Modular flavour symmetries from the **bottom up**

in collaboration with S.T. Petcov, P.P. Novichkov

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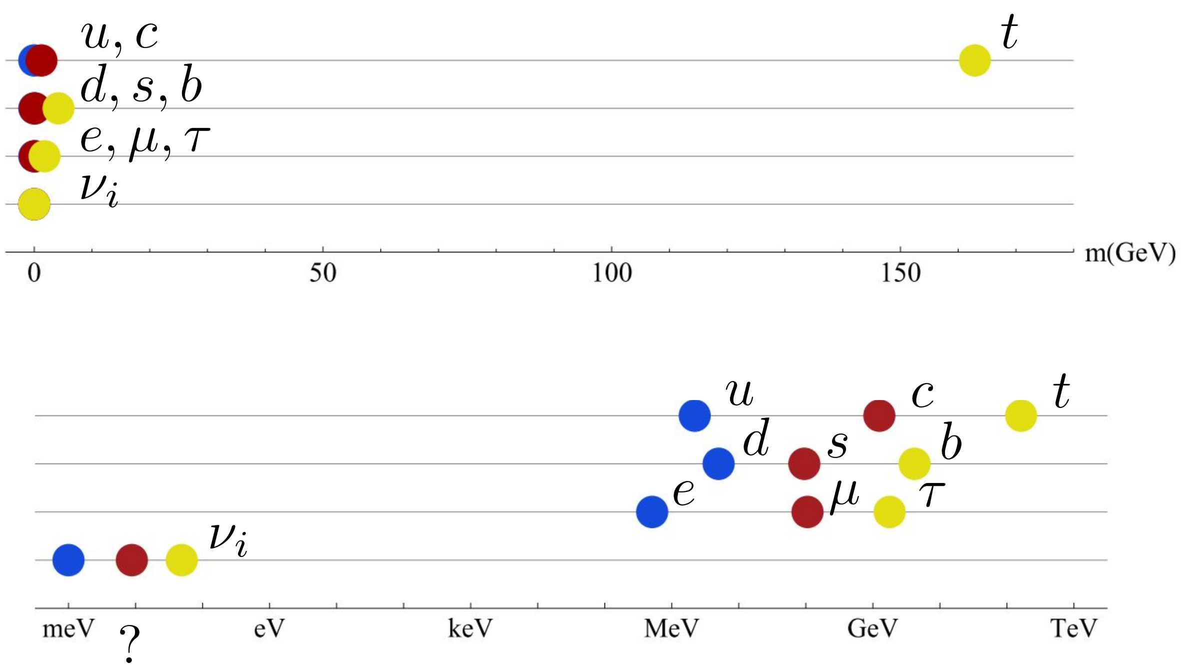
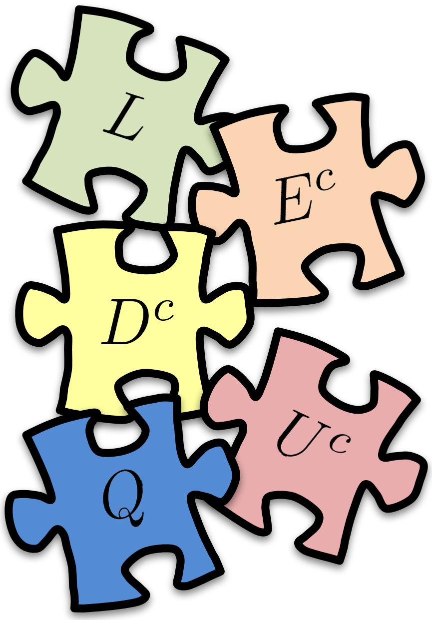
8 November 2022

Outline

- Why modular symmetries?
- How do modular symmetries work?
- Fermion mass hierarchies from mod. sym.
- Modulus stabilisation

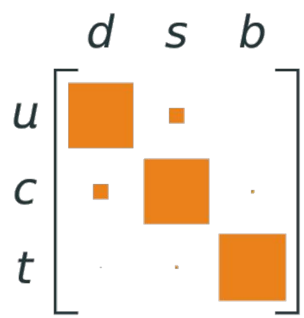
The flavour puzzle

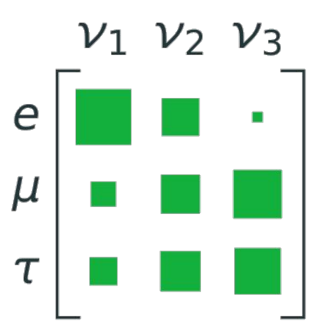


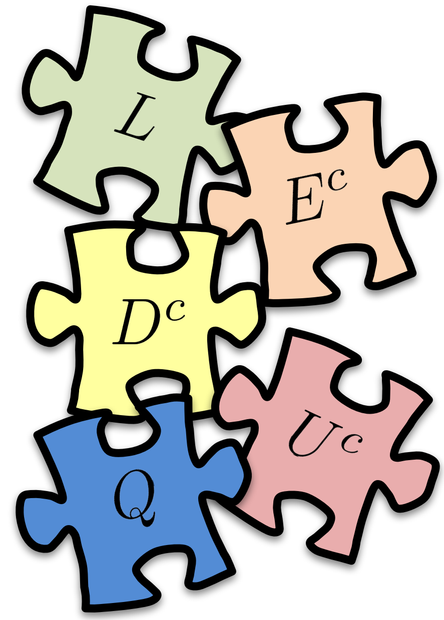


adapted from R. Toorop's PhD thesis

The flavour puzzle

$V_{\mathrm{CKM}} \sim$ 

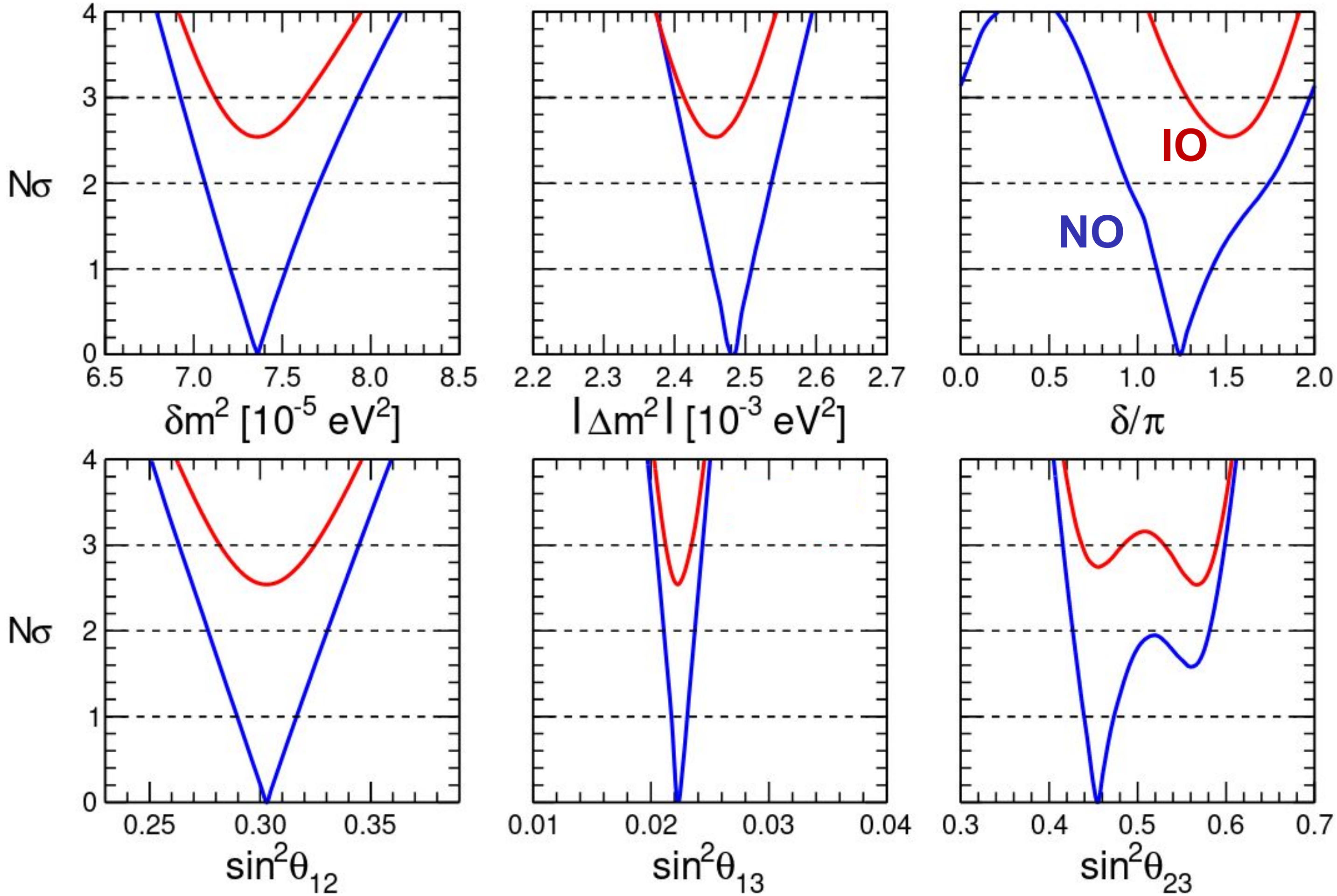
$U_{\mathrm{PMNS}} \sim$ 



adapted from P. Novichkov's slides at PASCOS 2021

3ν flavour paradigm

from Capozzi et al. 2107.00532,
see also València 2006.11237, NuFIT 2007.14792



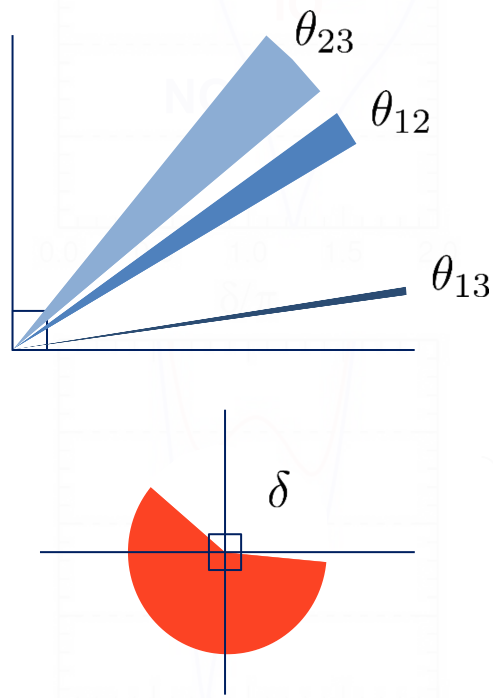
$\sim 2.5\sigma$

3ν flavour paradigm

from Capozzi et al. 2107.00532,
see also València 2006.11237, NuFIT 2007.14792

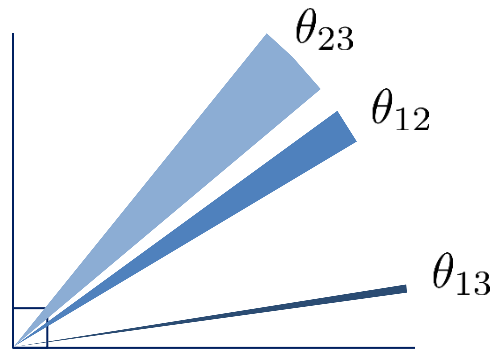
For a spectrum with NO:

