY MATCHING

Seiberg duality

	$SU(N)$	$U(1)_R$	$SU(F)_Q$	$SU(F)_{\tilde{Q}}$
W_α	**adj**	$+1$	$\mathbf{1}$	$\mathbf{1}$
Q	$\square$	$1-\frac{N}{F}$	$\square$	$\mathbf{1}$
$\tilde{Q}$	$\overline{\square}$	$1-\frac{N}{F}$	$\mathbf{1}$	$\square$

Electric SU(N)

	$SU(F-N)$	$U(1)_R$	$SU(F)_Q$	$SU(F)_{\tilde{Q}}$
W_α	**adj**	$+1$	$\mathbf{1}$	$\mathbf{1}$
q	$\square$	$1-\frac{\tilde{N}}{F}$	$\overline{\square}$	$\mathbf{1}$
$\tilde{q}$	$\overline{\square}$	$1-\frac{\tilde{N}}{F}$	$\mathbf{1}$	$\overline{\square}$
$M=\tilde{Q}Q$	$\mathbf{1}$	$2-\frac{2N}{F}$	$\square$	$\square$

Magnetic SU(F-N)

Follow Csáki and Murayama, and usual ’t Hooft argument

Fermion measure not invar under discrete sym.

Promoting global non-abelian* to gauge in usual way (spectators, very weakly gauged)

Anomalies must match

A rare and powerful probe of strong dynamics

Go through one example..

* also abelian up to min-charge caveats

IR ANOMALY MATCHING

Seiberg duality

	$SU(N)$	$U(1)_R$	$SU(F)_Q$	$SU(F)_{\tilde{Q}}$
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$\tilde{q}$	$\overline{\square}$	$1-\frac{\tilde{N}}{F}$	$\mathbf{1}$	$\overline{\square}$
$M=\tilde{Q}Q$	$\mathbf{1}$	$2-\frac{2N}{F}$	$\square$	$\square$

Magnetic SU(F-N)

Charge conjugation interchanges $Q \leftrightarrow \tilde{Q}$

Again consider anomalies associated with subgroup $S\check{U}(F)_C \subset SU(F)_Q \times SU(F)_{\tilde{Q}}$

that commutes with C such that $V_L = V_R^*$

$\mathcal{C}_S SU(F)_C^2$

C ◀ Global / Global

Electric

$Q \quad \tilde{Q}$ both in fundamental of diagonal subgroup

$Q - \tilde{Q}$ Odd eigenstate of C

$$N\,\mathrm{Tr}\,T^a_{fund}T^b_{fund} = N\cdot\frac{1}{2}\delta^{ab}$$

Magnetic

Same analysis for the $q,\ \tilde{q}$

$M^{ij} = \tilde{Q}^i Q^j$ Transposed under C.

Anti-symmetric piece contributes to anomaly

$$\mathrm{Tr}\,T^a_{anti}T^b_{anti} = (F-2)\cdot\frac{1}{2}\delta^{ab}$$

IR ANOMALY MATCHING

Seiberg duality

	$SU(N)$	$U(1)_R$	$SU(F)_Q$	$SU(F)_{\tilde{Q}}$
W_α	**adj**	$+1$	$\mathbf{1}$	$\mathbf{1}$
Q	$\square$	$1-\frac{N}{F}$	$\square$	$\mathbf{1}$
$\tilde{Q}$	$\overline{\square}$	$1-\frac{N}{F}$	$\mathbf{1}$	$\square$

Electric SU(N)

	$SU(F-N)$	$U(1)_R$	$SU(F)_Q$	$SU(F)_{\tilde{Q}}$
W_α	**adj**	$+1$	$\mathbf{1}$	$\mathbf{1}$
q	$\square$	$1-\frac{\tilde{N}}{F}$	$\overline{\square}$	$\mathbf{1}$
$\tilde{q}$	$\overline{\square}$	$1-\frac{\tilde{N}}{F}$	$\mathbf{1}$	$\overline{\square}$
$M=\tilde{Q}Q$	$\mathbf{1}$	$2-\frac{2N}{F}$	$\square$	$\square$

Magnetic SU(F-N)

Charge conjugation interchanges $Q \leftrightarrow \tilde{Q}$

Again consider anomalies associated with subgroup $S\check{U}(F)_C \subset SU(F)_Q \times SU(F)_{\tilde{Q}}$

that commutes with C such that $V_L = V_R^*$

$\mathcal{C}_S SU(F)_C^2$

$$N = \tilde{N} + (F-2) \mod 2$$

C ◀ **Global**
Global

Anomalies match (would have been a huge surprise if not! But its nevertheless a new, non-trivial check)

