Modular Flavor Symmetries and CP, from the top down

Andreas Trautner

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based on:

PLB 786 (2018) 283-287	1808.07060	w/ H.P. Nilles, M. Ratz, P. Vaudrevange
PLB 795 (2019) 7-14	1901.03251	w/ A. Baur, H.P. Nilles, P. Vaudrevange
NPB 947 (2019) 114737	1908.00805	w/ A. Baur, H.P. Nilles, P. Vaudrevange
NPB 971 (2021) 115534	2105.08078	w/ H.P. Nilles, S. Ramos-Sánchez, P. Vaudrevange
PRD 105 (2022) 5 055018	2112.06940	w/ A.Baur, H.P. Nilles, S. Ramos-Sánchez, P. Vaudrevange
JHEP 09 (2022) 224	2207.10677	w/ A.Baur, H.P. Nilles, S. Ramos-Sánchez, P. Vaudrevange

DISCRETE
2022

Baden-Baden
8.11.22

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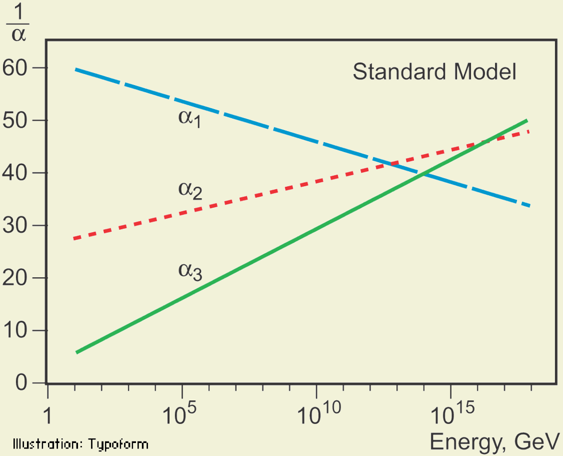
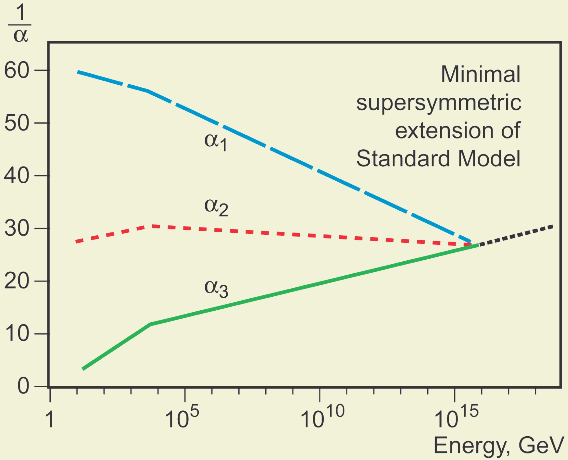
Outline: Flavor symmetry

- [Modular] Flavor symmetries from the top-down
- The eclectic flavor symmetry
- Breaking of the eclectic flavor symmetry
- Phenomenology of a concrete top-down example model
- Summary

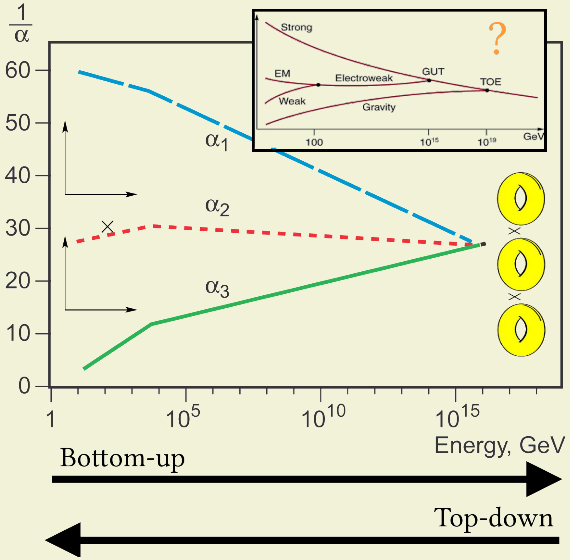
Unification – bottom-up vs. top-down



Unification – bottom-up vs. top-down

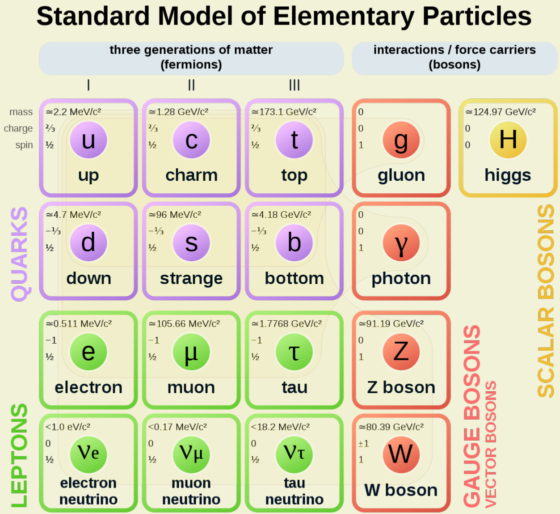


Unification – bottom-up vs. top-down



See e.g. “Supersymmetric standard model from the heterotic string”
[Buchmüller, Hamaguchi, Lebedev, Ratz ’05]

Is everything unified?



No "theory of everything" without a theory of flavor!

Modular Flavor Symmetries

Even w/o thoughts about UV completions: Very attractive framework.

Predictivity (few parameters), CP violation & hierarchies “built in”, . . .

- Neutrinos/Leptons

  [Feruglio ’17], [Kobayashi, Tanaka, Tatsuishi ’18], [Penedo, Petcov ’18], [Criado, Feruglio ’18], [Kobayashi, Omoto, Shimizu, Takagi, Tanimoto, Tatsuishi ’18], [Novichkov, Penedo, Petcov, Titov ’18 (2x)], [Novichkov, Petcov, Tanimoto ’18], [Nomura, Okada ’19], [de Medeiros Varzielas, King, Zhou ’19], [Liu, Ding ’19], [Criado, Feruglio, S.J.D.King ’19], . . .

- Quark sector

  [Okada, Tanimoto ’18 &’19], [Kobayashi, Shimizu, Takagi, Tanimoto, Tatsuishi, Uchida ’18], [Lu, Liu, Ding ’19], . . .

- Combination of modular transformations with CP

  [Baur, Nilles, AT, Vaudrevange ’19], [Novichkov, Penedo, Petcov, Titov ’19], [Kobayashi, Shimizu, Takagi, Tanimoto, Tatsuishi ’19]

- Within GUTs

  [de Anda, King, Perdomo ’18], [Kobayashi, Shimizu, Takagi, Tanimoto, Tatsuishi ’19], [Zhao, Zhang ’21], [Chen, Ding, King ’21], [Ding, King, Lu ’21],. . .

$\rightarrow$ See talk by Penedo

Modular Flavor Symmetries

Modular flavor symmetry is strongly motivated from top-down viewpoint of UV completions of the Standard Model.

Setting: compactified heterotic string theory. [Gross, Harvey, Martinec, Rohm '85] [Dixon, Harvey, Vafa, Witten '85 &'86]
Compactifications are controlled by modular invariance:

- Couplings among twisted-sector states are modular forms.
  [Ibañez '86],[Hamdi, Vafa '87],[Dixon, Friedan, Martinec, Shenker '87], [Lauer, Mas, Nilles '89 &'91]
- Effective 4D SUSY (sugra) theory controlled by modular invariance... [Ferrara, Lüst, (Shapere), Theisen '89(x2)]
- ...in particular, the Yukawa couplings.
  [Casas, Gomez, Munoz '91], [Lebedev '01], [Kobayashi, Lebedev '03], . . .
- Twisted-sector gives rise to chiral matter, can host 3 generations of (supersymmetric) SM. [Ibañez, (Kim), Nilles, Quevedo '87 (x2)]
- Flavor symmetries are a generic feature.
  [Lauer, Mas, Nilles '89 '91], [Kobayashi, Nilles, Plöger, Raby, Ratz '06]

This talk: Unambiguous derivation of unified flavor symmetry
& Example for explicit model with correct low energy pheno.

Types of (discrete) flavor symmetries

Schematically for the example of $\mathcal{N}=1$ SUSY.

x: spacetime, θ: superspace, Φ : (Super-)fields, T: modulus.
$K(T,\Phi)$: Kähler potential, $W(T,\Phi)$: Superpotential

$$\mathcal{S}=\int d^4x\,d^2\theta\,d^2\overline{\theta}K(T,\overline{T},\Phi,\overline{\Phi})+\int d^4x\,d^2\theta\,W(T,\Phi)+\int d^4x\,d^2\overline{\theta}\,\overline{W}(\overline{T},\overline{\Phi})\;.$$

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- **“traditional” Flavor symmetries** $\Phi \mapsto \rho(\mathrm{g})\Phi\,, \quad \mathrm{g} \in G$

  for a review, see e.g. [King & Luhn ’13]

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- "traditional" Flavor symmetries $G_{\mathrm{traditional}}$
- **modular Flavor symmetries** **[Feruglio '17]**

$$\Phi \xmapsto{\gamma} (c\,T+d)^n \rho(\gamma)\Phi\;,\quad T \xmapsto{\gamma} \frac{a\,T+b}{c\,T+d}\;,\quad \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\;.$$

Couplings are modular forms: $Y=Y(T)$, $Y(\gamma T) = (c\,T+d)^{k_Y}\,\rho_Y(\gamma)\,Y(T)$.

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- "traditional" Flavor symmetries $G_{\mathrm{traditional}}$
- modular Flavor symmetries G_{modular}
- **R symmetries** for non-Abelian discrete R flavor symmetries see [Chen, Ratz, AT '13]

$$\Phi(x,\theta) = \boldsymbol{\phi}(x) + \sqrt{2}\boldsymbol{\theta}\,\boldsymbol{\psi}(x) + \theta\theta F(x)\;, \implies \phi \mapsto e^{iq_\Phi\alpha}\phi,\ \psi \mapsto e^{i(q_\Phi - q_\theta)\alpha}\psi\,.$$

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- modular Flavor symmetries G_{modular}
- R symmetries G_R
- **general CP(-like) symmetries** [Novichkov, Penedo et al. '19],[Baur et al. '19]

$$\Phi \overset{\bar{\gamma}}{\longmapsto} (c\overline{T}+d)^n \rho(\bar{\gamma})\overline{\Phi} \; , \quad T \overset{\bar{\gamma}}{\longmapsto} \frac{a\overline{T}+b}{c\overline{T}+d} \; , \quad \det\left[\,\bar{\gamma} \in \mathrm{GL}(2,\mathbb{Z})\,\right] = -1 \; .$$

Types of (discrete) flavor symmetries

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- R symmetries G_R
- general $\mathcal{CP}$(-like) symmetries $\mathcal{CP}$

From the bottom-up: All kinds known and used, individually!