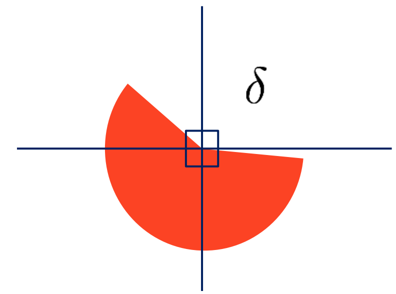
Parameter	Best-fit value
$\Delta m^2_\odot$	$7.36 \times 10^{-5}\ \text{eV}^2$
$\lvert\Delta m^2_{\text{atm}}\rvert$	$2.49 \times 10^{-3}\ \text{eV}^2$
$\sin^2\theta_{12}$	0.303
$\sin^2\theta_{13}$	0.0223
$\sin^2\theta_{23}$	0.455
δ	$1.24\,\pi$

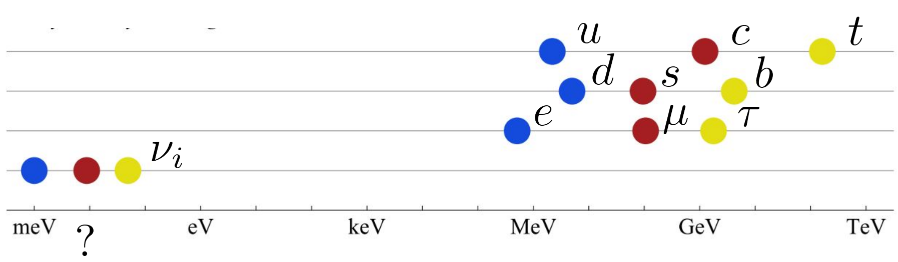


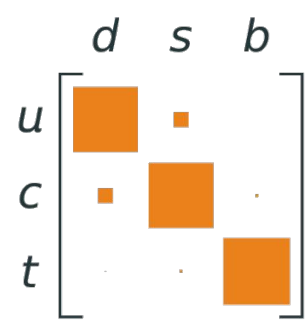
Is there an organizing principle behind this?

$$\frac{\Delta m^2_\odot}{|\Delta m^2_A|} \sim \frac{1}{30}$$

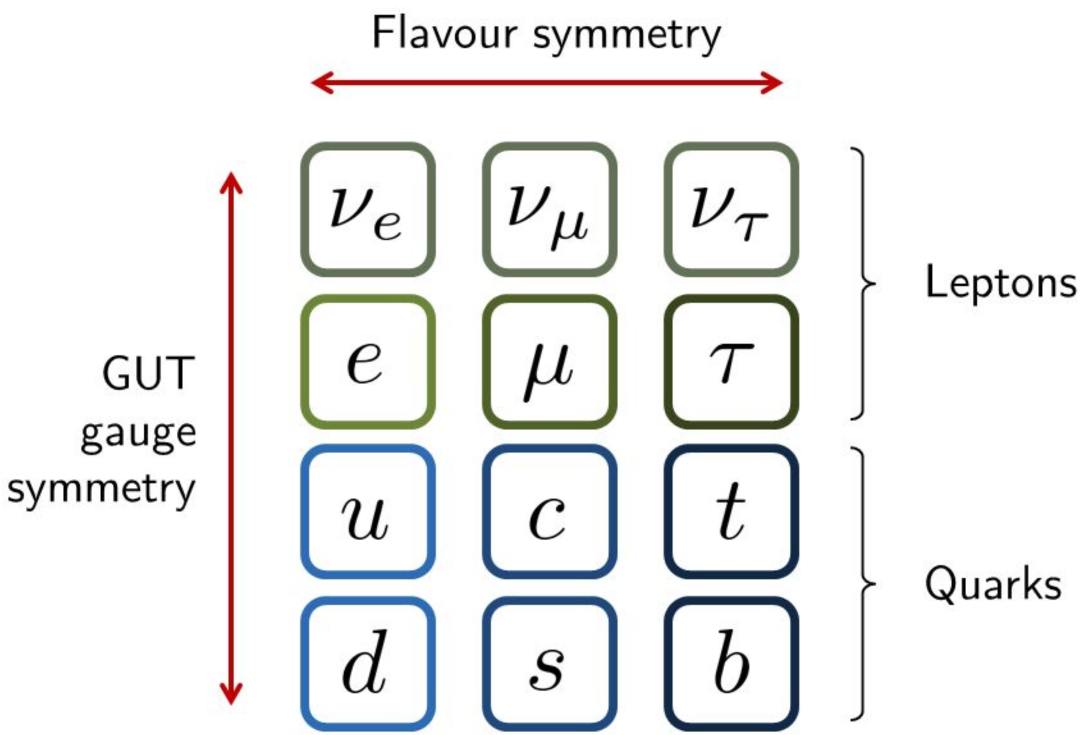








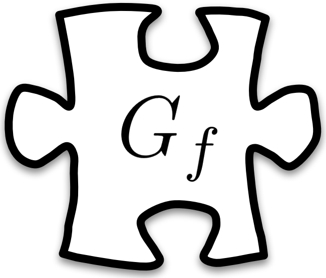
Flavour symmetries



For reviews, see: Altarelli and Feruglio (2010), Ishimori et al. (2010), King and Luhn (2013), Petcov (2017), Feruglio and Romanino (2019)

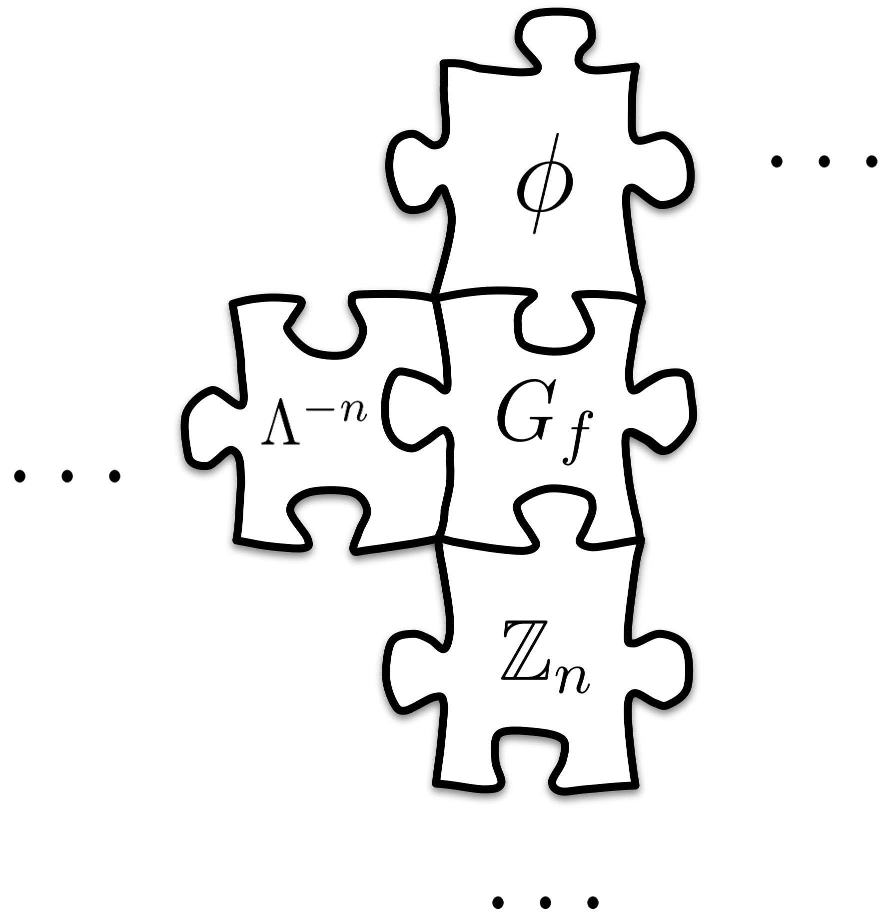
Problems with the usual approach

Non-Abelian discrete flavour symmetries

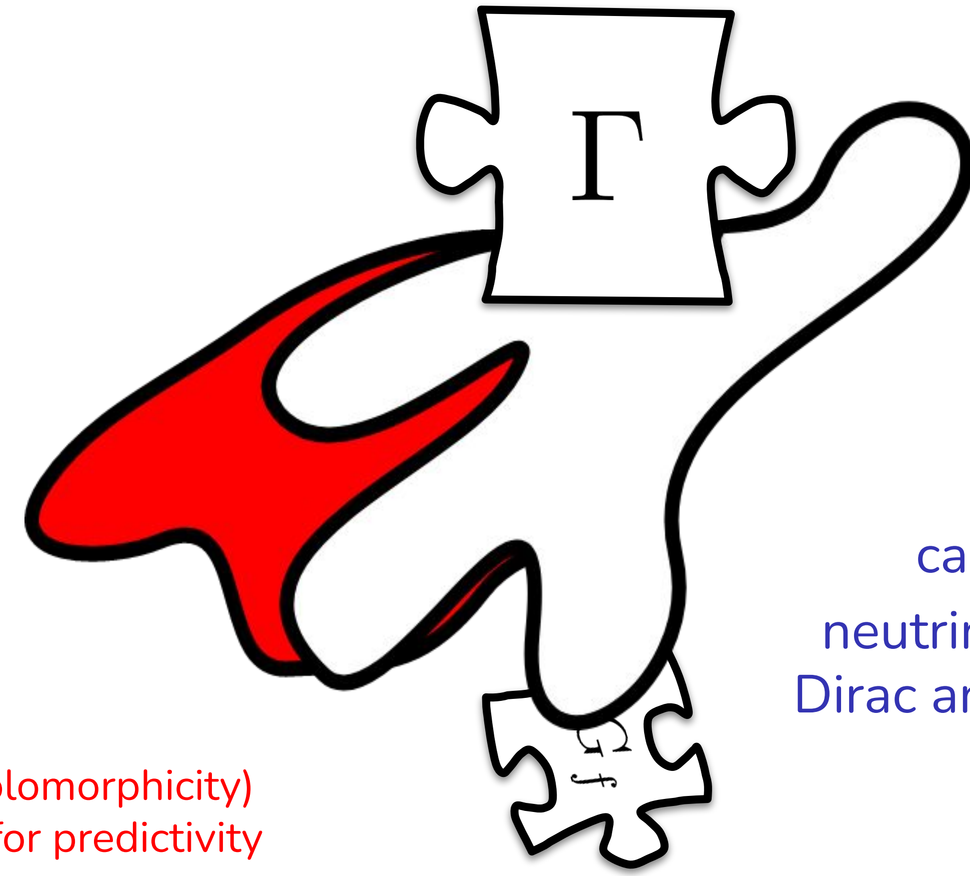


model-independent approaches relying on residual symmetries constrain mixing and the Dirac phase

Problems with the usual approach



Modular symmetry to the rescue!