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SPONTANEOUS BREAKING OF C, P

We can also consider anomalies of UV vs IR theories

If anomalies do not match it implies the symmetry is broken

SPONTANEOUS BREAKING OF C, P

Two N=1 SUSY theories

$SO(6)$ with two vectors

Intriligator and Seiberg (1995)

SO(6) breaks to SU(2)xSU(2)

		$SO(6)$	$U(1)_R$	$SU(2)_f$	$\mathbb{Z}_4$
UV	W_α	**adj**	$+1$	$\mathbf{1}$	0
	ϕ_i	$\mathbf{6}$	-1	$\mathbf{2}$	1
IR	M_{ij}	$\mathbf{1}$	-2	$\mathbf{3}$	2

$M_{ij} = \phi_i \phi_j$

Analysis of the Parity anomaly

	$SO(5)$	$\mathcal{P}\,R^2$	$\mathcal{P}\,(SU(2))^2$
W_α	$\mathbf{10}_+ \oplus \mathbf{5}_-$	$-$	$+$
ϕ_i	$\mathbf{5}_+ \oplus \mathbf{1}_-$	$+$	$-$
UV total		$-$	$-$
M_{ij}	$\mathbf{1}_+$	$+$	$+$

Not matched - P is spontaneously broken

Order parameter: $\langle \epsilon_{abcdef}\epsilon^{\alpha\beta} W^{ab}_\alpha W^{bc}_\beta \phi^e_i \phi^f_j \epsilon^{ij} \rangle \neq 0$

$SU(6)$ with one rank-three anti-symmetric tensor

Csáki, Schmaltz and Skiba (1996)

SU(6) breaks to SU(3)xSU(3)

		$SU(6)$	$U(1)_R$	$\mathbb{Z}_6$
UV	W_α	**adj**	$+1$	0
pseudo-real	A	$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	-1	1
IR	A^4	$\mathbf{1}$	-4	4

	$SO(6)$	$\mathcal{C}_S R^2$
W_α	$\square\square_- \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}_+$	$+$
A	$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}_+ \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}_-$	$+$
UV total		$+$
A^4	$\mathbf{1}_+$	$+$

Matched - C is unbroken

SPONTANEOUS BREAKING OF C, P

$SO(6)$ **with two v**

Intriligator and Seiber

SO(6) breaks to SU(2)

		$SO(6)$	$U(1)_R$	SU
UV	W_α	**adj**	$+1$	
	ϕ_i	**6**	-1	
IR	M_{ij}	**1**	-2	

$M_{ij} = \phi_i \phi_j$

Analysis of the Parity anomaly

	$SO(5)$	$\mathcal{P}\, R^2$	$\mathcal{P}\,(SU(2))^2$
W_α	$\mathbf{10}_+ \oplus \mathbf{5}_-$	$-$	$+$
ϕ_i	$\mathbf{5}_+ \oplus \mathbf{1}_-$	$+$	$-$
UV total		$-$	$-$
M_{ij}	$\mathbf{1}_+$	$+$	$+$

Not matched - P is spontaneously broken

Order parameter: $\langle \epsilon_{abcdef}\epsilon^{\alpha\beta} W^{ab}_\alpha W^{bc}_\beta \phi^e_i \phi^f_j \epsilon^{ij} \rangle \neq 0$

The anomaly matching is new

Two theories with similar dynamics on the face of it, have different fates

Two ground states in SO(6) theory, leading to possibility of domain walls

ank-three anti-symmetric tensor

maltz and Skiba (1996)

eaks to SU(3)xSU(3)

	(6)	$U(1)_R$	$\mathbb{Z}_6$
	j	$+1$	0
		-1	1
		-4	4

	$SO(6)$	$\mathcal{C}_S R^2$
W_α	$\square\square_- \oplus \boxminus_+$	$+$
A	$\boxminus_+ \oplus \boxminus_-$	$+$
UV total		$+$
A^4	$\mathbf{1}_+$	$+$