$\rightarrow$ See talk by Penedo.

for an up-to-date review see [Feruglio&Romanino '19]

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From the top-down: *all, at the same time*!

$$G_{\mathbf{eclectic}} \;=\; G_{\mathrm{traditional}} \,\cup\, G_{\mathrm{modular}} \,\cup\, G_{\mathrm{R}} \,\cup\, \mathcal{CP}\ ,$$

see works by [Baur, Nilles, AT, Vaudrevange ’19; Nilles, Ramos-Sánchez, Vaudrevange ’20]

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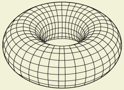
How to compute $G_{\mathbf{eclectic}}$?

Flavor symmetries from top-down perspective

- Setting is compactified heterotic string theory.

  Focus on $2D$ compact space, e.g. a torus: $\mathbb{T}^2$.

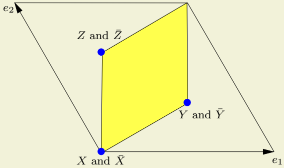


- **Example case:** $\mathbb{T}^2/\mathbb{Z}_3$ orbifold. Space group S (rot. & transl.).

$$g \in S \qquad g = (\theta^k, e\,n) \qquad \text{with} \quad k \in \{0,1,2\} \text{ and } n \in \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix} .$$

- we identify points $y \sim gy \Rightarrow$ fixed points.
- g constitutes boundary condition for closed strings; e.g. closed-string worldsheet boson [Dixon, Harvey, Vafa, Witten '85,'86]

$\Rightarrow$ Strings are "localized" at fixed points.



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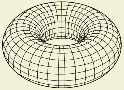

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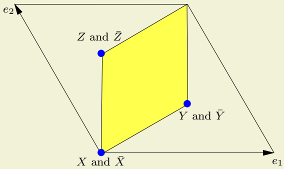
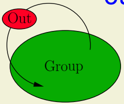


$\Rightarrow$ Strings are "localized" at fixed points.

- *New insight*: we can obtain flavor symmetries from outer automorphisms of the space group! [Baur, Nilles, AT, Vaudrevange '19]



  - inner auts: map fixed points to themselves $\Rightarrow$ trivial.
  - outer auts: permutation of fixed points $\Rightarrow$ non-trivial maps between strings at different f.p.'s!

$$h := (\sigma, t) \notin S\,, \quad \text{with} \quad g \overset{h}{\mapsto} h\,g\,h^{-1} \overset{!}{\in} S\,.$$

Origin of eclectic flavor symmetry in heterotic orbifolds

The *"whole"* story: **Narain lattice** formulation of heterotic string theory:

[Narain '86], [Narain, Samardi, Witten '87], [Narain, Sarmadi, Vafa'87], [Groot Nibbelink, Vaudrevange '17]

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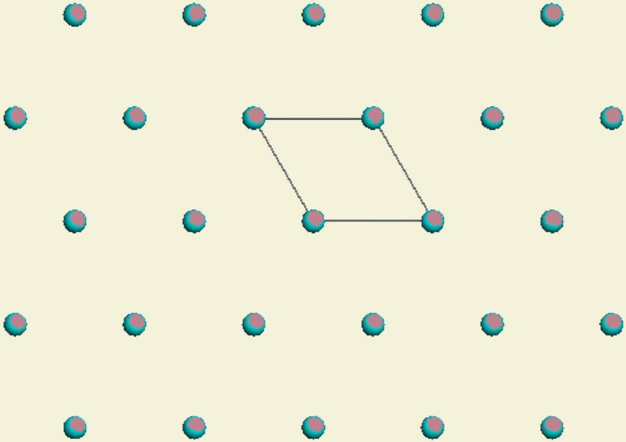
Lattice can have symmetries.



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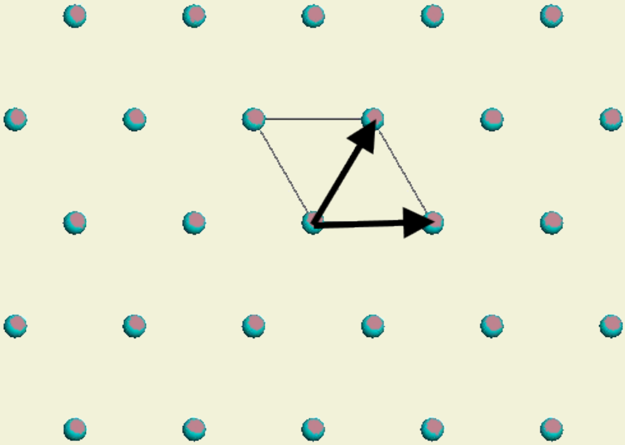

discrete translations

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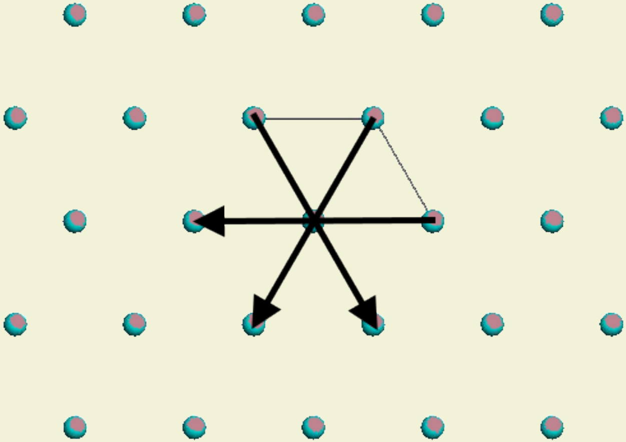

reflections / inversions

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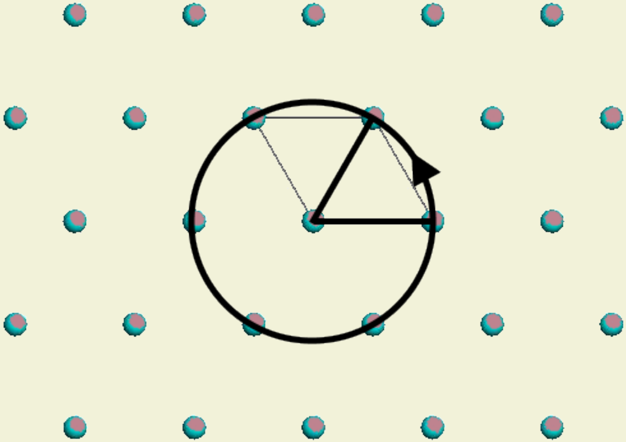

discrete rotations

Flavor symmetries from top-down perspective

Specializing to $D=2 \;\Rightarrow\; 4$-dim lattice w/ $E^{\mathrm{T}}E \equiv \mathcal{H} = \mathcal{H}(T,U)$

$\rightarrow$ **Kähler T** and **complex structure modulus U**.

Reflection and rotation outer automorphisms of Narain space group:

Modular transformations

$$\mathrm{O}_{\hat\eta}(D,D,\mathbb{Z}) := \left\langle\, \hat\Sigma \;\middle|\; \hat\Sigma \;\in\; \mathrm{GL}(2D,\mathbb{Z}) \quad \text{with} \quad \hat\Sigma^{\mathrm{T}}\hat\eta\,\hat\Sigma \;=\; \hat\eta \,\right\rangle .$$

$$\boxed{\mathrm{O}_{\hat\eta}(2,2,\mathbb{Z}) \cong \left[(\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)\right]/\mathbb{Z}_2 \;.}$$

Action on moduli $M \equiv \{T, U\}$ includes

$$\mathsf{s} : M \mapsto -\frac{1}{M}\;, \qquad \mathsf{t} : M \mapsto M{+}1\;, \qquad \mathsf{u} : M \mapsto -\overline{M}\;, \qquad \mathsf{d} : U \leftrightarrow T\;.$$

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Translationary outer automorphisms of Narain space group:

Traditional flavor symmetries

$M \;\overset{h}{\longmapsto}\; M' \;\neq\; M$ $\quad\leftrightarrow$ “modular flavor trafo”

$M \;\overset{h}{\longmapsto}\; M' \;=\; M$ $\quad\leftrightarrow$ “traditional flavor trafo”

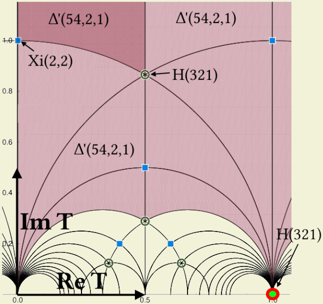
$$\mathrm{Out}_{\mathrm{Narain}} = \left\{(\hat\Sigma_1,0),\ (\hat\Sigma_2,0),\ \ldots,\ (\mathbb{1},\hat T_1),\ (\mathbb{1},\hat T_2),\ \ldots\right\}\ .$$

The **eclectic** flavor symmetry of $\mathbb{T}^2/\mathbb{Z}_3$

For this specific orbifold, $\langle U\rangle = \exp(2\pi\mathrm{i}/3)$.

<table>
<tr><th colspan="2">nature of symmetry</th><th>outer automorphism of Narain space group</th><th colspan="5">flavor groups</th></tr>
<tr><td rowspan="6">eclectic</td><td rowspan="2">modular</td><td>rotation S $\in$ $\mathrm{SL}(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_4$</td><td colspan="3" rowspan="2">T'</td><td rowspan="6">$\Omega(2)$</td></tr>
<tr><td>rotation T $\in$ $\mathrm{SL}(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td rowspan="4">traditional flavor</td><td>translation A</td><td>$\mathbb{Z}_3$</td><td rowspan="2">$\Delta(27)$</td><td rowspan="3">$\Delta(54)$</td><td rowspan="4">$\Delta'(54,2,1)$</td></tr>
<tr><td>translation B</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td>rotation $\mathrm{C}=\mathrm{S}^2\in\mathrm{SL}(2,\mathbb{Z})_T$</td><td colspan="2">$\mathbb{Z}_2^R$</td></tr>
<tr><td>rotation $\mathrm{R}\in\mathrm{SL}(2,\mathbb{Z})_U$</td><td colspan="3">$\mathbb{Z}_9^R$</td></tr>
</table>

table from [Nilles, Ramos-Sánchez, Vaudrevange '20]



Action on the T modulus as

$$\mathrm{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,,$$

$$\mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,,$$

$$\mathrm{K}_*^{\mathcal{CP}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,.$$

A, B, C, R : trivial!

The **eclectic** flavor symmetry of $\mathbb{T}^2/\mathbb{Z}_3$

For this specific orbifold, $\langle U \rangle = \exp({}^{2\pi \mathrm{i}}/{}_3)$.