F. Feruglio, 1706.08749

can constrain all:

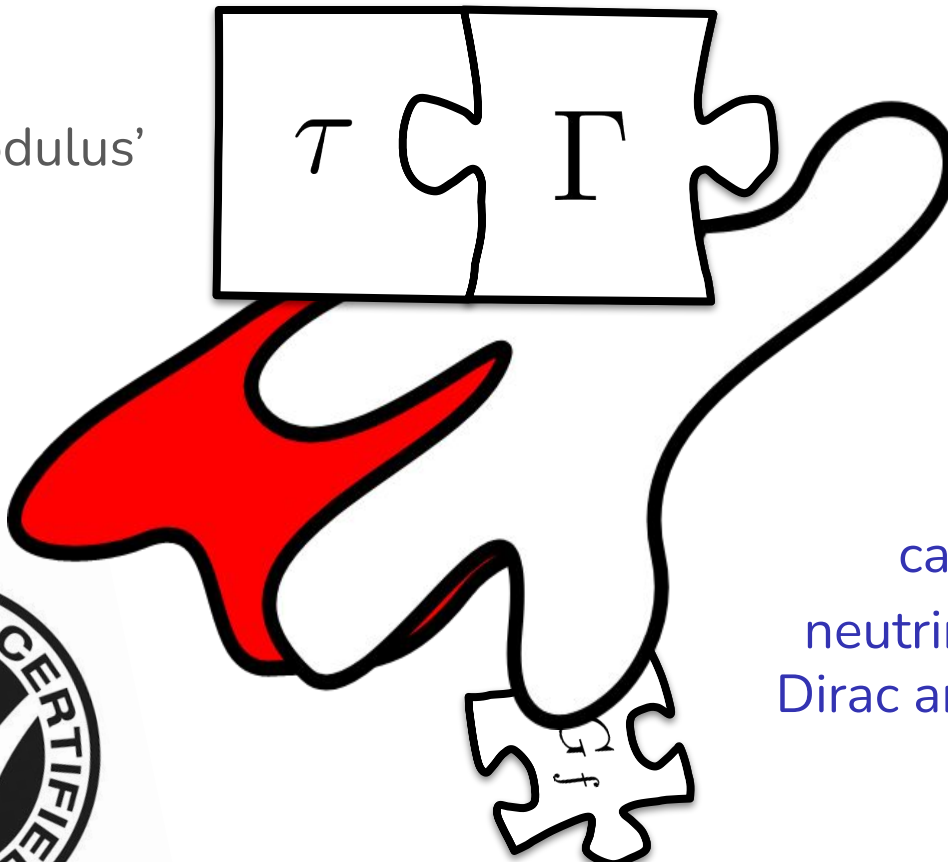
neutrino masses, mixing, Dirac and Majorana phases

SUSY (holomorphicity) **required** for predictivity

see also 2010.07952

Modular symmetry to the rescue!

'modulus'



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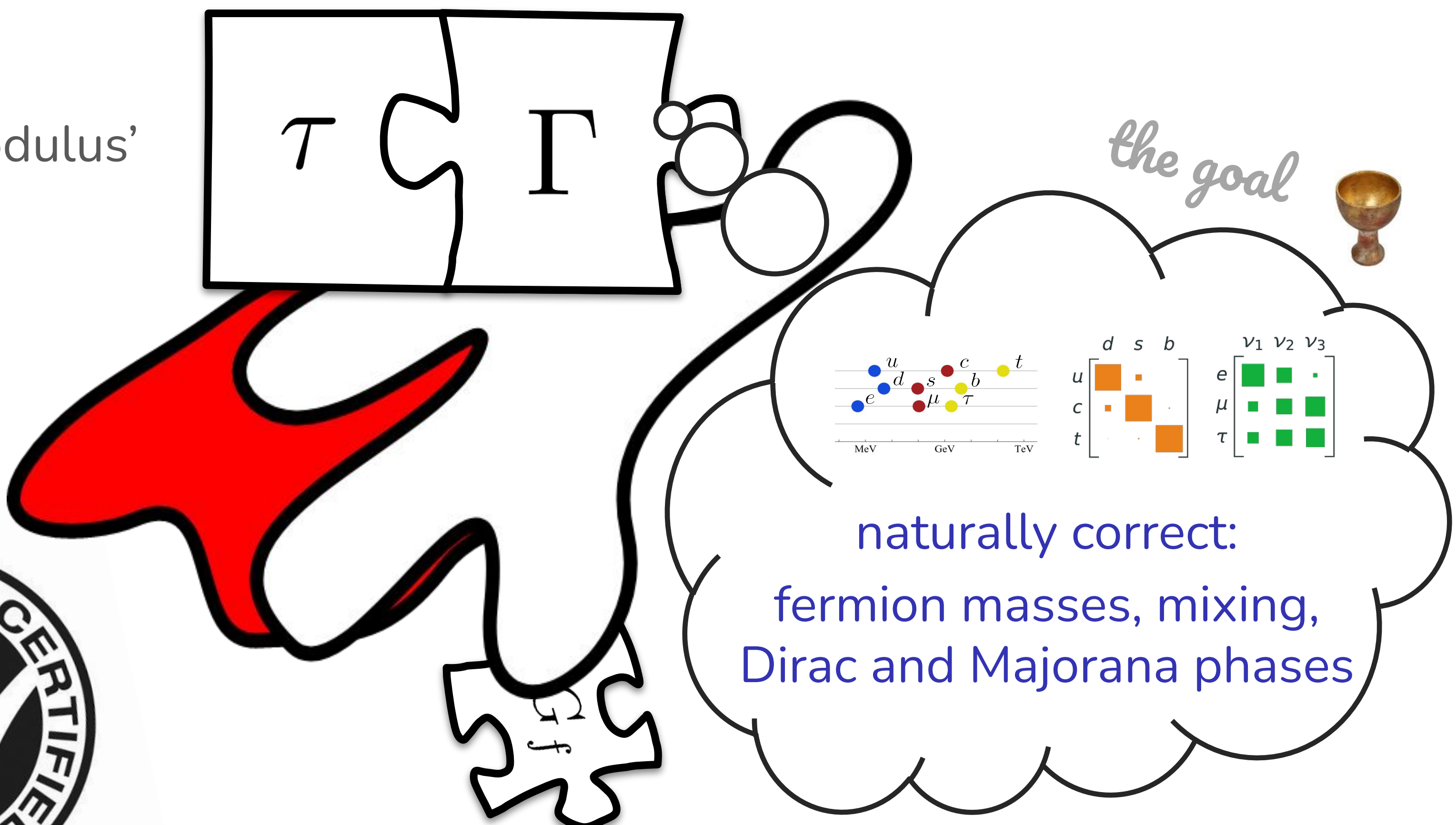
‘modulus’



How?

The modulus

τ

$\neq$

τ'

τ may describe a torus compactification

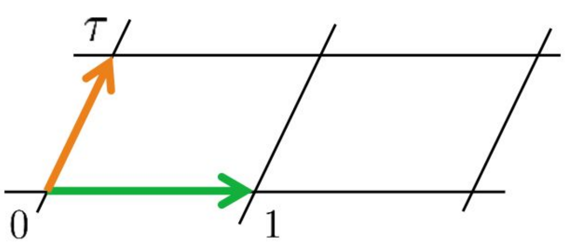
In the **bottom-up** modular approach τ is a dimensionless **spurion**

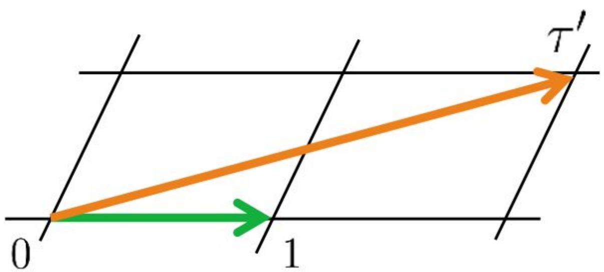
The modulus

τ

$=$

$$\tau' = \frac{a\tau+b}{c\tau+d}$$





$\tau \in \mathcal{H}$

$ad - bc = 1 \qquad a, b, c, d \in \mathbb{Z}$

The modulus

$=$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

$$\det \gamma = 1$$

$$a, b, c, d \in \mathbb{Z}$$

the ***modular group***

$$\Gamma \equiv SL(2, \mathbb{Z}) = \{\gamma\}$$

The modular group

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

$$\Gamma \equiv SL(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z},\ \det\gamma = 1 \right\}$$

Presentation in terms of generators S, T, R:

$$S^2 = R, \quad (ST)^3 = R^2 = \mathbb{1}, \quad RT = TR$$

The modular group

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$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} :$$

$$\tau \to -1/\tau$$

inver**S**ion

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} :$$

$$\tau \to \tau + 1$$

Translation

$$R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} :$$

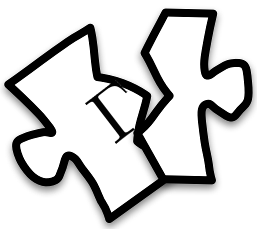
$$\tau \to \tau$$

Redundant

but can affect fields...

The modular group

$$\langle\tau\rangle \not\to \frac{a\tau+b}{c\tau+d}$$



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but can affect fields...

The field transformations

$$\psi \;\rightarrow\; (c\tau + d)^{-k}\; \rho(\gamma)\, \psi$$

The field transformations

NEW!

automorphy factor

$$\psi \;\to\; \boxed{(c\tau + d)^{-k}}\;\rho(\gamma)\,\psi$$

Weight $k \in \mathbb{Z}$

The field transformations

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$$\psi \to (c\tau + d)^{-k} \rho(\gamma) \psi$$

Weight $k \in \mathbb{Z}$

"Almost trivial" representation of the modular group

$$\rho\big(\Gamma(N)\big) = \mathbb{1}$$
$$\rho\big(T\,\Gamma(N)\big) = \rho(T)$$
$$\rho\big(S\,\Gamma(N)\big) = \rho(S)$$
$$\ldots$$

Feruglio, 1706.08749

The field transformations

NEW!

automorphy factor

$$\psi \to (c\tau + d)^{-k} \rho(\gamma) \psi$$

Weight $k \in \mathbb{Z}$

"Almost trivial" representation of the modular group

$$\Gamma(N) \subset \Gamma$$

Principal congruence subgroup of level *N*

$$\Gamma(N) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

$$\rho\big(\Gamma(N)\big) = \mathbb{1}$$

$$\rho\big(T\,\Gamma(N)\big) = \rho(T)$$

$$\rho\big(S\,\Gamma(N)\big) = \rho(S)$$

$$\ldots$$

Feruglio, 1706.08749

$\rho(\gamma)$ **is effectively a representation of** $\Gamma'_N \equiv \Gamma/\Gamma(N)$

The finite modular groups

$\Gamma'_N \equiv \Gamma/\Gamma(N)$ *behave like flavour groups*

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2,\mathbb{Z}_4)$	$A'_5 \equiv SL(2,\mathbb{Z}_5)$

← drop the **R** generator

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← drop the **R** generator

$\Gamma_2 \simeq S_3$

Kobayashi et al., 1803.10391

$\Gamma_3 \simeq A_4$

Feruglio, 1706.08749

$\Gamma_4 \simeq S_4$

JP, Petcov, 1806.11040

$\Gamma_5 \simeq A_5$

Novichkov et al., 1812.02158

summary in Appendices of Novichkov, JP, Petcov, Titov, 1905.11970

$\Gamma'_3 \simeq A'_4$

Liu, Ding, 1907.01488

$\Gamma'_4 \simeq S'_4$

Novichkov, JP, Petcov, 2006.03058

$\Gamma'_5 \simeq A'_5$

Wang, Yu, Zhou, 2010.10159

For top-down, see e.g.:

Kobayashi et al., 1804.06644; Kobayashi, Tamba, 1811.11384; de Anda et al., 1812.05620; Baur et al., 1901.03251, 1908.00805; Kariyazono et al., 1904.07546; Nilles et al., 2001.01736, 2004.05200, 2006.03059; Kobayashi, Otsuka, 2001.07972, 2004.04518; Abe et al., 2003.03512; Ohki et al., 2003.04174; Kikuchi et al., 2005.12642

& many more... (>150)

Need modular forms

$$\psi \sim (\mathbf{r}, k)$$

$$W \sim g(\psi_1 \ldots \psi_n)_{\mathbf{1}}$$

$$\psi \to (c\tau + d)^{-k} \rho_{\mathbf{r}}(\gamma)\, \psi$$

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$$\psi \sim (\mathbf{r}, k)$$

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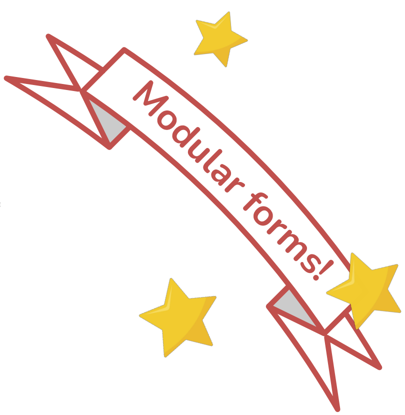
$$\psi \rightarrow (c\tau + d)^{-k}\,\rho_{\mathbf{r}}(\gamma)\,\psi$$

$$Y(\tau) \rightarrow (c\tau + d)^{k_Y}\,\rho_Y(\gamma)\,Y(\tau)$$



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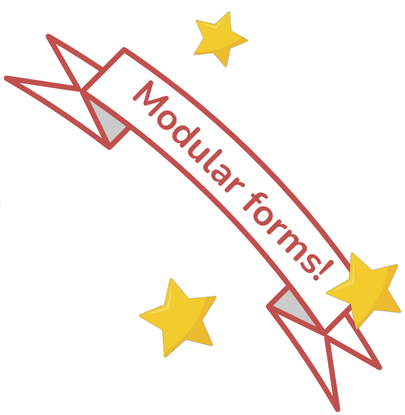
$$Y(\tau) \;\to\; (c\tau + d)^{k_Y}\,\rho_Y(\gamma)\,Y(\tau)$$

$$\begin{cases} k_Y = k_1 + \ldots + k_n \\ \rho_Y \otimes \rho_1 \otimes \ldots \otimes \rho_n \supset \mathbf{1} \end{cases}$$



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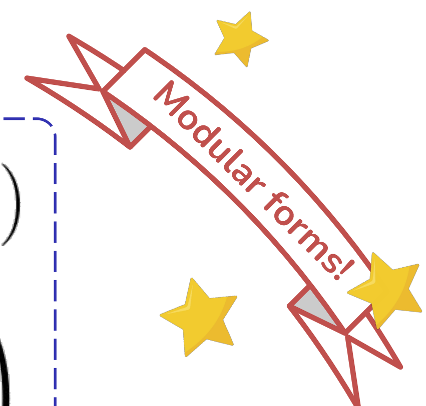
$$Y(\tau) \to (c\tau + d)^{k_Y}\,\rho_Y(\gamma)\,Y(\tau)$$

$$= Y\left(\frac{a\tau + b}{c\tau + d}\right)$$



The modular forms

N	2	3	4	5
Γ_N Γ'_N	S_3 S_3	A_4 $A'_4 \equiv T'$	S_4 $S'_4 \equiv SL(2,\mathbb{Z}_4)$	A_5 $A'_5 \equiv SL(2,\mathbb{Z}_5)$
$\dim \mathcal{M}_k(\Gamma(N))$	$k/2+1$	$k+1$	$2k+1$	$5k+1$

Not so many available!

A ***finite set*** *of functions for each k_Y*

Lowest-weight *k* modular forms for each group:

$\Gamma_N^{(\prime)}$	$Y_{\mathbf{r}}^{(k)}$
$\Gamma_2 \simeq S_3$	$Y_{\mathbf{2}}^{(2)}$
$\Gamma'_3 \simeq A'_4$	$Y_{\hat{\mathbf{2}}}^{(1)}$
$\Gamma_3 \simeq A_4$	$Y_{\mathbf{3}}^{(2)}$
$\Gamma'_4 \simeq S'_4$	$Y_{\hat{\mathbf{3}}}^{(1)}$
$\Gamma_4 \simeq S_4$	$Y_{\mathbf{2}}^{(2)}, Y_{\mathbf{3'}}^{(2)}$
$\Gamma'_5 \simeq A'_5$	$Y_{\hat{\mathbf{6}}}^{(1)}$
$\Gamma_5 \simeq A_5$	$Y_{\mathbf{3}}^{(2)}, Y_{\mathbf{3'}}^{(2)}, Y_{\mathbf{5}}^{(2)}$

Example

$$W \supset N N$$

Let's build a modular-invariant term!

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Let's build a modular-invariant term!

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$$W \supset \Lambda \left(N \otimes N \otimes Y_{\mathbf{3}}^{(2)} \right)_{\mathbf{1}}$$

Example

Let's build a modular-invariant term!

$$W \supset N N$$

$$\Gamma_3 \simeq A_4$$

$$N \sim (\mathbf{3}, 1)$$

$$W \supset \Lambda \left(N \otimes N \otimes Y_{\mathbf{3}}^{(2)} \right)_{\mathbf{1}}$$

$$Y_{\mathbf{3}}^{(2)}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}$$

$$M_N = \Lambda \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

so now we can build models...

Example: an S_4 lepton model

Novichkov, JP, Petcov, Titov, 1811.04933

Ingredients: **Choose group, field content**

$$\psi \sim (\mathbf{r}, k)$$

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$$N^c \sim (\mathbf{3}', 0)\,, \quad L \sim (\mathbf{3}, 2)$$
$$E^c \sim (\mathbf{1}', 0) \oplus (\mathbf{1}, 2) \oplus (\mathbf{1}', 2)$$

Ingredients: Choose group, field content $\psi \sim (\mathbf{r}, k)$

$$W = \alpha \left(E_1^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3}'}^{(4)}\right)_{\mathbf{1}} H_d$$
$$+\, g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda\, (N^c N^c)_{\mathbf{1}}\ ,$$

Procedure: Fit couplings and τ $\min \chi^2(\tau,\, g'/g,\, g^2/\Lambda,\, \alpha, \beta, \gamma)$

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$g' \in \mathbb{C}$ only physical phase

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$$+ g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}}\,,$$

Procedure: **Fit couplings and τ**

$$\min \chi^2(\tau,\, g'/g,\, g^2/\Lambda,\, \alpha, \beta, \gamma)$$

$$\text{gCP} \Rightarrow g' \in \mathbb{R}$$

Novichkov, JP, Petcov, Titov, 1905.11970

τ can be the only source of CPV

Example: an S4 lepton model (results)

Novichkov, JP, Petcov, Titov, 1811.04933

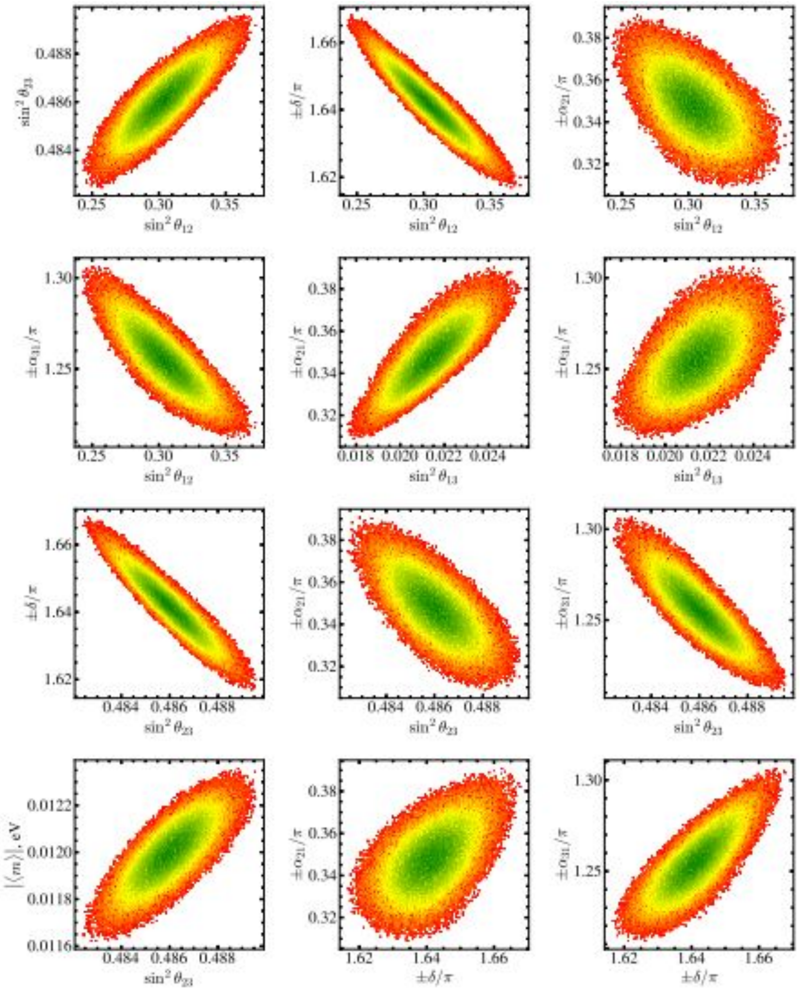
$$\sin^2\theta_{23} \sim 0.49$$
$$\delta \sim 1.6\pi$$
$$\alpha_{21} \sim 0.3\pi$$
$$\alpha_{31} \sim 1.3\pi$$

$$|\langle m \rangle|_{\beta\beta} \sim 12 \text{ meV}$$
$$\textstyle\sum_i m_i \sim 0.08 \text{ eV}$$

7 (4) parameters

vs.