Matched - C is unbroken

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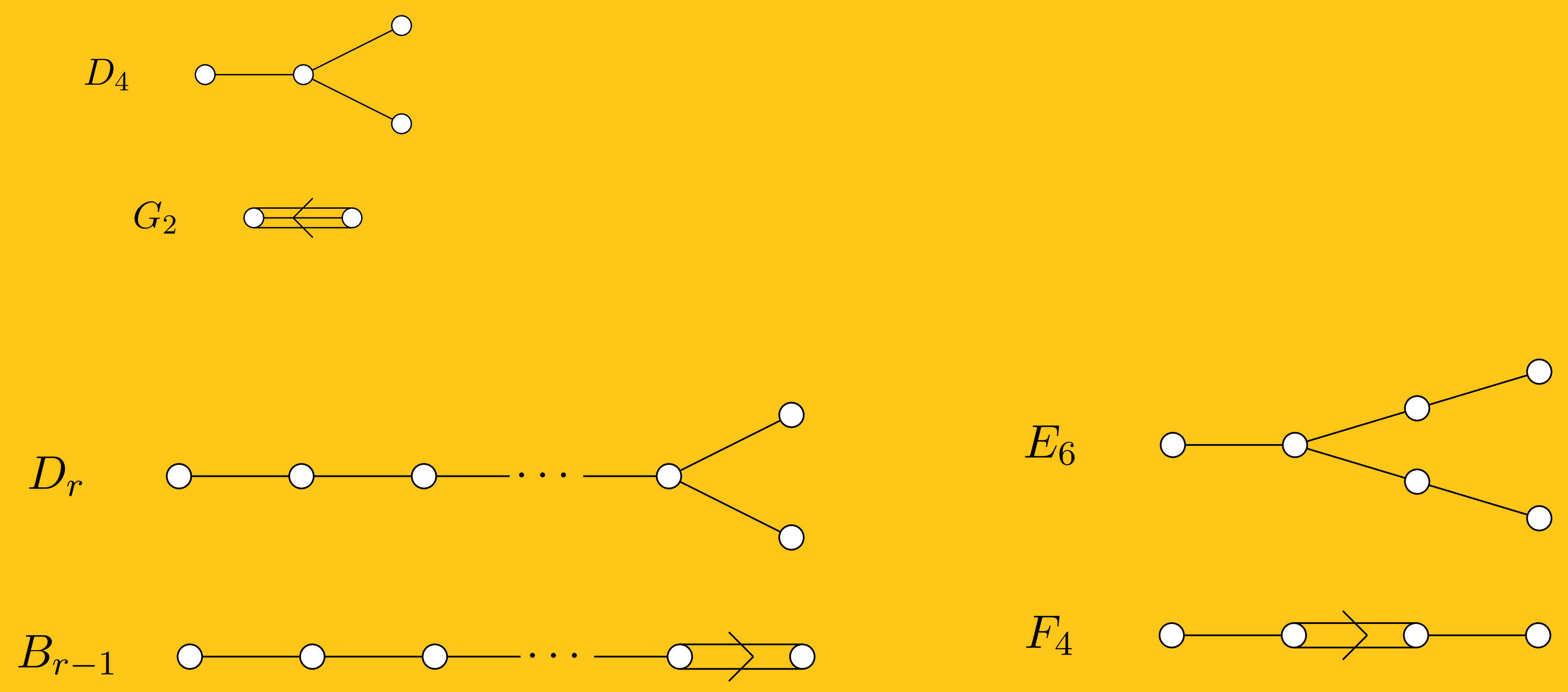
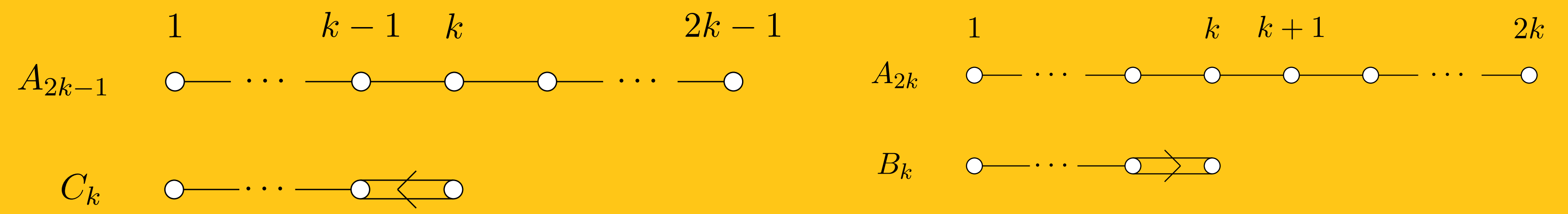
OTHER RESULTS



Other outer automorphisms



Numerous other examples of IR matching conditions (satisfied)

OUTLOOK

If symmetry found to be anomalous, constraints on model building (obstruction to gauging)

Anomaly inflow understanding? Edge states, similar to recent studies of time reversal?

Generalized / non-invertible symmetries

THANKS FOR LISTENING