The outer automorphisms of the corresponding Narain space group yield the following symmetries:

[Baur, Nilles, AT, Vaudrevange ’19; Nilles, Ramos-Sánchez, Vaudrevange ’20]

- a $\Delta(54)$ traditional flavor symmetry,
- an $\mathrm{SL}(2,\mathbb{Z})_T$ modular symmetry which acts as a $\Gamma_3' \cong T'$ finite modular symmetry on matter fields and their couplings,
- a $\mathbb{Z}_9^R$ discrete R-symmetry as remnant of $\mathrm{SL}(2,\mathbb{Z})_U$, and
- a $\mathbb{Z}_2^{\mathcal{CP}}$ $\mathcal{CP}$-like transformation.

$$G_{\textbf{eclectic}} \;=\; G_{\text{traditional}} \,\cup\, G_{\text{modular}} \,\cup\, G_{\text{R}} \,\cup\, \mathcal{CP}\;,$$

Together, the full eclectic group of this setting is of order 3888 given by

$$G_{\text{eclectic}} \;=\; \Omega(2) \rtimes \mathbb{Z}_2^{\mathcal{CP}}\,, \qquad \text{with} \quad \Omega(2) \cong [1944, 3448]\,.$$

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- a $\Delta(54)$ traditional flavor symmetry,
- an $\mathrm{SL}(2,\mathbb{Z})_T$ modular symmetry which acts as a $\Gamma'_3 \cong T'$ finite modular symmetry on matter fields and their couplings,
- a $\mathbb{Z}_9^R$ discrete R-symmetry as remnant of $\mathrm{SL}(2,\mathbb{Z})_U$, and
- a $\mathbb{Z}_2^{\mathcal{CP}}$ $\mathcal{CP}$-like transformation.

$$G_{\textbf{eclectic}} \;=\; G_{\text{traditional}} \,\cup\, G_{\text{modular}} \,\cup\, G_{\text{R}} \,\cup\, \mathcal{CP}\,,$$

Together, the full eclectic group of this setting is of order 3888 given by

$$G_{\text{eclectic}} \;=\; \Omega(2) \rtimes \mathbb{Z}_2^{\mathcal{CP}}\,, \qquad \text{with} \quad \Omega(2) \cong [1944, 3448]\,.$$

Explicit $\mathbb{T}^2/\mathbb{Z}_3$ models: charge assignments

Model	ℓ	$\bar{e}$	$\bar{\nu}$	q	$\bar{u}$	$\bar{d}$	H_u	H_d	flavons
A	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	Φ_0	Φ_0	$\Phi_{-2/3,-1}$
B	$\Phi_{-1/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-1/3}$	Φ_{-1}	Φ_0	$\Phi_{-2/3,-1}$
C	$\Phi_{-2/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-2/3}$	Φ_{-1}	Φ_{-1}	$\Phi_{-1/3,-1}$
D	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{\pm 2/3,0}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	Φ_0	$\Phi_{-1,0}$	$\Phi_{\pm 2/3,-1}$
E	$\Phi_{-2/3,-1/3}$	$\Phi_{-2/3,0}$	$\Phi_{0,-2/3,-1/3,-5/3}$	$\Phi_{-1,-2/3}$	$\Phi_{-2/3}$	$\Phi_{0,-2/3}$	Φ_0	Φ_0	$\Phi_{-2/3,-1/3,-5/3,-1}$

for methodology, see [Carballo-Pérez, Peinado, Ramos-Sánchez '16; Ramos-Sánchez '17]
[Olguin-Trejo, Perez-Martinez, Ramos-Sanchez '18]

sector	matter fields Φ_n	eclectic flavor group $\Omega(2)$: modular T' subgroup: irrep $\boldsymbol{s}$	$\rho_{\boldsymbol{s}}(\mathrm{S})$	$\rho_{\boldsymbol{s}}(\mathrm{T})$	n	traditional $\Delta(54)$ subgroup: irrep $\boldsymbol{r}$	$\rho_{\boldsymbol{r}}(\mathrm{A})$	$\rho_{\boldsymbol{r}}(\mathrm{B})$	$\rho_{\boldsymbol{r}}(\mathrm{C})$	$\mathbb{Z}_9^R$ R
bulk	Φ_0	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	$+1$	0
	Φ_{-1}	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3
θ	$\Phi_{-2/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-2/3$	$\mathbf{3}_2$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$+\rho(\mathrm{C})$	1
	$\Phi_{-5/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-5/3$	$\mathbf{3}_1$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$-\rho(\mathrm{C})$	-2
θ^2	$\Phi_{-1/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$-1/3$	$\bar{\mathbf{3}}_1$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$-\rho(\mathrm{C})$	2
	$\Phi_{+2/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$+2/3$	$\bar{\mathbf{3}}_2$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$+\rho(\mathrm{C})$	5
super-potential	$\mathcal{W}$	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3

table from [Nilles, Ramos-Sánchez, Vaudrevange '20]

Explicit $\mathbb{T}^2/\mathbb{Z}_3$ models: charge assignments

Model	ℓ	$\bar{e}$	$\bar{\nu}$	q	$\bar{u}$	$\bar{d}$	H_u	H_d	flavons
A	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	Φ_0	Φ_0	$\Phi_{-2/3,-1}$
B	$\Phi_{-1/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-1/3}$	Φ_{-1}	Φ_0	$\Phi_{-2/3,-1}$
C	$\Phi_{-2/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-2/3}$	Φ_{-1}	Φ_{-1}	$\Phi_{-1/3,-1}$
D	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{\pm 2/3,0}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	Φ_0	$\Phi_{-1,0}$	$\Phi_{\pm 2/3,-1}$
E	$\Phi_{-2/3,-1/3}$	$\Phi_{-2/3,0}$	$\Phi_{0,-2/3,-1/3,-5/3}$	$\Phi_{-1,-2/3}$	$\Phi_{-2/3}$	$\Phi_{0,-2/3}$	Φ_0	Φ_0	$\Phi_{-2/3,-1/3,-5/3,-1}$

for methodology, see [Carballo-Pérez, Peinado, Ramos-Sánchez ’16; Ramos-Sánchez ’17]
[Olguin-Trejo, Perez-Martinez, Ramos-Sanchez ’18]

sector	matter fields	eclectic flavor group $\Omega(2)$: modular T' subgroup				traditional $\Delta(54)$ subgroup				$\mathbb{Z}_9^R$
	Φ_n	irrep $\boldsymbol{s}$	$\rho_{\boldsymbol{s}}(\mathrm{S})$	$\rho_{\boldsymbol{s}}(\mathrm{T})$	n	irrep $\boldsymbol{r}$	$\rho_{\boldsymbol{r}}(\mathrm{A})$	$\rho_{\boldsymbol{r}}(\mathrm{B})$	$\rho_{\boldsymbol{r}}(\mathrm{C})$	R
bulk	Φ_0	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	$+1$	0
	Φ_{-1}	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3
θ	$\Phi_{-2/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-2/3$	$\mathbf{3}_2$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$+\rho(\mathrm{C})$	1
	$\Phi_{-5/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-5/3$	$\mathbf{3}_1$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$-\rho(\mathrm{C})$	-2
θ^2	$\Phi_{-1/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$-1/3$	$\bar{\mathbf{3}}_1$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$-\rho(\mathrm{C})$	2
	$\Phi_{+2/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$+2/3$	$\bar{\mathbf{3}}_2$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$+\rho(\mathrm{C})$	5
super-potential	$\mathcal{W}$	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3

table from [Nilles, Ramos-Sánchez, Vaudrevange ’20]

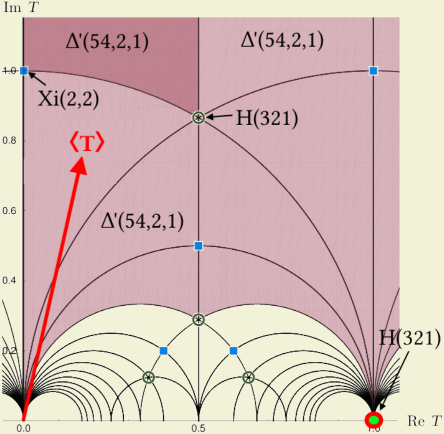
Sources of eclectic symmetry breaking

1. Modulus VEV $\langle T \rangle$.

Points w/ enhanced symmetry:

$$\langle T \rangle = \mathrm{i}, \qquad \Omega(2) \to \Xi(2,2)$$
$$\langle T \rangle = \omega, -\omega^2 \qquad \Omega(2) \to H(3,2,1)$$
$$\langle T \rangle = \mathrm{i}\infty, 1 \qquad \Omega(2) \to H(3,2,1)$$

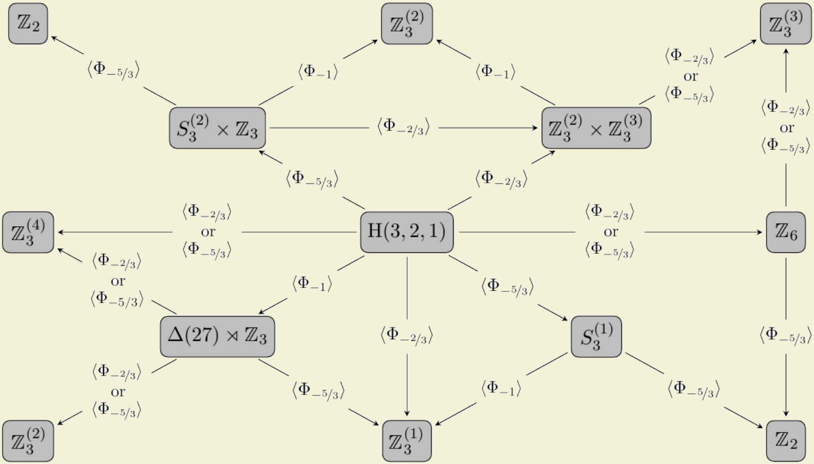
$$\Xi(2,2) \cong [324, 111]$$
$$H(3,2,1) \cong [486, 125]$$



2. Flavon VEVs

$$\langle \Phi_{-2/3} \rangle \sim \langle \mathbf{3}_2 \rangle, \qquad \langle \Phi_{-5/3} \rangle \sim \langle \mathbf{3}_1 \rangle, \qquad \langle \Phi_{-1} \rangle \sim \langle \mathbf{1}' \rangle\;,$$
$$\langle \Phi_{-1/3} \rangle \sim \langle \overline{\mathbf{3}}_1 \rangle, \qquad \langle \Phi_{+2/3} \rangle \sim \langle \overline{\mathbf{3}}_2 \rangle.$$

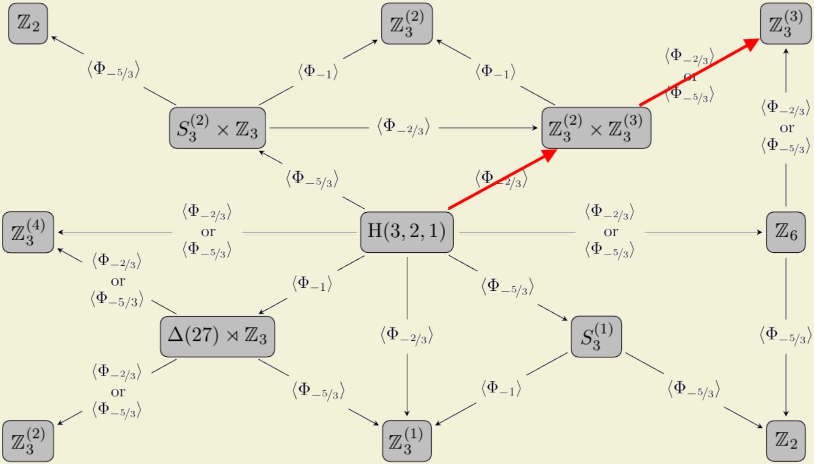
Example: Breakdown of $H(3,2,1)$ at $\langle T \rangle = \mathrm{i}\infty$



[Baur, Nilles, Ramos-Sánchez, AT, Vaudrevange '22]

Residual symmetries help to generate hierarchies in masses and mixing matrix elements.