12 (9) observables



Summary (1/3)

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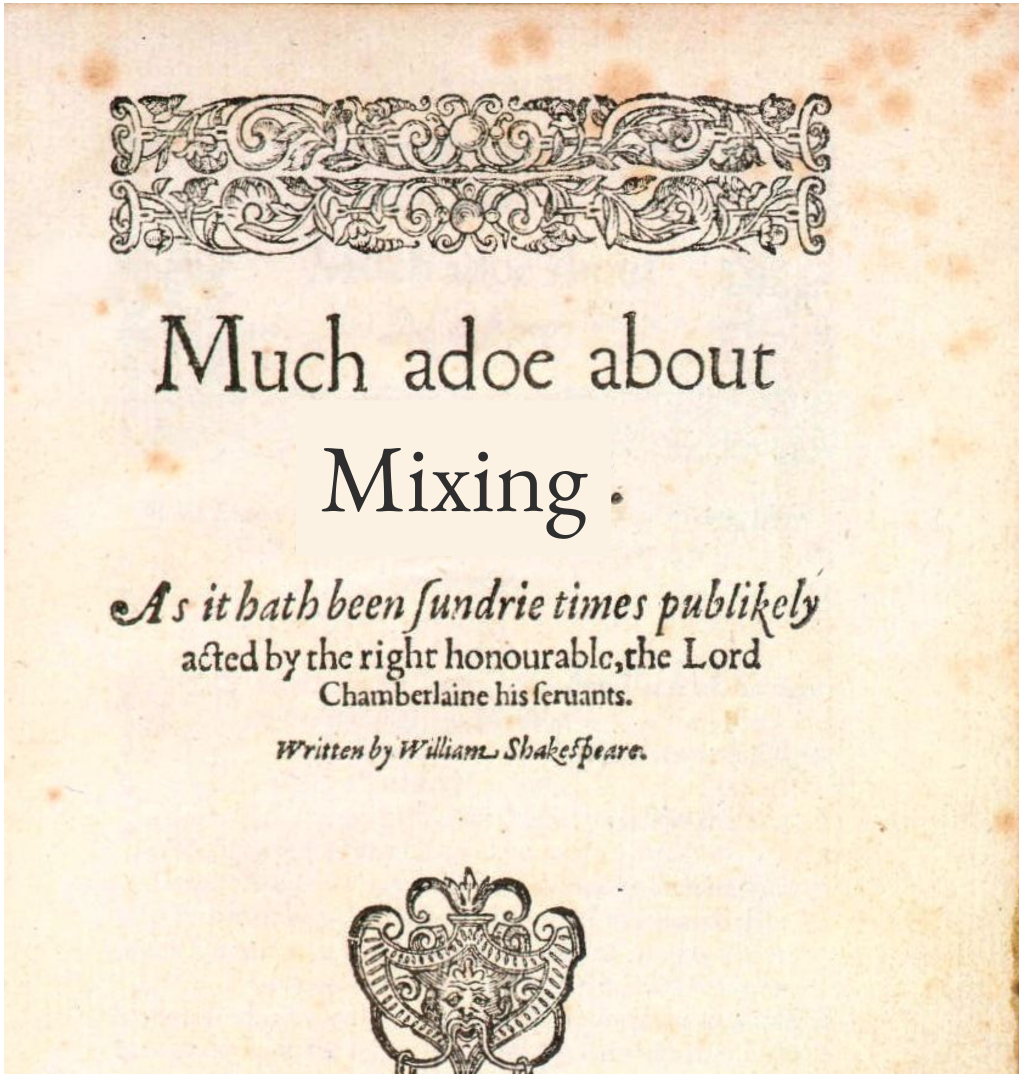
- **Modular symmetry** may strongly constrain masses and mixing.
- Fields carrying a non-trivial modular weight transform with a **scale factor** in addition to the usual unitary rotation.
- To build invariants one needs modular forms, which are functions of a **single complex parameter** τ.

Fermion mass hierarchies from residual modular symmetries

JHEP 04 (2021) 206 [2102.07488]

Mass hierarchies from modular symmetry?



Mass hierarchies from modular symmetry?

- Usually fermion mass hierarchies are put in **by hand**: hierarchies (or cancellations) between superpotential parameters

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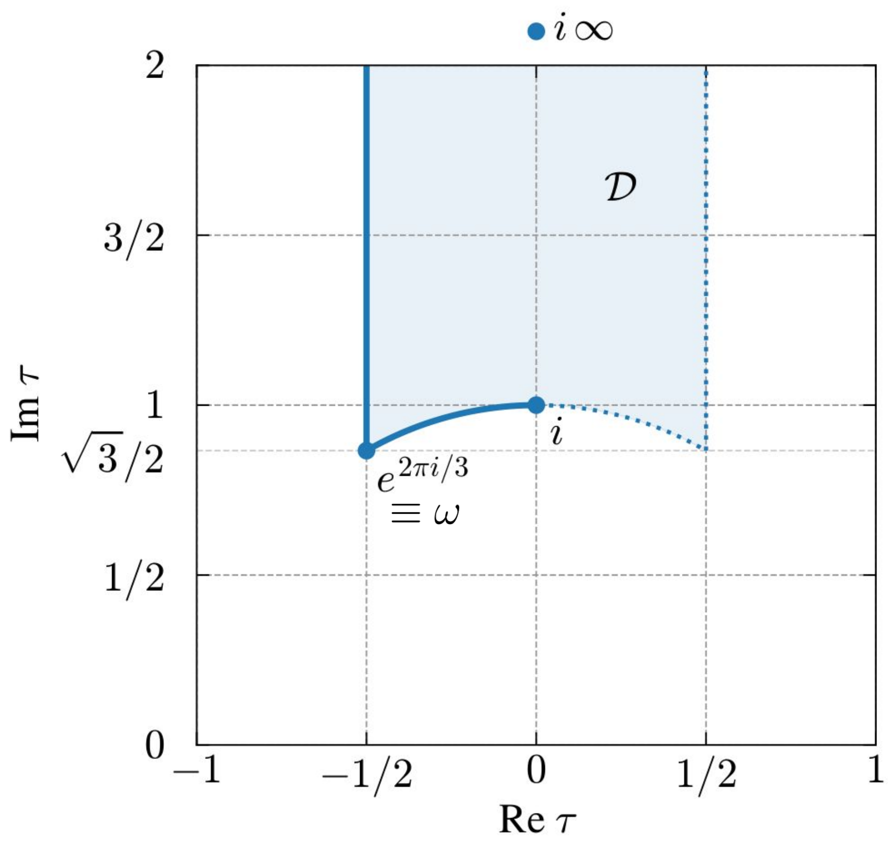
- **Other approaches** - new (weighted) scalars which enter the mass matrices a la Froggatt-Nielsen. Weights are analogous to FN charges

  Criado, Feruglio, King, 1908.11867
  King, King, 2002.00969

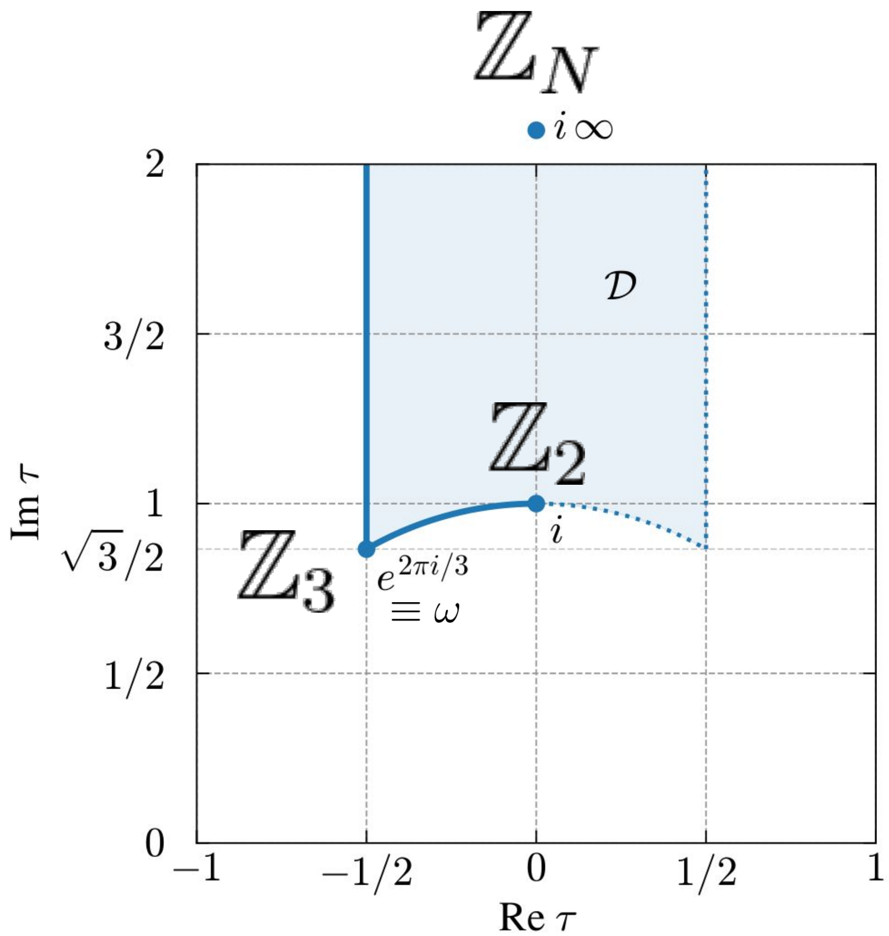
- **Our approach** - No new scalars, mechanism uses **only** τ, common weights across generations (unlike FN charges)

Residual modular symmetries



- The **fundamental domain** is enough
- **Any τ** breaks the modular symmetry

Residual modular symmetries



- The **fundamental domain** is enough
- **Any *τ*** breaks the modular symmetry
- At special values of *τ*, some **residual symmetry** remains

Key idea:

some couplings vanish as we approach a symmetric point

Corrections to vanishing couplings

$$\tau = \tau_{\text{sym}}$$

$$M \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\psi^c \, M \, \psi$$

Key idea:

some couplings vanish as we approach a symmetric point

Corrections to vanishing couplings

$$\tau = \tau_{\text{sym}}$$

$$\epsilon \sim |\tau - \tau_{\text{sym}}| > 0$$

$$M \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad M \sim \begin{pmatrix} 1 & \epsilon^{\cdots} & \epsilon^{\cdots} \\ \epsilon^{\cdots} & \epsilon^{\cdots} & \epsilon^{\cdots} \\ \epsilon^{\cdots} & \epsilon^{\cdots} & \epsilon^{\cdots} \end{pmatrix}$$

$$\psi^c \, M \, \psi$$

In the vicinity of the sym. point, the couplings are $\mathcal{O}(\epsilon^l)$

Key idea:

some couplings vanish as we approach a symmetric point

Decompositions under residual groups

(determine $\mathcal{O}(\epsilon^l)$)

τ_{sym}	Residual sym.	Possible powers ϵ^l
i	$\mathbb{Z}_2$	$l = 0, 1$
ω	$\mathbb{Z}_3$	$l = 0, 1, 2$
$i\infty$	$\mathbb{Z}_N$	$l = 0, 1, \ldots, N$

Decompositions under residual groups (determine $\mathcal{O}(\epsilon^l)$)

$\tau_{\rm sym}$	Residual sym.	Possible powers ϵ^l
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$i\infty$	$\mathbb{Z}_N$	$l = 0, 1, \ldots, N$

Feruglio, Gherardi, Romanino, Titov, 2101.08718 (for A_4, m_e=0)

$$\psi^c \, M \, \psi$$

$$\psi \xrightarrow{\gamma} (c\tau + d)^{-k} \rho(\gamma)\, \psi$$

$$\psi^c \xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c(\gamma)\, \psi^c$$

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger$$

$$\psi \rightsquigarrow \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots}$$

$$\psi^c \rightsquigarrow \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots}$$

In general, depend on weights

Determined for all $N \leqslant 5$

Example: hierarchical mass matrix (A5)

$$\psi \sim (\mathbf{3}, k)$$
$$\psi^c \sim (\mathbf{3}', k^c)$$

$$\Rightarrow$$

Under the residual group of

$$\tau_{\text{sym}} = i\infty$$

$$\psi \rightsquigarrow 1_0 \oplus \mathbf{1}_1 \oplus \mathbf{1}_4$$
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For $\psi^c\, M\, \psi$, we expect:

$$M \sim \begin{pmatrix} 1 & \epsilon^4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix} \Rightarrow$$

with $\epsilon = e^{-2\pi\, \text{Im}\, \tau/5}$

fermion spectrum

$$\sim (1, \epsilon, \epsilon^4)$$ ✔

Indeed the case, provided enough modular forms contribute to *M*

(otherwise, $m_e = 0$)

Example: hierarchical mass matrix (A5)

$$\psi \sim (\mathbf{3}, k) \quad \Rightarrow$$

Under the residual group of

$$\tau_{\text{sym}} = i\infty$$

Not like Froggatt-Nielsen. Instead, it is an **improvement**!

