The cyclic symmetries in the representations of unitary discrete subgroups

Darius Jurčiukonis[(1)], and Luís Lavoura[(2)]

[(1)] Vilnius University, Institute of Theoretical Physics and Astronomy
darius.jurciukonis@tfai.vu.lt

[(2)] Universidade de Lisboa, Instituto Superior Técnico, CFTP
balio@cftp.tecnico.ulisboa.pt

based on e-Print: 2210.12133

November 8, 2022

Introduction

- The lightest Dark Matter (DM) particle must be (almost) stable on cosmological timescales.
- If it is stable, then there is an unbroken cyclic $\mathbb{Z}_N$ symmetry such that standard matter is invariant under $\mathbb{Z}_N$ while DM is not; the $\mathbb{Z}_N$ charge different from 1 of the lightest DM particle prevents it from decaying to standard matter, which has $\mathbb{Z}_N$ charge 1.
- The simplest possibility consists in G being a discrete group of order O that is isomorphic to the direct product $\mathbb{Z}_N \times G'$, where G' is a group of order O/N. Standard matter must be placed in the trivial representation of $\mathbb{Z}_N$ while DM is placed in non-trivial representations of $\mathbb{Z}_N$ [e-Print: 1412.5600], [e-Print: 1911.05515],...
- However, also discrete groups G that cannot be written as the direct product of a cyclic group and a smaller group may have a non-trivial $\mathbb{Z}_N$ center. If $\mathbb{Z}_N$ remains unbroken when G is broken, and if there are particles with $\mathbb{Z}_N$ charge different from 1, then those particles play the role of DM, while the particles with $\mathbb{Z}_N$ value 1 are standard matter [e-Print: 1205.3442], [e-Print: 2204.12517],...

SU(3)

The defining representation of $SU(3)$ consists of the 3×3 unitary matrices with determinant 1 and includes the matrix

$$A_3 = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} = \omega \times \mathbb{1}_3, \tag{1}$$

where $\omega = \exp(2i\pi/3)$. The Abelian group

$$\mathbb{Z}_3 = \left\{ \mathbb{1}_3,\ A_3,\ (A_3)^2 \right\}. \tag{2}$$

forms the center of $SU(3)$ in the defining representation. The matrix A_3 commutes with all the matrices in the defining representation of $SU(3)$ and satisfies $(A_3)^3 = \mathbb{1}_3$. Therefore, in a D-dimensional irrep of $SU(3)$

$$A_3 \mapsto \omega^{q_D} \times \mathbb{1}_D, \tag{3}$$

where q_D is an integer that depends on the irrep and may be either 0, 1, or 2 modulo 3.

SU(3)

- Irreps with $q_D = 0$ (like the octet and the decaplet) have A_3 represented by $\mathbb{1}_D$ and are unfaithful representations of $SU(3)$.
- Irreps with either $q_D = 1$ (like the triplet) or $q_D = 2$ (like the sextet and the anti-triplet) are faithful.
- The q_D-values help determine the tensor products of irreps of the subgroup. This may be used to explain the stability of DM:
  - if Nature had an internal symmetry that was a discrete subgroup of $SU(3)$, that contained the matrix A_3 in its defining representation, and that stayed unbroken, then standard matter would sit in irreps of that subgroup with $q_D = 0$,
  - while DM would be in irreps with either $q_D = 1$ or $q_D = 2$; the lightest DM particle would then automatically be stable.

Motivation

- The defining representation of $SU(D)$ consists of the $D \times D$ unitary matrices with determinant 1. It is obvious that, in this representation, the center of $SU(D)$ is formed by the D diagonal matrices

$$\Delta \times \mathbb{1}_D,\ \Delta^2 \times \mathbb{1}_D,\ \Delta^3 \times \mathbb{1}_D,\ \ldots,\ \Delta^D \times \mathbb{1}_D = \mathbb{1}_D, \tag{4}$$

where $\Delta = \exp(2i\pi/D)$. Thus, the center of $SU(D)$ is a $\mathbb{Z}_D$ group.

- On the other hand, discrete subgroups of $U(D)$ do not bear the constraint that the determinants of the matrices in their defining representations should be 1. As a consequence, if

$$\mathbb{Z}_t = \left\{\theta \times \mathbb{1}_D,\ \theta^2 \times \mathbb{1}_D,\ \theta^3 \times \mathbb{1}_D,\ \ldots,\ \theta^t \times \mathbb{1}_D = \mathbb{1}_D\right\}, \tag{5}$$

where $\theta = \exp(2i\pi/t)$, is the center of a discrete subgroup of $U(D)$, then there appears to be *a priori* no restriction on t.