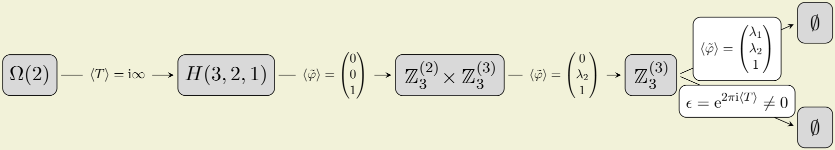
Example: Breakdown of $H(3,2,1)$ at $\langle T \rangle = \mathrm{i}\infty$



[Baur, Nilles, Ramos-Sánchez, AT, Vaudrevange '22]

Residual symmetries help to generate hierarchies in masses and mixing matrix elements.

Example: Breakdown of $H(3,2,1)$ at $\langle T\rangle \approx \mathrm{i}\infty$



$$\langle\tilde\varphi_{\mathbf{3}_2}\rangle = (\lambda_1, \lambda_2, 1)\ , \qquad \epsilon := \mathrm{e}^{2\pi\mathrm{i}\langle T\rangle}\ .$$

$\mathbb{Z}_3^{(2)} \subset G_{\mathrm{traditional}}$ generated by $\rho_{\mathbf{3}_2,\mathrm{i}\infty}(\mathrm{ABA}^2) = \begin{pmatrix}\omega & 0 & 0\\ 0 & \omega^2 & 0\\ 0 & 0 & 1\end{pmatrix}$,

$\mathbb{Z}_3^{(3)} \subset G_{\mathrm{modular}}$ generated by $\rho_{\mathbf{3}_2,\mathrm{i}\infty}(\mathrm{T}) = \begin{pmatrix}\omega^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$.

Spontaneous breaking of eclectic symmetry controlled by ***technically natural*** small parameters

$$\epsilon, \lambda_1 \ll \lambda_2 \ll 1\ .$$

A concrete "Model A" example

Superpotential predicted, and tightly constrained: $(M_{\rm Pl}=1)$

$$W = \phi^0 \left[\left(\phi_u^0\,\varphi_u\right) Y_u\,H_u\,\overline{u}\,q + \left(\phi_d^0\,\varphi_d\right) Y_d\,H_d\,\overline{d}\,q \right] + \phi^0 \left[\left(\phi_e^0\,\varphi_d\right) Y_\ell\,H_d\,\overline{e}\,\ell + \left(\varphi_\nu\right) Y_\nu\,H_u\,\overline{\nu}\,\ell \right] + \phi_M^0\,\varphi_d\,\overline{\nu}\,\overline{\nu}\,.$$

All superpotential terms have the generic structure

$$\Phi_0 \ldots \Phi_0\,\hat{Y}^{(1)}(T)\,\Phi^{(1)}_{-2/3}\,\Phi^{(2)}_{-2/3}\,\Phi^{(3)}_{-2/3}\;,$$

"singlet flavon(s) × modular form × **triplet** matter × **triplet** matter × **triplet** flavon" .

$\Longrightarrow$ *All* mass terms can be written as [Nilles, Ramos-Sanchez, Vaudrevange '20]

$$\left(\Phi^{(1)}_{-2/3}\right)^{\rm T}\;M\left(T,c,\Phi^{(3)}_{-2/3}\right)\quad\Phi^{(2)}_{-2/3}\;,$$

$$M\left(T,c,\Phi^{(3)}_{-2/3}\right) \;=\; c\begin{pmatrix} \hat{Y}_2(T)\,X & -\dfrac{\hat{Y}_1(T)}{\sqrt{2}}\,Z & -\dfrac{\hat{Y}_1(T)}{\sqrt{2}}\,Y \\ -\dfrac{\hat{Y}_1(T)}{\sqrt{2}}\,Z & \hat{Y}_2(T)\,Y & -\dfrac{\hat{Y}_1(T)}{\sqrt{2}}\,X \\ -\dfrac{\hat{Y}_1(T)}{\sqrt{2}}\,Y & -\dfrac{\hat{Y}_1(T)}{\sqrt{2}}\,X & \hat{Y}_2(T)\,Z \end{pmatrix}.$$

with $\Phi^{(3)}_{-2/3} \equiv (X,Y,Z)$, and $\hat{Y}^{(1)}(T) \equiv \begin{pmatrix}\hat{Y}_1(T)\\ \hat{Y}_2(T)\end{pmatrix} \equiv \dfrac{1}{\eta(T)}\begin{pmatrix} -3\sqrt{2}\,\eta^3(3\,T) \\ 3\eta^3(3\,T)+\eta^3(T/3)\end{pmatrix}.$

A concrete "Model A" example

Superpotential predicted, and tightly constrained: $(M_{\mathrm{Pl}} = 1)$

$$W = \phi^0 \left[\left(\phi_u^0\, \varphi_u\right) Y_u\, H_u\, \overline{u}\, q + \left(\phi_d^0\, \varphi_d\right) Y_d\, H_d\, \overline{d}\, q \right] \\ + \phi^0 \left[\left(\phi_e^0\, \varphi_d\right) Y_\ell\, H_d\, \overline{e}\, \ell + \left(\varphi_\nu\right) Y_\nu\, H_u\, \overline{\nu}\, \ell \right] + \phi_M^0\, \varphi_d\, \overline{\nu}\, \overline{\nu}\,.$$

All superpotential terms have the generic structure

$$\Phi_0 \ldots \Phi_0\, \hat{Y}^{(1)}(T)\, \Phi^{(1)}_{-2/3}\, \Phi^{(2)}_{-2/3}\, \Phi^{(3)}_{-2/3}\,,$$

"singlet flavon(s) × modular form × **triplet** matter × **triplet** matter × **triplet** flavon" .

$\Longrightarrow$ *All* mass terms can be written as [Nilles, Ramos-Sanchez, Vaudrevange '20]

$$\left(\Phi^{(1)}_{-2/3}\right)^{\mathrm{T}}\, M\left(T, c, \Phi^{(3)}_{-2/3}\right) \quad \Phi^{(2)}_{-2/3}\,,$$

$$M\left(\langle T\rangle, \Lambda, \langle\tilde{\varphi}\rangle\right) \;=\; \Lambda \begin{pmatrix} \lambda_1 & 3\,\epsilon^{1/3} & 3\,\lambda_2\,\epsilon^{1/3} \\ 3\,\epsilon^{1/3} & \lambda_2 & 3\,\lambda_1\,\epsilon^{1/3} \\ 3\,\lambda_2\,\epsilon^{1/3} & 3\,\lambda_1\,\epsilon^{1/3} & 1 \end{pmatrix} + \mathcal{O}(\epsilon)\,.$$

$\Longrightarrow$ **Analytic control** over hierarchies, e.g. mass ratios

$$\frac{m_1}{m_2} \approx \left|\frac{\lambda_1}{\lambda_2}\right| \qquad \frac{m_2}{m_3} \approx |\lambda_2| \qquad \text{for} \qquad |\epsilon^{2/3}| \ll |\lambda_1 \lambda_2| \ll |\lambda_2|^2\,,$$

$$\frac{m_1}{m_2} \approx 9\left|\frac{\epsilon^{2/3}}{\lambda_2^2}\right| \qquad \frac{m_2}{m_3} \approx |\lambda_2| \qquad \text{for} \qquad |\lambda_1 \lambda_2| \ll |\epsilon^{2/3}| \ll |\lambda_2|^2\,.$$

Numerical analysis: fit to data

Strategy: [Baur, Nilles, Ramos-Sánchez, AT, Vaudrevange 2207.10677]

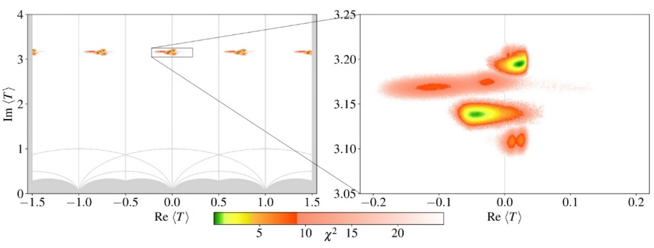
- Fit model to data as *proof-of-existence* of working consistent top-down models.
- Lepton data from $\mathrm{NuFITv5.1}$. [Esteban, Gonzalez-Garcia, Maltoni, Schwetz, Zhou ’20]
- Quark data from PDG.
- “State-of-the-art” handling of SUSY breaking and running. $(M_{\mathrm{SUSY}} \approx 10\,\mathrm{TeV}, \tan(\beta) \approx 10)$ [Ross, Serna ’08], [Antusch, Maurer ’13], [Feruglio ’17], [Ding, King, Yao ’21]
- Numerically minimize χ^2 using `lmfit`. [Newville et al.’21]
- Explore each minimum w/ MCMC sampler `emcee`. [Foreman-Mackey et al.’12]

Numerical analysis: fit to lepton data

Lepton sector fit: ***effectively*** only 7 parameters

$$x \;=\; \{\mathrm{Re}\,\langle T\rangle,\, \mathrm{Im}\,\langle T\rangle,\, \langle\tilde{\varphi}_{\mathrm{e},1}\rangle,\, \langle\tilde{\varphi}_{\mathrm{e},2}\rangle,\, \langle\tilde{\varphi}_{\nu,1}\rangle,\, \langle\tilde{\varphi}_{\nu,2}\rangle,\, \Lambda_\nu\}\;.$$

	right green region		left green region	
parameter	best-fit value	1σ interval	best-fit value	1σ interval
$\mathrm{Re}\,\langle T\rangle$	0.02279	$0.01345 \to 0.03087$	-0.04283	$-0.05416 \to -0.02926$
$\mathrm{Im}\,\langle T\rangle$	3.195	$3.191 \to 3.199$	3.139	$3.135 \to 3.142$
$\langle\tilde{\varphi}_{\mathrm{e},1}\rangle$	$-4.069\cdot 10^{-5}$	$-4.321\cdot 10^{-5} \to -3.947\cdot 10^{-5}$	$2.311\cdot 10^{-5}$	$2.196\cdot 10^{-5} \to 2.414\cdot 10^{-5}$
$\langle\tilde{\varphi}_{\mathrm{e},2}\rangle$	0.05833	$0.05793 \to 0.05876$	0.05826	$0.05792 \to 0.05863$
$\langle\tilde{\varphi}_{\nu,1}\rangle$	0.001224	$0.001201 \to 0.001248$	-0.001274	$-0.001304 \to -0.001248$
$\langle\tilde{\varphi}_{\nu,2}\rangle$	-0.9857	$-1.0128 \to -0.9408$	0.9829	$0.9433 \to 1.0122$
Λ_ν [eV]	0.05629	$0.05442 \to 0.05888$	0.05591	$0.05408 \to 0.05850$
χ^2	0.08		0.45	