Explicit example at weight 2

$$W \supset \sum_s \alpha_s \left(Y_{\mathbf{5}}^{(5,2)}(\tau)\psi^c\psi\right)_{\mathbf{1},s} \quad \Rightarrow \quad M(\tau) = \alpha \begin{pmatrix} \sqrt{3}Y_1 & Y_5 & Y_2 \\ Y_4 & -\sqrt{2}Y_3 & -\sqrt{2}Y_5 \\ Y_3 & -\sqrt{2}Y_2 & -\sqrt{2}Y_4 \end{pmatrix}_{Y_{\mathbf{5}}^{(5,2)}}$$

$$(Y_1, Y_2, Y_3, Y_4, Y_5) \propto \left(-1/\sqrt{6}, \epsilon, 3\epsilon^2, 4\epsilon^3, 7\epsilon^4\right)$$

with $\epsilon = e^{-2\pi \,\mathrm{Im}\, \tau/5}$

Indeed the case, provided enough modular forms contribute to *M*

(otherwise, $m_e = 0$)

Scan of possible mass patterns

Performed for 3 generations, for all $N \leqslant 5$

e.g. fermion spectra for multiplets of modular A5

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$, $k+k^c \equiv 0$	$\tau \simeq \omega$, $k+k^c \equiv 1$	$\tau \simeq \omega$, $k+k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{3}$	$\mathbf{3}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{3}$	$\mathbf{3}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon^4)$
$\mathbf{3}'$	$\mathbf{3}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{3}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^4)$
$\mathbf{3}'$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon^2,\epsilon^3)$
$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,1)$

Promising hierarchical patterns

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r} \otimes \mathbf{r}^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	
			$\tau \simeq i\infty$	
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	

Promising hierarchical patterns

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r} \otimes \mathbf{r}^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$
			$\tau \simeq i\infty$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$ with $\mathbf{1}_a \neq (\mathbf{1}_b)^*$
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a$, or $\mathbf{2} \oplus \mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes [\mathbf{2} \oplus \mathbf{1}$, or $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}']$, $\mathbf{3}' \otimes [\mathbf{2} \oplus \mathbf{1}'$, or $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}']$, $\hat{\mathbf{3}}' \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}']$, $\hat{\mathbf{3}} \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}']$
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes \mathbf{3}'$

Promising hierarchical patterns (try leptons)

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r}\otimes\mathbf{r}^c$
2	S_3	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	
3	A'_4	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$ $\tau\simeq i\infty$	
4	S'_4	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	
		$(1,\epsilon,\epsilon^3)$	$\tau\simeq i\infty$	$\hat{\mathbf{3}}'\otimes(\hat{\mathbf{2}}\oplus\hat{\mathbf{1}})$
5	A'_5	$(1,\epsilon,\epsilon^4)$	$\tau\simeq i\infty$	$\mathbf{3}\otimes\mathbf{3}'$

$L\sim(\hat{\mathbf{2}}\oplus\hat{\mathbf{1}},2),\ E^c\sim(\hat{\mathbf{3}}',2),\ N^c\sim(\mathbf{3},1)$

8 parameters

$L\sim(\mathbf{3},3),\ E^c\sim(\mathbf{3}',1),\ N^c\sim(\hat{\mathbf{2}},2)$

8 parameters

Masses are OK :)

Promising hierarchical patterns (try leptons)

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r} \otimes \mathbf{r}^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	
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4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	$\hat{\mathbf{3}}' \otimes (\hat{\mathbf{2}} \oplus \hat{\mathbf{1}})$
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes \mathbf{3}'$

$L \sim (\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}, 2),\ E^c \sim (\hat{\mathbf{3}}', 2),\ N^c \sim (\mathbf{3}, 1)$

8 parameters

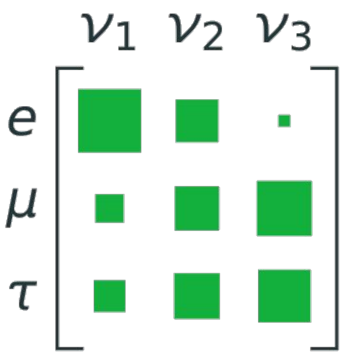
$L \sim (\mathbf{3}, 3),\ E^c \sim (\mathbf{3}', 1),\ N^c \sim (\hat{\mathbf{2}}, 2)$

8 parameters

Masses are OK, but mixing is tuned :(

Wrong PMNS in the symmetric limit:
parameters are driven into cancellations

How to avoid fine-tuning (in the lepton sector)



$$\xrightarrow{\tau \to \tau_{\mathrm{sym}}} \quad \begin{bmatrix} \star & \star & 0 \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix} \text{ or } \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}$$

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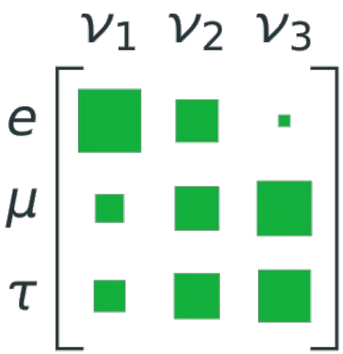

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Reyimuaji, Romanino, 1801.10530

$$1. \begin{cases} L \sim 1 \oplus 1 \oplus 1 \\ E^c \sim 1 \oplus \mathbf{r} \not\supset 1 \end{cases} \qquad 2. \begin{cases} L \sim \mathbf{1} \oplus \mathbf{1} \oplus \bar{\mathbf{1}} \\ E^c \sim \bar{\mathbf{1}} \oplus \mathbf{r} \not\supset \mathbf{1}, \bar{\mathbf{1}} \end{cases}$$

$$3.\ m_e = m_\mu = m_\tau = 0$$

$$4.\ m_{\nu_1} = m_{\nu_2} = m_{\nu_3} = 0$$

for mixing near symmetric points, see also Okada, Tanimoto, 2009.14242

Promising hierarchical patterns (leptons)

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r}_{E^c} \otimes \mathbf{r}_L$	Case
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$	1 or 4
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$	2
			$\tau \simeq i\infty$	$[\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}'] \otimes [\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}']$, $[\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}''] \otimes [\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'']$	2
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a$, or $\mathbf{2} \oplus \mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$	1 or 4
5	A'_5	–	–	–	–

1. $\begin{cases} L \sim 1 \oplus 1 \oplus 1 \\ E^c \sim 1 \oplus \mathbf{r} \not\supset 1 \end{cases}$

2. $\begin{cases} L \sim \mathbf{1} \oplus \mathbf{1} \oplus \bar{\mathbf{1}} \\ E^c \sim \bar{\mathbf{1}} \oplus \mathbf{r} \not\supset \mathbf{1}, \bar{\mathbf{1}} \end{cases}$

3. $m_e = m_\mu = m_\tau = 0$

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3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$		2
			$\tau \simeq i\infty$		2
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a \quad] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$	1 or 4
5	A'_5	–	–	–	–

1. $\begin{cases} L \sim 1 \oplus 1 \oplus 1 \\ E^c \sim 1 \oplus \mathbf{r} \not\supset 1 \end{cases}$

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3. $m_e = m_\mu = m_\tau = 0$

4. $m_{\nu_1} = m_{\nu_2} = m_{\nu_3} = 0$

Example: lepton model close to ω

Only S_4' model from a scan requiring minimal # params., $m_e > 0$, and Dirac phase within 2σ range (otherwise unconstrained):

$$L \sim (\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}', 2),\ E^c \sim (\hat{\mathbf{3}}, 4),\ N^c \sim (\mathbf{3}', 1)$$

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Superpotential:

$$W = \left[\alpha_1 \left(Y^{(4,6)}_{\mathbf{3}',1} E^c L_1\right)_{\mathbf{1}} + \alpha_3 \left(Y^{(4,6)}_{\mathbf{3}',1} E^c L_2\right)_{\mathbf{1}} + \alpha_4 \left(Y^{(4,6)}_{\mathbf{3}',2} E^c L_2\right)_{\mathbf{1}} + \alpha_5 \left(Y^{(4,6)}_{\mathbf{3}} E^c L_3\right)_{\mathbf{1}}\right] H_d$$
$$+ \left[g_1 \left(Y^{(4,3)}_{\hat{\mathbf{3}}} N^c L_1\right)_{\mathbf{1}} + g_2 \left(Y^{(4,3)}_{\hat{\mathbf{3}}} N^c L_2\right)_{\mathbf{1}} + g_3 \left(Y^{(4,3)}_{\hat{\mathbf{3}}'} N^c L_3\right)_{\mathbf{1}}\right] H_u$$
$$+ \Lambda \left(Y^{(4,2)}_{\mathbf{2}} (N^c)^2\right)_{\mathbf{1}} .$$

with gCP imposed

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$$M_e \propto \begin{pmatrix} 1 & \alpha - 2\beta & 2\sqrt{3} i\gamma \\ \sqrt{3}\epsilon & \sqrt{3}(\alpha + 2\beta)\epsilon & 2i\gamma\epsilon \\ \frac{5}{2}\epsilon^2 & \left(\frac{5}{2}\alpha - \beta\right)\epsilon^2 & -\frac{5}{\sqrt{3}} i\gamma\epsilon^2 \end{pmatrix}$$

$$|\epsilon| \simeq 2.8 \left|\frac{\tau - \omega}{\tau - \omega^2}\right| \sim \left|\tau - e^{2\pi i/3}\right|$$

$$M_\nu \propto \epsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & a \\ 1 & a & 2i\sqrt{\frac{2}{3}}b \end{pmatrix}$$

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$$M_\nu \propto \epsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & a \\ 1 & a & 2i\sqrt{\frac{2}{3}} b \end{pmatrix}$$

$$u \equiv \frac{\tau - \omega}{\tau - \omega^2}$$

$|u|$ quantifies the deviation of τ from the left cusp

Example: lepton model close to ω

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$$M_\nu \propto \epsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & a \\ 1 & a & 2i\sqrt{\frac{2}{3}} b \end{pmatrix}$$