- Motivated by this observation that discrete subgroups of $U(D)$ may in general have diverse centers, in our work we have surveyed many discrete groups in order to find out their centers and also which groups $U(D)$ they are subgroups of.

Procedure

- We have surveyed all the discrete groups of order $O \leq 2000$ in the SmallGroups library, except the groups of order either 512, 1024, or 1536.
- We have discarded all the groups that are isomorphic to the direct product of a smaller (*i.e.* of lower order) group and a cyclic group.
- We have used GAP to find out all the irreps of each remaining group, and then to ascertain whether those irreps are faithful or not.
- We have discarded all the groups that do not have any faithful irreducible representation.
- We have thus obtained 87,349 non-isomorphic groups, that are all listed in our tables available at the site https://github.com/jurciukonis/GAP-group-search.

Intersections

- There are relatively few groups that have firreps with different dimensions. For instance:
  - A_5 has firreps of dimensions three, four, and five.
  - On the other hand, the group $\Sigma\,(36 \times 3)$, that has `SmallGroups` identifier [108, 15], has irreps of dimensions 1, 3, and 4, but the 1- and 4-dimensional irreps are unfaithful—all the firreps have dimension 3.
- We have found 2787 such discrete groups, out of the total 87,349 groups that we have surveyed.

Example

- The discrete group $\mathrm{GL}(2,\,3)$ has order 48 and `SmallGroups` identifier [48, 29].
- There is a faithful representation through 2×2 unitary matrices:

$$a \mapsto \frac{1}{3}\begin{pmatrix} i\sqrt{3} & \sqrt{6}\omega \\ -\sqrt{6}\omega^2 & -i\sqrt{3} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

  The first two matrices have determinant 1 while the third one has determinant -1; hence, we classify $\mathrm{GL}(2,\,3)$ as a subgroup of $U(2)$, but it is not a subgroup of $SU(2)$.
- On the other hand, there is another faithful irrep of $\mathrm{GL}(2,\,3)$, through 4×4 unitary matrices, all of them with determinant 1:

$$a \mapsto \frac{1}{9}\begin{pmatrix} -3\sqrt{3}i & 0 & 6i & -3\sqrt{2} \\ 0 & 3\sqrt{3}i & 3\sqrt{2} & -6i \\ 6i & -3\sqrt{2} & i\sqrt{3} & -2\sqrt{6} \\ 3\sqrt{2} & -6i & 2\sqrt{6} & -i\sqrt{3} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & -\omega^2 & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

  and $b \mapsto \mathrm{diag}\left(\omega,\ \omega^2,\ 1,\ 1\right)$. Therefore, we classify $\mathrm{GL}(2,\,3)$ as a subgroup of both $U(2)$ and $SU(4)$, but $\mathrm{GL}(2,\,3)$ earns these two classifications *through different irreps.*

Computing time

- The scan over the `SmallGroups` library to find the firreps of all possible dimensions constituted a computationally very expensive task.
- Our computations with `GAP` took about three months.
- Most of the time was consumed in the computation of the irreps of the groups.
- For example, the computation for group [1320, 15], *viz.* $\mathrm{SL}(2, 11)$, took about 320 CPUH running on Intel Xeon CPU @ 1.60GHz or about 46 CPUH in the newer Intel i9-10850K CPU @ 3.60GHz.
- Orders 768, 1280, and 1792 have more than one million non-isomorphic groups of each order and therefore require many CPUH to scan over all of them.

Database

We have obtained 87,349 non-isomorphic groups, that are all listed in our tables available at the site https://github.com/jurciukonis/GAP-group-search.

Group order (O)	Group number	SmallGroups identifier	N	D	# of firreps in U(D)	# of firreps in SU(D)	Structure description
6	1	[6, 1]	1	2	1	0	S3
8	3	[8, 3]	2	2	1	0	D8
8	4	[8, 4]	2	2	1	1	Q8
10	1	[10, 1]	1	2	2	0	D10
12	1	[12, 1]	2	2	1	1	C3:C4
12	3	[12, 3]	1	3	1	1	A4
14	1	[14, 1]	1	2	3	0	D14
16	6	[16, 6]	4	2	2	0	C8:C2
16	7	[16, 7]	2	2	2	0	D16
16	8	[16, 8]	2	2	2	0	QD16
16	9	[16, 9]	2	2	2	2	Q16
16	13	[16, 13]	4	2	2	0	(C4xC2):C2
18	1	[18, 1]	1	2	3	0	D18
20	1	[20, 1]	2	2	2	2	C5:C4
20	3	[20, 3]	1	4	1	0	C5:C4
21	1	[21, 1]	1	3	2	2	C7:C3
22	1	[22, 1]	1	2	5	0	D22
24	1	[24, 1]	4	2	2	0	C3:C8
24	3	[24, 3]	2	2	3	1	SL(2,3)

Database

The groups that have faithful irreducible representations of more than one different dimensions.