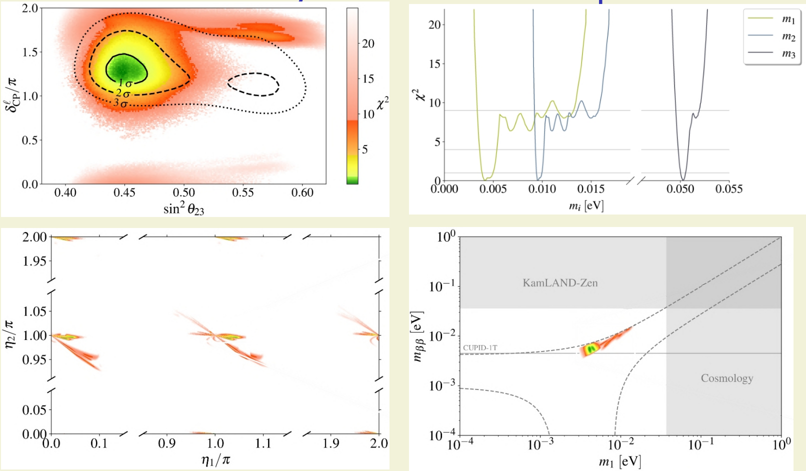


Numerical analysis: fit to lepton data

Neutrino sector fit:

	model			experiment		
observable	best fit	1σ interval	3σ interval	best fit	1σ interval	3σ interval
m_e/m_μ	0.00473	$0.00470 \to 0.00477$	$0.00462 \to 0.00485$	0.00474	$0.00470 \to 0.00478$	$0.00462 \to 0.00486$
m_μ/m_τ	0.0586	$0.0581 \to 0.0590$	$0.0572 \to 0.0600$	0.0586	$0.0581 \to 0.0590$	$0.0572 \to 0.0600$
$\sin^2\theta_{12}$	0.303	$0.294 \to 0.315$	$0.275 \to 0.335$	0.304	$0.292 \to 0.316$	$0.269 \to 0.343$
$\sin^2\theta_{13}$	0.02254	$0.02189 \to 0.02304$	$0.02065 \to 0.02424$	0.02246	$0.02184 \to 0.02308$	$0.02060 \to 0.02435$
$\sin^2\theta_{23}$	0.449	$0.436 \to 0.468$	$0.414 \to 0.593$	0.450	$0.434 \to 0.469$	$0.408 \to 0.603$
$\delta^\ell_{\mathcal{CP}}/\pi$	1.28	$1.15 \to 1.47$	$0.81 \to 1.94$	1.28	$1.14 \to 1.48$	$0.80 \to 1.94$
$\eta_1/\pi \mod 1$	0.029	$0.018 \to 0.048$	$-0.031 \to 0.090$	-	-	-
$\eta_2/\pi \mod 1$	0.994	$0.992 \to 0.998$	$0.935 \to 1.004$	-	-	-
$J_{\mathcal{CP}}$	-0.026	$-0.033 \to -0.015$	$-0.035 \to 0.019$	-0.026	$-0.033 \to -0.016$	$-0.033 \to 0.000$
$J_{\mathcal{CP}}^{\max}$	0.0335	$0.0330 \to 0.0341$	$0.0318 \to 0.0352$	0.0336	$0.0329 \to 0.0341$	$0.0317 \to 0.0353$
$\Delta m^2_{21}/10^{-5}$ [eV^2]	7.39	$7.35 \to 7.49$	$7.21 \to 7.65$	7.42	$7.22 \to 7.63$	$6.82 \to 8.04$
$\Delta m^2_{31}/10^{-3}$ [eV^2]	2.508	$2.488 \to 2.534$	$2.437 \to 2.587$	2.521	$2.483 \to 2.537$	$2.430 \to 2.593$
m_1 [eV]	0.0042	$0.0039 \to 0.0049$	$0.0034 \to 0.0131$	< 0.037	-	-
m_2 [eV]	0.0095	$0.0095 \to 0.0099$	$0.0092 \to 0.0157$	-	-	-
m_3 [eV]	0.0504	$0.0501 \to 0.0505$	$0.0496 \to 0.0519$	-	-	-
$\sum_i m_i$ [eV]	0.0641	$0.0636 \to 0.0652$	$0.0628 \to 0.0806$	< 0.120	-	-
$m_{\beta\beta}$ [eV]	0.0055	$0.0045 \to 0.0064$	$0.0040 \to 0.0145$	< 0.036	-	-
m_β [eV]	0.0099	$0.0097 \to 0.0102$	$0.0094 \to 0.0159$	< 0.8	-	-
χ^2	0.08					

Numerical analysis: fit to data $\rightarrow$ predictions



The fit predicts: [Baur, Nilles, Ramos-Sánchez, AT, Vaudrevange PRD'22 & JHEP'22]

- θ^{ℓ}_{23} lies in lower octant,
- Normal ordering of neutrino masses,
- Majorana phases $\eta_{1,2} \approx \pi$ close to CP conserving values.

Importance of Kähler corrections

- Kähler corrections are important, because they are unconstrained in generic bottom-up modular flavor models.
  [Chen, Ramos-Sánchez, Ratz '19]
- Unlike pure modular flavor theories, the traditional flavor symmetry helps to control the Kaehler potential.

  $\curvearrowright$ K is canonical at leading order. [Nilles, Ramos-Sanchez, Vaudrevange '20]
- However, there are higher-order Kähler corrections due to VEV of flavon fields that break the traditional flavor symmetry.
- **No** Kähler corrections included in our lepton sector fit.
- **Must** include Kähler corrections for quark sector (Note: this might be specific to Model A: $\varphi_d \equiv \varphi_\ell$).

Schematically:

$$K_{\mathrm{LO}} \supset -\log(-\mathrm{i}T + \mathrm{i}\overline{T}) + \sum_{\Phi} \left[(-\mathrm{i}T + \mathrm{i}\overline{T})^{-2/3} + (-\mathrm{i}T + \mathrm{i}\overline{T})^{1/3} |\hat{Y}^{(1)}(T)|^2 \right] |\Phi|^2 \;,$$

$$K_{\mathrm{NLO}} \supset \sum_{\Psi,\varphi} \left[(-\mathrm{i}T + \mathrm{i}\overline{T})^{-4/3} \sum_a |\Psi\varphi|^2_{\mathbf{1},a} + (-\mathrm{i}T + \mathrm{i}\overline{T})^{-1/3} \sum_a |\hat{Y}^{(1)}(T)\Psi\varphi|^2_{\mathbf{1},a} \right] \;.$$

Kähler corrections – parametrization

For a given quark flavor $f = \{\mathrm{u}, \mathrm{d}, \mathrm{q}\}$,

$$K_{ij}^{(f)} \approx \chi^{(f)} \left[\delta_{ij} + \lambda_{\varphi_{\mathrm{eff}}}^{(f)} \left(A_{ij}^{(f)} + \kappa_{\varphi_{\mathrm{eff}}}^{(f)} B_{ij}^{(f)}\right)\right] ,$$

with flavor space structures $A = A(\varphi, T)$ and $B = B(\varphi, T)$ that are fixed by group theory and depend on *all* flavon fields. We can define “effective flavons” such that

$$\sum_{\varphi} \lambda_{\varphi}^{(f)} A_{ij}(\varphi) =: \lambda_{\varphi_{\mathrm{eff}}}^{(f)} A_{ij}(\tilde{\varphi}_{\mathrm{eff}}^{(A,f)}) \equiv \lambda_{\varphi_{\mathrm{eff}}}^{(f)} A_{ij}^{(f)} ,$$

$$\sum_{\varphi} \lambda_{\varphi}^{(f)} \kappa_{\varphi}^{(f)} B_{ij}(\varphi) =: \lambda_{\varphi_{\mathrm{eff}}}^{(f)} \kappa_{\varphi_{\mathrm{eff}}}^{(f)} B_{ij}(\tilde{\varphi}_{\mathrm{eff}}^{(B,f)}) \equiv \lambda_{\varphi_{\mathrm{eff}}}^{(f)} \kappa_{\varphi_{\mathrm{eff}}}^{(f)} B_{ij}^{(f)} .$$

Tilde means we took the scale out of the flavon directions

$$\tilde{\varphi}_{\mathrm{eff}}^{(A,B)} := \varphi_{\mathrm{eff}}^{(A,B)} / \Lambda_{\varphi_{\mathrm{eff}}^{(A,B)}} \quad \text{such that} \quad \tilde{\varphi}_{\mathrm{eff}}^{(A,B)} := \left(\tilde{\varphi}_{\mathrm{eff},1}^{(A,B)}, \tilde{\varphi}_{\mathrm{eff},2}^{(A,B)}, 1\right)^{\mathrm{T}} .$$

Finally, we can define the parameters

$$\alpha_i^{(f)} := \sqrt{\lambda_{\varphi_{\mathrm{eff}}}^{(f)}} \langle \tilde{\varphi}_{\mathrm{eff},i}^{(A,f)} \rangle , \qquad \beta_i^{(f)} := \sqrt{\lambda_{\varphi_{\mathrm{eff}}}^{(f)}} \langle \tilde{\varphi}_{\mathrm{eff},i}^{(B,f)} \rangle ,$$

and one can show that

$$\lambda_{\varphi_{\mathrm{eff}}}^{(f)} A_{ij}^{(f)} = \alpha_i^{(f)} \alpha_j^{(f)} , \qquad \lambda_{\varphi_{\mathrm{eff}}}^{(f)} B_{ij}^{(f)} \approx \beta_i^{(f)} \beta_j^{(f)} .$$

Note: All this is very specific to Models of type A.