$|\epsilon| \simeq 0.02$

$a = 1.5 \pm 0.15$

$b = 2.22 \pm 0.17$

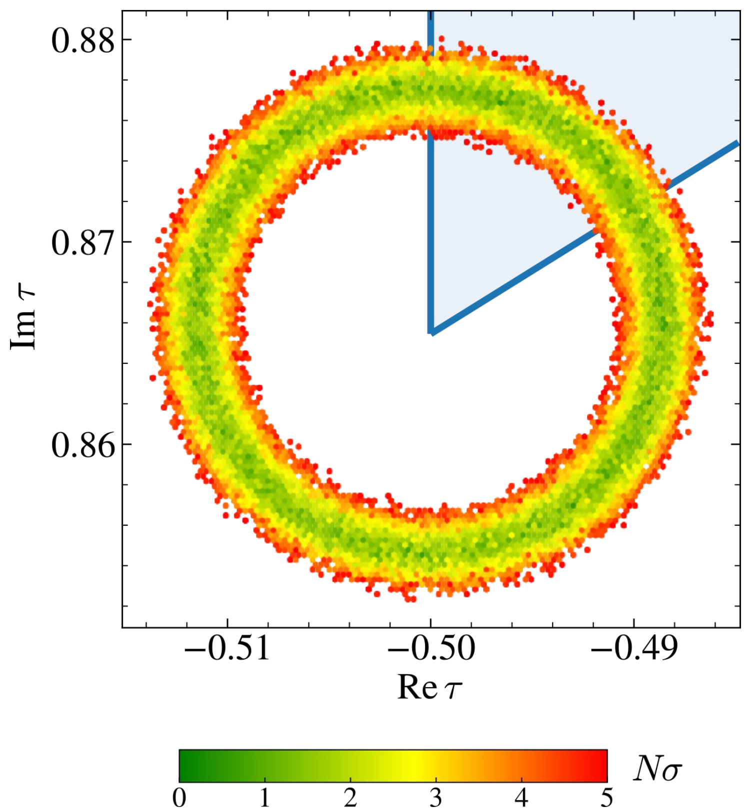
$\alpha = 2.45 \pm 0.44$

$\beta = 2.14 \pm 0.32$

$\gamma = 0.91 \pm 0.05$

Example: lepton model close to ω

$$|\epsilon| \simeq 0.02 \;\Leftrightarrow\; |u| \simeq 0.007$$



$$m_e = \mathcal{O}(\epsilon^2)$$
$$m_\mu = \mathcal{O}(\epsilon)$$
$$m_\tau = \mathcal{O}(1)$$

$$\text{NO}, \quad m_{\nu_1} = 0 \qquad \delta \simeq \pi$$
$$m_{\beta\beta} = (1.44 \pm 0.33)\ \text{meV}$$

Naturally allows for **hierarchies**, **large mixing**, and some **predictivity**

Summary (2/3)

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- Fermion **mass hierarchies** can naturally arise if τ is in the vicinity of a point of residual symmetry,

$$\tau_{\mathrm{sym}} = \omega, i\infty, (i)$$

- This mechanism works without flavons.

- **Natural lepton mixing** can also arise in such models. Requiring no fine-tuning in the whole lepton sector is remarkably restrictive.

- As seen in the model and anticipated from the hierarchical patterns, $|u| \simeq 0.007$ is required. Ad hoc?

Modulus stabilisation

JHEP 03 (2022) 149 [2201.02020]

Simplest modular-invariant potentials?

- Studied by Cvetič, Font, Ibáñez, Lüst and Quevedo (1991) $\mathcal{N}=1$ SUGRA

$$K(\tau,\bar{\tau}) = -\Lambda_K^2 \log(2\ \mathrm{Im}\,\tau)$$
$$G(\tau,\bar{\tau}) = \kappa^2 K(\tau,\bar{\tau}) + \log\left|\kappa^3 W(\tau)\right|^2$$

$$\kappa^2 = 8\pi/M_P^2$$

- Superpotential has modular weight $-\mathfrak{n} = -1, -2, -3, \ldots$

$$W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^{2\mathfrak{n}}}$$

$$\mathfrak{n} = \kappa^2 \Lambda_K^2$$

- Simplified model, independent of the level *N*

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- Simplified model, independent of the level *N*

Modular-invariant potentials

$$W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^{2\mathfrak{n}}}$$

$$V = e^{\kappa^2 K} \left(K^{i\bar{j}} D_i W D_{\bar{j}} W^* - 3\kappa^2 |W|^2 \right)$$

Modular-invariant potentials

$$W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^{2\mathfrak{n}}}$$

$$V = e^{\kappa^2 K}\left(K^{i\bar{j}} D_i W D_{\bar{j}} W^* - 3\kappa^2 |W|^2\right)$$

$$V(\tau,\bar{\tau}) = \frac{\Lambda_V^4}{(2\,\mathrm{Im}\,\tau)^{\mathfrak{n}}|\eta(\tau)|^{4\mathfrak{n}}}\left[\left|iH'(\tau) + \frac{\mathfrak{n}}{2\pi} H(\tau)\hat{G}_2(\tau,\bar{\tau})\right|^2 \frac{(2\,\mathrm{Im}\,\tau)^2}{\mathfrak{n}} - 3|H(\tau)|^2\right]$$

Modular-invariant potentials

$$W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^{2\mathfrak{n}}}$$

$$V = e^{\kappa^2 K}\left(K^{i\bar{j}} D_i W D_{\bar{j}} W^* - 3\kappa^2 |W|^2\right)$$

$$\Lambda_V = \left(\kappa^2 \Lambda_W^6\right)^{1/4}$$

$$V(\tau,\bar{\tau}) = \frac{\Lambda_V^4}{(2\,\mathrm{Im}\,\tau)^{\mathfrak{n}} |\eta(\tau)|^{4\mathfrak{n}}} \left[\left| i H'(\tau) + \frac{\mathfrak{n}}{2\pi} H(\tau) \hat{G}_2(\tau,\bar{\tau})\right|^2 \frac{(2\,\mathrm{Im}\,\tau)^2}{\mathfrak{n}} - 3|H(\tau)|^2\right]$$

$$\hat{G}_2(\tau,\bar{\tau}) = G_2(\tau) - \frac{\pi}{\mathrm{Im}\,\tau}$$

$$\frac{\eta'(\tau)}{\eta(\tau)} = \frac{i}{4\pi} G_2(\tau)$$

Modular-invariant potentials

$$W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^{2\mathfrak{n}}}$$

$$V = e^{\kappa^2 K}\left(K^{i\bar{j}} D_i W D_{\bar{j}} W^* - 3\kappa^2 |W|^2\right)$$

$$V(\tau,\bar{\tau}) = \frac{\Lambda_V^4}{(2\,\mathrm{Im}\,\tau)^{\mathfrak{n}} |\eta(\tau)|^{4\mathfrak{n}}} \left[\left| iH'(\tau) + \frac{\mathfrak{n}}{2\pi} H(\tau) \hat{G}_2(\tau,\bar{\tau})\right|^2 \frac{(2\,\mathrm{Im}\,\tau)^2}{\mathfrak{n}} - 3|H(\tau)|^2\right]$$

$$\mathfrak{n} = 3$$

$$V(\tau,\bar{\tau}) = \frac{\Lambda_V^4}{8(\mathrm{Im}\,\tau)^3 |\eta|^{12}} \left[\frac{4}{3}\left| iH' + \frac{3}{2\pi} H \hat{G}_2\right|^2 (\mathrm{Im}\,\tau)^2 - 3|H|^2\right]$$

The superpotential

$$W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^6}$$

$$V(\tau,\bar\tau) = \frac{\Lambda_V^4}{8(\mathrm{Im}\,\tau)^3|\eta|^{12}}\left[\frac{4}{3}\left|iH' + \frac{3}{2\pi}H\hat{G}_2\right|^2(\mathrm{Im}\,\tau)^2 - 3|H|^2\right]$$

- Most general holomorphic $H(\tau)$ (except at $i\infty$) Cvetič et al (1991)

$$H(\tau) = (j(\tau) - 1728)^{m/2}\, j(\tau)^{n/3}\, \mathcal{P}(j(\tau))$$

$m, n = 0, 1, 2, \ldots$

$$j = \left(\frac{72}{\pi^2}\frac{\eta\eta'' - 3\eta'^2}{\eta^{10}}\right)^3 = \left[\frac{72}{\pi^2\eta^6}\left(\frac{\eta'}{\eta^3}\right)'\right]^3$$

The superpotential

$$W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^6}$$

$$V(\tau,\bar\tau) = \frac{\Lambda_V^4}{8(\operatorname{Im}\tau)^3|\eta|^{12}} \left[\frac{4}{3}\left|iH' + \frac{3}{2\pi}H\hat{G}_2\right|^2 (\operatorname{Im}\tau)^2 - 3|H|^2\right]$$

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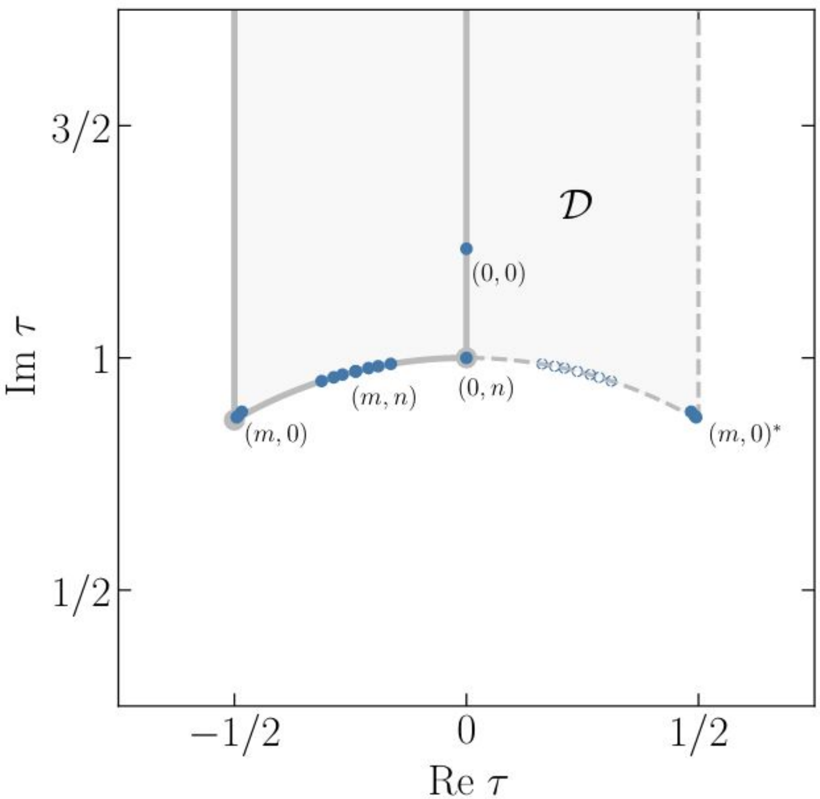
$$\mathcal{P}(j) = 1$$

simplest choice

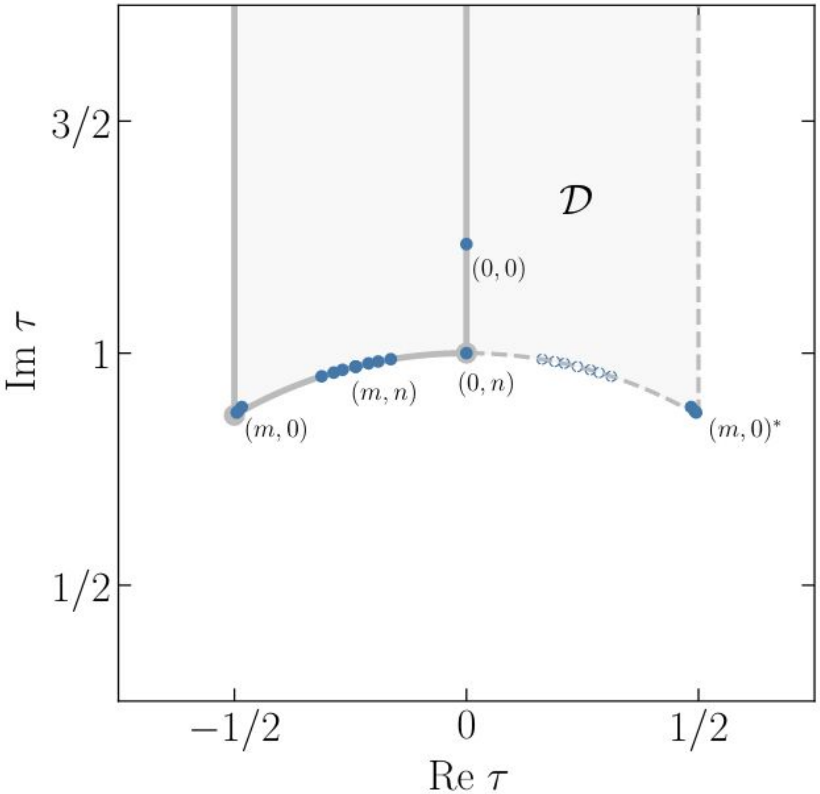
- This potential is **modular-** and **CP-invariant** (also for some other *P*(*j*)'s)
- Everything can be expressed in terms of η and its derivatives...