Group order (O)	Group number	SmallGroups identifier	N	dimensions of a firreps (D): 2	3	4	5	6	7	8	Structure description
48	28	[48, 28]	2	U_s		U_s					C2.S4=SL(2,3).C2
48	29	[48, 29]	2	U		U_s					GL(2,3)
60	5	[60, 5]	1		U_s	U_s	U_s				A5
96	64	[96, 64]	1		UU_s			U			((C4xC4):C3):C2
96	67	[96, 67]	4	U		U					SL(2,3):C4
96	192	[96, 192]	4	U		U_s					(((C4xC2):C2):C3):C2
120	5	[120, 5]	2	U_s		U_s		U_s			SL(2,5)
120	34	[120, 34]	1			U	UU_s	U			S5
150	5	[150, 5]	1		UU_s			U			(C5xC5):S3
168	42	[168, 42]	1		U_s			U_s	U_s	U_s	PSL(3,2)
192	182	[192, 182]	2		U			U_s			((C4xC4):C3):C4
192	187	[192, 187]	8	U		U					C8.S4=SL(2,3).C8
192	963	[192, 963]	8	U		U					(((C8xC2):C2):C3):C2
192	987	[192, 987]	2			U_s				U_s	(SL(2,3):C4):C2
192	988	[192, 988]	2			U_s				U_s	(((C2xC2xC2):(C2xC2)):C3):C2
192	989	[192, 989]	2			U_s				U_s	(SL(2,3):C4):C2
192	990	[192, 990]	2			U_s				U_s	(((C2xQ8):C2):C3):C2

Database

Number of discrete groups with center $\mathbb{Z}_N$ that have at least one faithful irreducible representation of dimension D

	dimensions of a firreps (D):																																
N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	20	21	22	23	24	26	27	28	29	30	31	32	36	40	42	Total
1	499	154	643	41	848	26	468	72	118	5	692	6	33	28	133	1	345	42	14	5	1	174	2	4	6	1	8	1	5	5	1	1	4382
2	993	7	9174	5	1592	2	29382	14	205		4417		37	2	10587		153	135		1		507	1		8		8						57230
3		173	1		487	1	2	458			131			9			346		6			14											1628
4	493	5	5709	4	572	1	12734	3	62		722		7	1	489		14	14															20830
5				30					20					3																			53
6	1	12	40		847		33	24			573						82																1612
7						7																											7
8	242	3	1636	2	172		931		13		55		2																				3056
9		66			104	1	1	65			9																						246
10			1	4					33																								38
11										2																							2
12	1	9	20		258			7			90																						385
14						1																											1
15				1																													1
16	117	2	366	1	39		27		1																								553
18	1	7	10		126																												144
20			1	3					7																								11
24	1	8	6		49																												64
25				6																													6
27		25			10																												35
32	54	1	52		4																												111
36	1	6	4		21																												32
40				1																													1
48	1	3			10																												14
54	1	2			10																												13
64	19	1	2																														22
72	1	1																															2
81		5																															5
96	1	1																															2
108	1	1																															2
128	3																																3
144	1																																1
162	1																																1
Total	2432	492	17665	98	5149	39	43578	643	459	7	6689	6	79	43	11209	1	940	191	20	6	1	695	3	4	14	1	16	1	5	5	1	1	90493

Conclusions

e-Print: 2210.12133

- In this work we have pointed out that Dark Matter may be stabilized by a $\mathbb{Z}_N$ cyclic group under which it has a non-trivial charge – contrary to standard matter.
- Then we have performed an extensive search for the centers of discrete groups that cannot be written in the form $\mathbb{Z}_N \times G'$ and that have faithful irreducible representations. The following are our conclusions:
  - We have found groups with centers $\mathbb{Z}_N$ for $N \leq 162$.
  - We have found groups with $N = 2^p \times 3^q$ for *all* the integers p and q such that $N \leq 162$.
  - We have found groups with $N = 2^p \times 5$ for $0 \leq p \leq 3$.
  - We have also found groups with $N = 7$, $N = 11$, $N = 14$, $N = 15$, and $N = 25$.
  - The number N always divides the order O of the group. The integer O/N always has at least two prime factors.

Thanks for your attention!