Numerical analysis: fit to quark & lepton data

Parameters:

	parameter	best-fit value
superpotential	$\mathrm{Im}\,\langle T\rangle$	3.195
	$\mathrm{Re}\,\langle T\rangle$	0.02279
	$\langle\tilde{\varphi}_{\mathrm{u},1}\rangle$	$2.0332\cdot 10^{-4}$
	$\langle\vartheta_{\mathrm{u},1}\rangle$	1.6481
	$\langle\tilde{\varphi}_{\mathrm{u},2}\rangle$	$6.3011\cdot 10^{-2}$
	$\langle\vartheta_{\mathrm{u},2}\rangle$	-1.5983
	$\langle\tilde{\varphi}_{\mathrm{e},1}\rangle$	$-4.069\cdot 10^{-5}$
	$\langle\tilde{\varphi}_{\mathrm{e},2}\rangle$	$5.833\cdot 10^{-2}$
	$\langle\tilde{\varphi}_{\nu,1}\rangle$	$1.224\cdot 10^{-3}$
	$\langle\tilde{\varphi}_{\nu,2}\rangle$	-0.9857
	Λ_ν [eV]	0.05629
Kähler potential	α_1^{u}	-0.94917
	α_2^{u}	0.0016906
	α_3^{u}	0.31472
	α_1^{d}	0.95067
	α_2^{d}	0.0077533
	α_3^{d}	0.30283
	α_1^{q}	-0.96952
	α_2^{q}	-0.20501
	α_3^{q}	0.041643

Observables:

	observable	model best fit	exp. best fit	exp. 1σ interval
quark sector	$m_{\mathrm{u}}/m_{\mathrm{c}}$	0.00193	0.00193	$0.00133 \to 0.00253$
	$m_{\mathrm{c}}/m_{\mathrm{t}}$	0.00280	0.00282	$0.00270 \to 0.00294$
	$m_{\mathrm{d}}/m_{\mathrm{s}}$	0.0505	0.0505	$0.0443 \to 0.0567$
	$m_{\mathrm{s}}/m_{\mathrm{b}}$	0.0182	0.0182	$0.0172 \to 0.0192$
	ϑ_{12} [deg]	13.03	13.03	$12.98 \to 13.07$
	ϑ_{13} [deg]	0.200	0.200	$0.193 \to 0.207$
	ϑ_{23} [deg]	2.30	2.30	$2.26 \to 2.34$
	$\delta_{\mathcal{CP}}^{\mathrm{q}}$ [deg]	69.2	69.2	$66.1 \to 72.3$
lepton sector	m_{e}/m_μ	0.00473	0.00474	$0.00470 \to 0.00478$
	m_μ/m_τ	0.0586	0.0586	$0.0581 \to 0.0590$
	$\sin^2\theta_{12}$	0.303	0.304	$0.292 \to 0.316$
	$\sin^2\theta_{13}$	0.0225	0.0225	$0.0218 \to 0.0231$
	$\sin^2\theta_{23}$	0.449	0.450	$0.434 \to 0.469$
	$\delta_{\mathcal{CP}}^{\ell}/\pi$	1.28	1.28	$1.14 \to 1.48$
	η_1/π	0.029	-	-
	η_2/π	0.994	-	-
	$J_{\mathcal{CP}}$	-0.026	-0.026	$-0.033 \to -0.016$
	$J_{\mathcal{CP}}^{\max}$	0.0335	0.0336	$0.0329 \to 0.0341$
	$\Delta m_{21}^2/10^{-5}$ [eV^2]	7.39	7.42	$7.22 \to 7.63$
	$\Delta m_{31}^2/10^{-3}$ [eV^2]	2.521	2.510	$2.483 \to 2.537$
	m_1 [eV]	0.0042	<0.037	-
	m_2 [eV]	0.0095	-	-
	m_3 [eV]	0.0504	-	-
	$\sum_i m_i$ [eV]	0.0641	<0.120	-
	$m_{\beta\beta}$ [eV]	0.0055	<0.036	-
	m_β [eV]	0.0099	<0.8	-
	χ^2	0.11		

Imposed constraints:

- $\kappa^{(f)}_{\varphi_{\mathrm{eff}}} = 1 \;\forall\, f$,
- $\alpha_i^{(f)} = \beta_i^{(f)} \;\forall f, i$,
- $\alpha_i^{(f)} \in \mathbb{R}$.

Possible lessons for bottom-up model building

Empirical observations:

- Modular flavor symmetries do not arise alone;
  They are generically accompanied by (partly overlapping!)
  - "traditional" discrete flavor symmetries (& flavons),
  - discrete (non-Abelian) R symmetries,
  - $\mathcal{CP}$-type symmetries.

$$G_{\textbf{eclectic}} = G_{\text{traditional}} \cup G_{\text{modular}} \cup G_{\text{R}} \cup \mathcal{CP}\,.$$

- Modular weights of matter fields are fractional,
  Modular weights of (Yukawa) couplings are integer.
- Modular weights are $1:1$ "locked" to all other flavor symmetry representations.

Conjecture: This may be a general top-down feature !?

for other known examples, see
[Ishiguro, Kobayashi, Otsuka '21], [Kikuchi, Kobayahsi, Uchida '21]
[Almumin, Chen, Knapp-Pérez, Ramos-Sánchez, Ratz, Shukla '21]

Many open questions

- Extra tori? $\rightarrow$ Metaplectic groups

  [Ding, Feruglio, Liu '20 &'21], [Nilles, Ramos-Sanchez, AT, Vaudrevange '21]

- Other possible realistic string configurations?

  "Size of the 'landscape' "?

- Moduli stabilization?
- Flavon potential?
- Restrictions on Kähler potential?

  see [Chen, Ramos-Sanchez, Ratz '19]
  [Chen, Knapp-Perez, Ramos-Hamud, Ramos-Sanchez, Ratz, Shukla '21]

Summary

- There are explicit models of heterotic string theory that reproduce, at low energies, the

  MSSM + (modular) flavor symmetry + flavons.

- The complete flavor symmetry can unambiguously be derived by the outer automorphisms of the Narain space group.
- One finds an "eclectic" flavor symmetry that non-trivially unifies:

$$G_{\textbf{eclectic}} = G_{\text{traditional}} \cup G_{\text{modular}} \cup G_{\text{R}} \cup \mathcal{CP}\,.$$

- This symmetry is broken by
  - Expectation values of the moduli, e.g. $\langle U \rangle$, $\langle T \rangle$.
  - Expectation values of the flavon fields.
- (Approximate) residual symmetries are common, and can help to naturally generate hierachies in masses and mixing matrix elements.
- We have identified one example for a model that can give a successfull fit to the observed SM flavor structure.

Thank You

Backup slides

What is an outer automorphism?

Example: $\mathbb{Z}_3$ symmetry, generated by $\mathsf{a}^3 = \mathsf{id}$.

- All elements of $\mathbb{Z}_3$: $\{\mathsf{id},\, \mathsf{a},\, \mathsf{a}^2\}$.
- Outer automorphism group ("Out") of $\mathbb{Z}_3$: generated by

$$u(\mathsf{a}) : \mathsf{a} \mapsto \mathsf{a}^2. \quad \big(\text{think: } \mathsf{u}\,\mathsf{a}\,\mathsf{u}^{-1} \;=\; \mathsf{a}^2\big)$$

$\mathbb{Z}_3$	id	a	a^2
$\mathbf{1}$	1	1	1
$\mathbf{1}'$	1	ω	ω^2
$\mathbf{1}''$	1	ω^2	ω

$(\omega := \mathrm{e}^{2\pi\,\mathrm{i}/3})$

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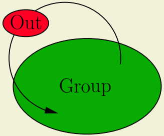

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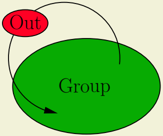
Abstract: Out is a reshuffling of symmetry elements.



In words: Out is a **"symmetry of the symmetry"**.

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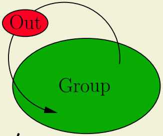

Concrete: Out is a 1:1 mapping of representations $\boldsymbol{r} \mapsto \boldsymbol{r}'$.
Comes with a transformation matrix U, which is given by

$$U\rho_{\boldsymbol{r}'}(\mathrm{g})U^{-1} = \rho_{\boldsymbol{r}}(u(\mathrm{g}))\,, \qquad \forall \mathrm{g} \in G\,.$$

(consistency condition)

[Fallbacher, AT, '15]
[Holthausen, Lindner, Schmidt, '13]

- $\rho_{\boldsymbol{r}}(g)$: representation matrix for group element $g \in G$
- $u : g \mapsto u(g)$: **outer** automorphism

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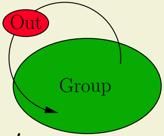

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E.g.: Physical CP trafo
is a special case of this!
$\boldsymbol{r} \mapsto \boldsymbol{r}' = \boldsymbol{r}^*$

er, AT, '15]
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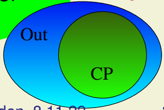


Flavor and Modular Symmetries

Feruglio: “Are neutrino masses modular forms?” [Feruglio ’17]

General (bottom-up) idea:

- Supersymmetric (say $N = 1$) theory.
- Ask for **modular invariance**: [Ferrara, Lüst, (Shapere), Theisen ’89(x2)]

$$\tau \mapsto \gamma\tau = \frac{a\tau + b}{c\tau + d}\,, \qquad \varphi^{(I)} \mapsto (c\tau + d)^{-k_I}\,\rho^{(I)}(\gamma)\varphi^{(I)}\,.$$

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)\,, \quad \{a, b, c, d\} \in \mathbb{Z}\,, \qquad \Phi := (\tau, \varphi)\,.$$

- *EITHER* $W(\Phi)$, $K(\Phi, \overline{\Phi})$ invariant (K up to Kähler transf.), $\longrightarrow$global SUSY
  OR compensating against each other. $\longrightarrow$SUGRA
- In any case, Yukawa couplings must be *modular forms*:

$$W(\Phi) = \sum_n Y_{I_1 \dots I_n}(\tau)\,\varphi^{(I_1)} \dots \varphi^{(I_n)}\,,$$

$$Y_{I_1 \dots I_n}(\tau) \mapsto Y_{I_1 \dots I_n}(\gamma\tau) \overset{!}{=} \left[\mathrm{e}^{\mathrm{i}\alpha(\gamma)}\right](c\tau + d)^{k_Y(n)}\,Y_{I_1 \dots I_n}(\tau)$$

- $\tau \to \langle\tau\rangle$ breaks modular symmetry $\Longleftrightarrow$ τ takes rôle of flavon!

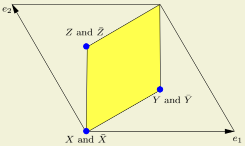
Origin of eclectic flavor symmetry in heterotic orbifolds

Narain lattice formulation of heterotic string theory: [Narain '86]

[Narain, Samardi, Witten '87],[Narain, M. H. Sarmadi, and C. Vafa,'87],[Groot Nibbelink & Vaudrevange '17]

Lattice can have symmetries. Symmetries can have fixed points.

e.g. $\mathbb{T}^2/\mathbb{Z}_3$ (with $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$)



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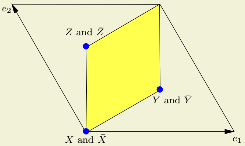

Symmetries can have **outer automorphisms**.

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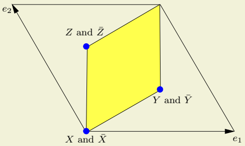

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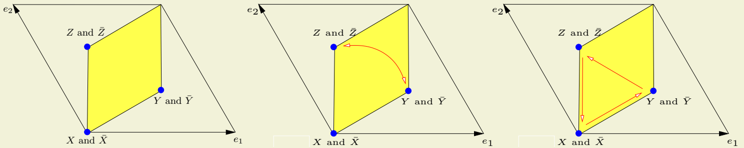

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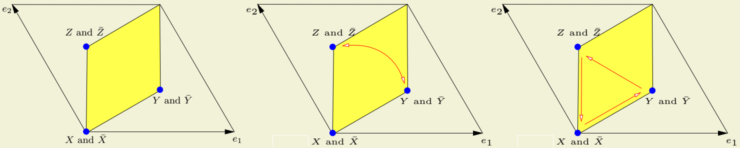

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New insight: Flavor symmetries are given by outer automorphisms of the Narain lattice space group! [Baur, Nilles, AT, Vaudrevange '19]

In this way we can unambiguously compute them in the top-down approach.

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- Bosonic string coordinates, D right- and D left-moving, $y_{\mathrm{R,L}}$, *compactified* on $2D$ torus:

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- $\Theta^K = \mathbb{1}$, is an "orbifold twist" with $\theta_{\mathrm{R,L}} \in \mathrm{SO}(D)$.
- "Narain lattice":

$$\Gamma = \{E\,\hat{N} \mid \hat{N} \in \mathbb{Z}^{2D}\}$$

(Γ is even, self-dual lattice with metric $\eta = \mathrm{diag}(-\mathbb{1}_D, \mathbb{1}_D)$.)

- $\hat{N} = (n,m) \in \mathbb{Z}^{2D}$, n: winding number, m: Kaluza-Klein number of string boundary condition.
- E: "Narain vielbein", depends on moduli of the torus; $E^{\mathrm{T}}E \equiv \mathcal{H} = \mathcal{H}(T,U)$.

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$$\mathcal{H}(T,U) \;=\; \frac{1}{\mathrm{Im}\,T\,\mathrm{Im}\,U}\begin{pmatrix} |T|^2 & |T|^2\,\mathrm{Re}\,U & \mathrm{Re}\,T\,\mathrm{Re}\,U & -\mathrm{Re}\,T \\ |T|^2\,\mathrm{Re}\,U & |T\,U|^2 & |U|^2\,\mathrm{Re}\,T & -\mathrm{Re}\,T\,\mathrm{Re}\,U \\ \mathrm{Re}\,T\,\mathrm{Re}\,U & |U|^2\,\mathrm{Re}\,T & |U|^2 & -\mathrm{Re}\,U \\ -\mathrm{Re}\,T & -\mathrm{Re}\,T\,\mathrm{Re}\,U & -\mathrm{Re}\,U & 1 \end{pmatrix}.$$

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Narain space group $g = (\Theta^k, E\, \hat{N}) \in S_{\mathrm{Narain}}$ is given by multiplicative closure of all twist and shifts

$$S_{\mathrm{Narain}} \ := \ \langle\ (\Theta, 0)\ \ ,\ \ (\mathbb{1}, E_i)\ \ \text{for}\ \ i \in \{1, \ldots, 2D\}\ \rangle\ .$$

Flavor symmetries from top-down perspective

The *"whole"* story: **Narain lattice** formulation of heterotic string theory:

[Narain '86], [Narain, Samardi, Witten '87], [Narain, Sarmadi, Vafa'87], [Groot Nibbelink, Vaudrevange '17]

- Winding $\leftrightarrow$ momentum duality $\Rightarrow D \curvearrowright 2D$ lattice.
- "Narain lattice": $\Gamma = \{E\,\hat{N} \mid \hat{N} = (n,m) \in \mathbb{Z}^{2D}\}$
  even, self-dual, metric $\eta = \mathrm{diag}(-\mathbb{1}_D, \mathbb{1}_D)$, n: winding #, m: Kaluza-Klein #
  E: "Narain vielbein", depends on moduli of the torus; $E^{\mathrm{T}}E \equiv \mathcal{H} = \mathcal{H}(T,U)$.
- Narain lattice space group $S_{\mathrm{Narain}} \ni g = (\Theta^k, E\,\hat{N})$.
- Outs of S_{Narain}, $h := (\hat{\Sigma}, \hat{T}) \notin S_{\mathrm{Narain}}$,

$$g \overset{h}{\mapsto} h\,g\,h^{-1} \overset{!}{\in} S_{\mathrm{Narain}}\,, \qquad \hat{\Sigma}^{\mathrm{T}}\hat{\eta}\,\hat{\Sigma} = \hat{\eta}\,.$$

  $\curvearrowright$ Solve **consistency conditions** to find *all* Outs.

The outer automorphisms of S_{Narain} include: [Baur, Nilles, AT, Vaudrevange '19 (2x)]

(i) fixed-point permutation symmetry (S_3 in previous example),

(ii) "space group selection rules" [Hamidi and Vafa '86]

(iii) target space modular transformations (inkl. T-duality),

(iv) "$\mathcal{CP}$-like" transformations.

Details of representations

	quarks and leptons						Higgs fields		flavons							
label	q	$\bar{u}$	$\bar{d}$	ℓ	$\bar{e}$	$\bar{\nu}$	H_u	H_d	φ_e	φ_u	φ_ν	ϕ^0	ϕ^0_M	ϕ^0_e	ϕ^0_u	ϕ^0_d
$\mathrm{SU}(3)_c$	$\mathbf{3}$	$\overline{\mathbf{3}}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathrm{SU}(2)_L$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathrm{U}(1)_Y$	$1/6$	$-2/3$	$1/3$	$-1/2$	1	0	$1/2$	$-1/2$	0	0	0	0	0	0	0	0
$\Delta(54)$	$\mathbf{3}_2$	$\mathbf{3}_2$	$\mathbf{3}_2$	$\mathbf{3}_2$	$\mathbf{3}_2$	$\mathbf{3}_2$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}_2$	$\mathbf{3}_2$	$\mathbf{3}_2$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
T'	$\mathbf{2}'\oplus\mathbf{1}$	$\mathbf{2}'\oplus\mathbf{1}$	$\mathbf{2}'\oplus\mathbf{1}$	$\mathbf{2}'\oplus\mathbf{1}$	$\mathbf{2}'\oplus\mathbf{1}$	$\mathbf{2}'\oplus\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}'\oplus\mathbf{1}$	$\mathbf{2}'\oplus\mathbf{1}$	$\mathbf{2}'\oplus\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbb{Z}_9^R$	1	1	1	1	1	1	0	0	1	1	1	0	0	0	0	0
n	$-2/3$	$-2/3$	$-2/3$	$-2/3$	$-2/3$	$-2/3$	0	0	$-2/3$	$-2/3$	$-2/3$	0	0	0	0	0
$\mathbb{Z}_3$	1	1	ω	ω	1	1	1	1	1	ω	ω^2	1	1	ω^2	ω^2	ω^2
$\mathbb{Z}_3$	ω^2	ω^2	1	1	ω^2	ω^2	1	1	ω^2	1	ω	1	1	ω^2	ω^2	ω^2
$\mathbb{Z}_3$	1	1	ω	1	1	1	1	1	1	ω^2	1	1	1	1	ω	ω^2

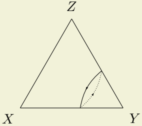
Vectorlike exotic matter:

#	irrep	labels	#	irrep	labels
101	$(\mathbf{1},\mathbf{1})_0$	s_i			
51	$(\mathbf{1},\mathbf{1})_{-1/3}$	V_i	51	$(\mathbf{1},\mathbf{1})_{1/3}$	$\overline{V}_i$
14	$(\mathbf{1},\mathbf{1})_{-2/3}$	X_i	14	$(\mathbf{1},\mathbf{1})_{2/3}$	$\overline{X}_i$
10	$(\mathbf{1},\mathbf{2})_{-1/2}$	L_i	10	$(\mathbf{1},\mathbf{2})_{1/2}$	$\overline{L}_i$
9	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$	$\overline{D}_i$	9	$(\mathbf{3},\mathbf{1})_{-1/3}$	D_i
8	$(\mathbf{1},\mathbf{2})_{-1/6}$	W_i	8	$(\mathbf{1},\mathbf{2})_{1/6}$	$\overline{W}_i$
2	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$	$\overline{U}_i$	2	$(\mathbf{3},\mathbf{1})_{2/3}$	U_i
4	$(\overline{\mathbf{3}},\mathbf{1})_0$	Z_i	4	$(\mathbf{3},\mathbf{1})_0$	$\overline{Z}_i$
1	$(\overline{\mathbf{3}},\mathbf{1})_{-1/3}$	Y	1	$(\mathbf{3},\mathbf{1})_{1/3}$	$\overline{Y}$

Transformation of massless matter fields

sector	matter fields Φ_n	eclectic flavor group $\Omega(2)$: modular T' subgroup: irrep $\boldsymbol{s}$	$\rho_{\boldsymbol{s}}(\mathrm{S})$	$\rho_{\boldsymbol{s}}(\mathrm{T})$	n	traditional $\Delta(54)$ subgroup: irrep $\boldsymbol{r}$	$\rho_{\boldsymbol{r}}(\mathrm{A})$	$\rho_{\boldsymbol{r}}(\mathrm{B})$	$\rho_{\boldsymbol{r}}(\mathrm{C})$	$\mathbb{Z}_9^R$ R
bulk	Φ_0	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	$+1$	0
	Φ_{-1}	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3
θ	$\Phi_{-2/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-2/3$	$\mathbf{3}_2$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$+\rho(\mathrm{C})$	1
	$\Phi_{-5/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-5/3$	$\mathbf{3}_1$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$-\rho(\mathrm{C})$	-2
θ^2	$\Phi_{-1/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$-1/3$	$\bar{\mathbf{3}}_1$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$-\rho(\mathrm{C})$	2
	$\Phi_{+2/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$+2/3$	$\bar{\mathbf{3}}_2$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$+\rho(\mathrm{C})$	5
super-potential	$\mathcal{W}$	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3

table from [Nilles, Ramos-Sánchez, Vaudrevange '20]



$$\left(\omega := \mathrm{e}^{2\pi\mathrm{i}/3}\right)$$

$$\rho(\mathrm{S}) = \frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{pmatrix}, \quad \rho(\mathrm{T}) = \begin{pmatrix}\omega^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

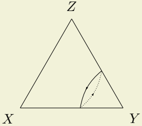
$$\rho(\mathrm{A}) = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}, \quad \rho(\mathrm{B}) = \begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix}, \quad \rho(\mathrm{C}) = -\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} = \rho(\mathrm{S})^2.$$

Transformation of massless matter fields

sector	matter fields Φ_n	eclectic flavor group $\Omega(2)$								
		modular T' subgroup				traditional $\Delta(54)$ subgroup				$\mathbb{Z}_9^R$
		irrep $\boldsymbol{s}$	$\rho_{\boldsymbol{s}}(\mathrm{S})$	$\rho_{\boldsymbol{s}}(\mathrm{T})$	n	irrep $\boldsymbol{r}$	$\rho_{\boldsymbol{r}}(\mathrm{A})$	$\rho_{\boldsymbol{r}}(\mathrm{B})$	$\rho_{\boldsymbol{r}}(\mathrm{C})$	R
bulk	Φ_0	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	$+1$	0
	Φ_{-1}	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3
θ	$\Phi_{-2/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-2/3$	$\mathbf{3}_2$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$+\rho(\mathrm{C})$	1
	$\Phi_{-5/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-5/3$	$\mathbf{3}_1$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$-\rho(\mathrm{C})$	-2
θ^2	$\Phi_{-1/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$-1/3$	$\bar{\mathbf{3}}_1$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$-\rho(\mathrm{C})$	2
	$\Phi_{+2/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$+2/3$	$\bar{\mathbf{3}}_2$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$+\rho(\mathrm{C})$	5
super-potential	$\mathcal{W}$	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3

$\left(\omega := \mathrm{e}^{2\pi\mathrm{i}/3}\right)$

table from [Nilles, Ramos-Sánchez, Vaudrevange '20]



$$\rho(\mathrm{S}) = \frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{pmatrix}, \quad \rho(\mathrm{T}) = \begin{pmatrix}\omega^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$\rho(\mathrm{A}) = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}, \quad \rho(\mathrm{B}) = \begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix}, \quad \rho(\mathrm{C}) = -\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} = \rho(\mathrm{S})^2.$$

Example: Breakdown of $H(3,2,1)$ at $\langle T\rangle = \omega$

$H(3,2,1)$ subgroup	branchings $\Phi_{-2/3}$	branchings $\Phi_{-5/3}$	subgroup generator(s)	corresponding vevs $\langle\Phi_{-2/3}\rangle$	corresponding vevs $\langle\Phi_{-5/3}\rangle$
$\mathrm{S}_3^{(2)}\times\mathbb{Z}_3^{(3)}$	$\mathbf{1}_1'\oplus\mathbf{2}_c$	$\mathbf{1}\oplus\mathbf{2}_c$	$\mathrm{C}, \mathrm{AB^2A}, \mathrm{AB^2AR(ST)}$	–	$(\omega^2,1,1)^\mathrm{T}$
$\mathbb{Z}_3^{(2)}\times\mathbb{Z}_3^{(3)}$	$\mathbf{1}\oplus\mathbf{1}_{\omega,1}\oplus\mathbf{1}_{\omega^2,\omega}$	$\mathbf{1}\oplus\mathbf{1}_{\omega^2,1}\oplus\mathbf{1}_{\omega,\omega^2}$	$\mathrm{AB^2A}, \mathrm{AB^2AR(ST)}$	$(\omega^2,1,1)^\mathrm{T}$	$(\omega^2,1,1)^\mathrm{T}\oplus\langle\Phi_{-1}\rangle$
$\mathbb{Z}_3^{(3)}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}_\omega$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}_{\omega^2}$	$\mathrm{AB^2AR(ST)}$	$(0,1,-\omega^2)^\mathrm{T}$ $+\alpha(1,0,-\omega^2)^\mathrm{T}$	$(1,-1,0)^\mathrm{T}$ $+\alpha(0,-\omega,1)^\mathrm{T}$
$\mathrm{S}_3^{(1)}$	$\mathbf{1}'\oplus\mathbf{2}$	$\mathbf{1}\oplus\mathbf{2}$	C, A	–	$(1,1,1)^\mathrm{T}$
$\mathbb{Z}_3^{(1)}$	$\mathbf{1}\oplus\mathbf{1}_\omega\oplus\mathbf{1}_{\omega^2}$	$\mathbf{1}\oplus\mathbf{1}_\omega\oplus\mathbf{1}_{\omega^2}$	A	$(1,1,1)^\mathrm{T}$	$(1,1,1)^\mathrm{T}\oplus\langle\Phi_{-1}\rangle$
$\mathbb{Z}_6$	$\mathbf{1}\oplus\mathbf{1}_{-1}\oplus\mathbf{1}_{-\omega}$	$\mathbf{1}\oplus\mathbf{1}_{-1}\oplus\mathbf{1}_\omega$	$\mathrm{B^2ACR^2(ST)^2}$	$(1,-1,0)^\mathrm{T}$	$(1,1,-2\omega^2)^\mathrm{T}$
$\mathbb{Z}_3^{(3)}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}_{\omega^2}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}_{\omega^2}$	$\mathrm{AB^2AR(ST)}$	$(0,1,-\omega^2)^\mathrm{T}$ $+\alpha(1,0,-\omega^2)^\mathrm{T}$	$(1,-1,0)^\mathrm{T}$ $+\alpha(0,-\omega,1)^\mathrm{T}$
$\mathbb{Z}_3^{(4)}$	$\mathbf{1}\oplus\mathbf{1}_\omega\oplus\mathbf{1}_{\omega^2}$	$\mathbf{1}\oplus\mathbf{1}_\omega\oplus\mathbf{1}_{\omega^2}$	$\mathrm{BR(ST)^2}$	$(1,a,b)^\mathrm{T}$	$(1,a,b)^\mathrm{T}$
$\mathbb{Z}_2$	$\mathbf{1}\oplus\mathbf{1}_{-1}\oplus\mathbf{1}_{-1}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}_{-1}$	C	$(0,1,-1)^\mathrm{T}$ (preserves $\mathbb{Z}_6^{(2)}$)	$(1,0,0)^\mathrm{T}$ $+\alpha(0,1,1)^\mathrm{T}$

Representation matrices of the flavor group of twisted matter fields $\Phi_{-2/3}$ and $\Phi_{-5/3}$

$$\Phi_{-2/3}: \quad \rho_{\mathbf{3}_2,\omega}(\mathrm{A}) = \rho(\mathrm{A}), \qquad \rho_{\mathbf{3}_2,\omega}(\mathrm{B}) = \rho(\mathrm{B}), \qquad \rho_{\mathbf{3}_2,\omega}(\mathrm{C}) = \rho(\mathrm{C}),$$

$$\rho_{\mathbf{3}_2,\omega}(\mathrm{R}) = \mathrm{e}^{2\pi\mathrm{i}/9}\,\mathbb{1}_3, \qquad \rho_{\mathbf{3}_2,\omega}(\mathrm{ST}) = \mathrm{e}^{2\pi\mathrm{i}\,2/9}\,\rho(\mathrm{ST}), \qquad \text{and}$$

$$\Phi_{-5/3}: \quad \rho_{\mathbf{3}_1,\omega}(\mathrm{A}) = \rho(\mathrm{A}), \qquad \rho_{\mathbf{3}_1,\omega}(\mathrm{B}) = \rho(\mathrm{B}), \qquad \rho_{\mathbf{3}_1,\omega}(\mathrm{C}) = -\rho(\mathrm{C}),$$

$$\rho_{\mathbf{3}_1,\omega}(\mathrm{R}) = \mathrm{e}^{-4\pi\mathrm{i}/9}\,\mathbb{1}_3, \qquad \rho_{\mathbf{3}_1,\omega}(\mathrm{ST}) = \mathrm{e}^{2\pi\mathrm{i}\,5/9}\,\rho(\mathrm{ST}).$$

Narain vielbein

The Narain vielbein can be parameterized as (in absence of Wilson lines)

$$E \ := \ \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{e^{-\mathrm{T}}}{\sqrt{\alpha'}}\,(G-B) & -\sqrt{\alpha'}\,e^{-\mathrm{T}} \\ \frac{e^{-\mathrm{T}}}{\sqrt{\alpha'}}\,(G+B) & \sqrt{\alpha'}\,e^{-\mathrm{T}} \end{pmatrix} .$$

In this definition of the Narain vielbein, e denotes the vielbein of the D-dimensional geometrical torus $\mathbb{T}^D$ with metric $G := e^{\mathrm{T}}e$, $e^{-\mathrm{T}}$ corresponds to the inverse transposed matrix of e, B is the anti-symmetric background B-field ($B = -B^{\mathrm{T}}$), and α' is called the Regge slope.

World-sheet modular invariance requires E to span even, self-dual lattice $\Gamma = \{E\,\hat{N} \mid \hat{N} \in \mathbb{Z}^{2D}\}$ with metric η of signature (D,D). Consequently, one can always choose E such that

$$E^{\mathrm{T}}\eta\,E \ = \ \hat{\eta}\,, \quad \text{where} \quad \eta \ := \ \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad \text{and} \quad \hat{\eta} \ := \ \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} .$$

Transformation of moduli

To compute the transformation properties of the moduli T and U we use the generalized metric $\mathcal{H} = E^{\mathrm{T}}E$. As the Narain vielbein depends on the moduli $E = E(T,U)$ so does the generalized metric $\mathcal{H} = \mathcal{H}(T,U)$. It transforms as

$$\mathcal{H}(T,U) \stackrel{\hat{\Sigma}}{\longmapsto} \mathcal{H}(T',U') = \hat{\Sigma}^{-\mathrm{T}}\mathcal{H}(T,U)\hat{\Sigma}^{-1} .$$

This equation can be used to read off the transformations of the moduli

$$T \stackrel{\hat{\Sigma}}{\longmapsto} T' = T'(T,U) \quad \text{and} \quad U \stackrel{\hat{\Sigma}}{\longmapsto} U' = U'(T,U) .$$

For a two-torus $\mathbb{T}^2$, the generalized metric in terms of the torus moduli reads

$$\mathcal{H}(T,U) = \frac{1}{\operatorname{Im} T \operatorname{Im} U} \begin{pmatrix} |T|^2 & |T|^2 \operatorname{Re} U & \operatorname{Re} T \operatorname{Re} U & -\operatorname{Re} T \\ |T|^2 \operatorname{Re} U & |T\,U|^2 & |U|^2 \operatorname{Re} T & -\operatorname{Re} T \operatorname{Re} U \\ \operatorname{Re} T \operatorname{Re} U & |U|^2 \operatorname{Re} T & |U|^2 & -\operatorname{Re} U \\ -\operatorname{Re} T & -\operatorname{Re} T \operatorname{Re} U & -\operatorname{Re} U & 1 \end{pmatrix} .$$

Explicit generators of $\Omega(2)$ for $\mathbb{T}^2/\mathbb{Z}_3$

$\mathrm{SL}(2,\mathbb{Z})_T$ modular generators S and T arise from rotational outer automorphisms and act on the modulus via

$$\mathrm{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

Reflectional outer automorphism coresponding to $\mathbb{Z}_2^{\mathcal{CP}}$ $\mathcal{CP}$-like transformation:

$$\mathrm{K}_* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\rho(\mathrm{S}) = \frac{\mathrm{i}}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \quad \text{and} \quad \rho(\mathrm{T}) = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

The traditional flavor symmetry $\Delta(54)$ is generated by two translational outer automorphisms of the Narain space group A and B, together with the $\mathbb{Z}_2$ rotational outer automorphism $\mathrm{C} := \mathrm{S}^2$.

$$\rho(\mathrm{A}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \ \rho(\mathrm{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \text{ and } \rho(\mathrm{C}) = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \rho(\mathrm{S})^2,$$