Global minima for (*m*,*n*)-potentials

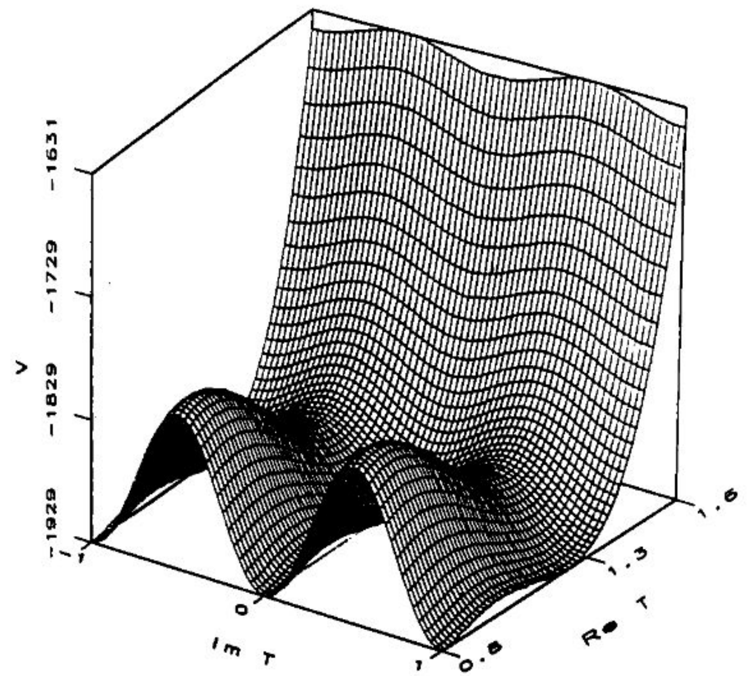


"(...) we conjecture that all extrema of V entirely lie on [the boundary]." — Cvetič et al.

Global minima for (*m*,*n*)-potentials



(0,0) ✓



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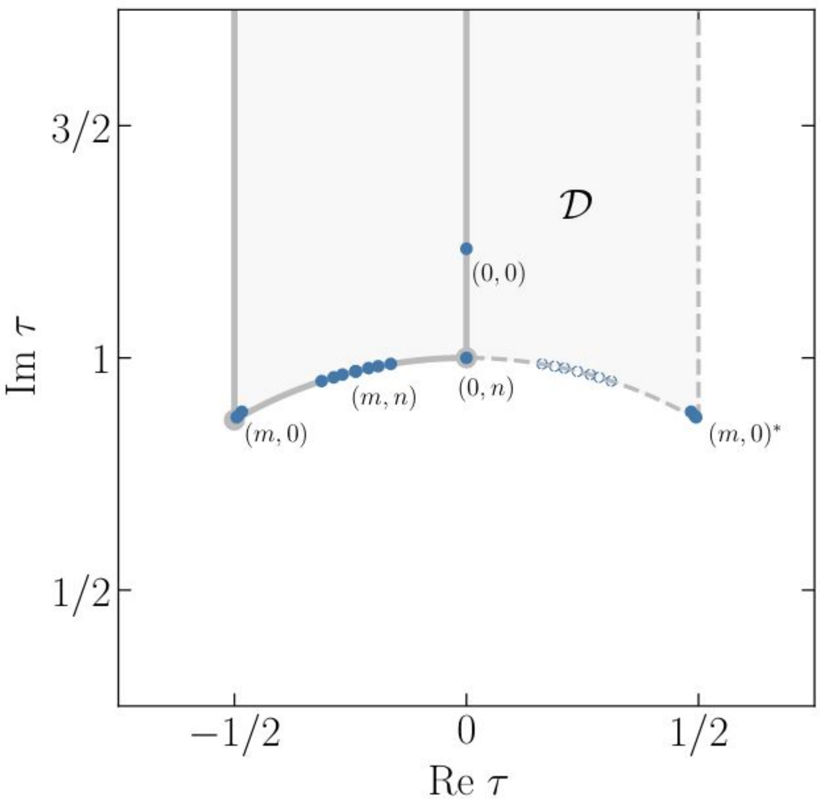
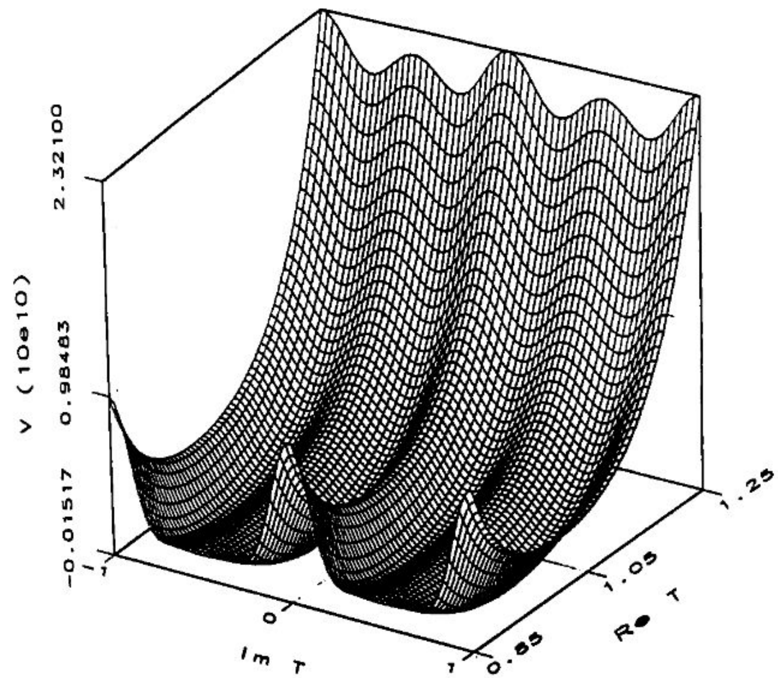
Global minima for (*m*,*n*)-potentials



(1,1) ✓



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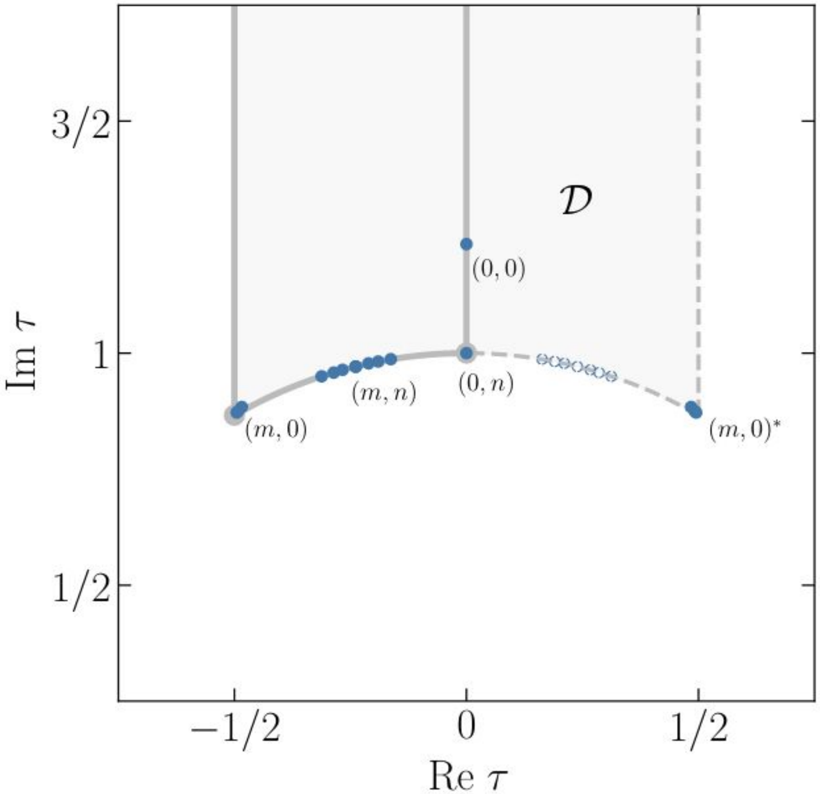
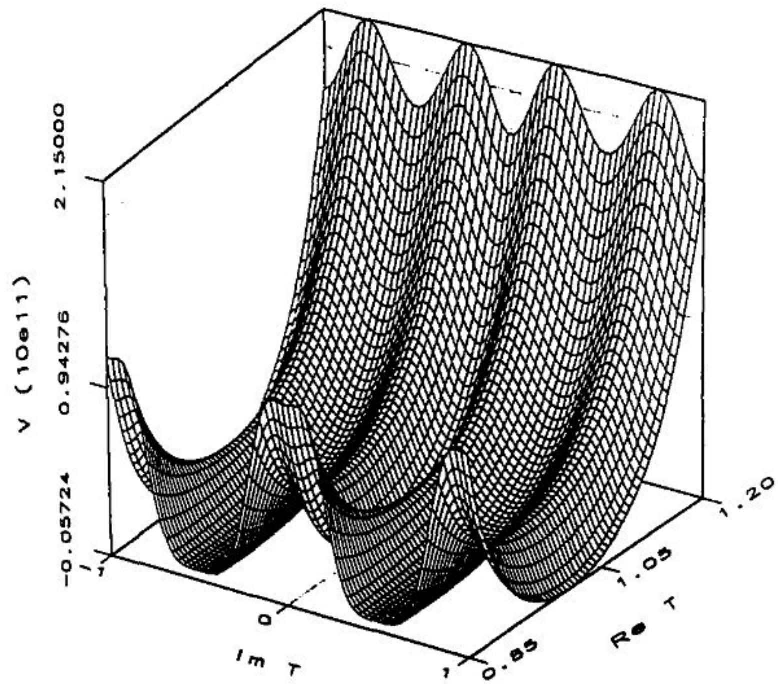
Global minima for (*m*,*n*)-potentials



(0,3)



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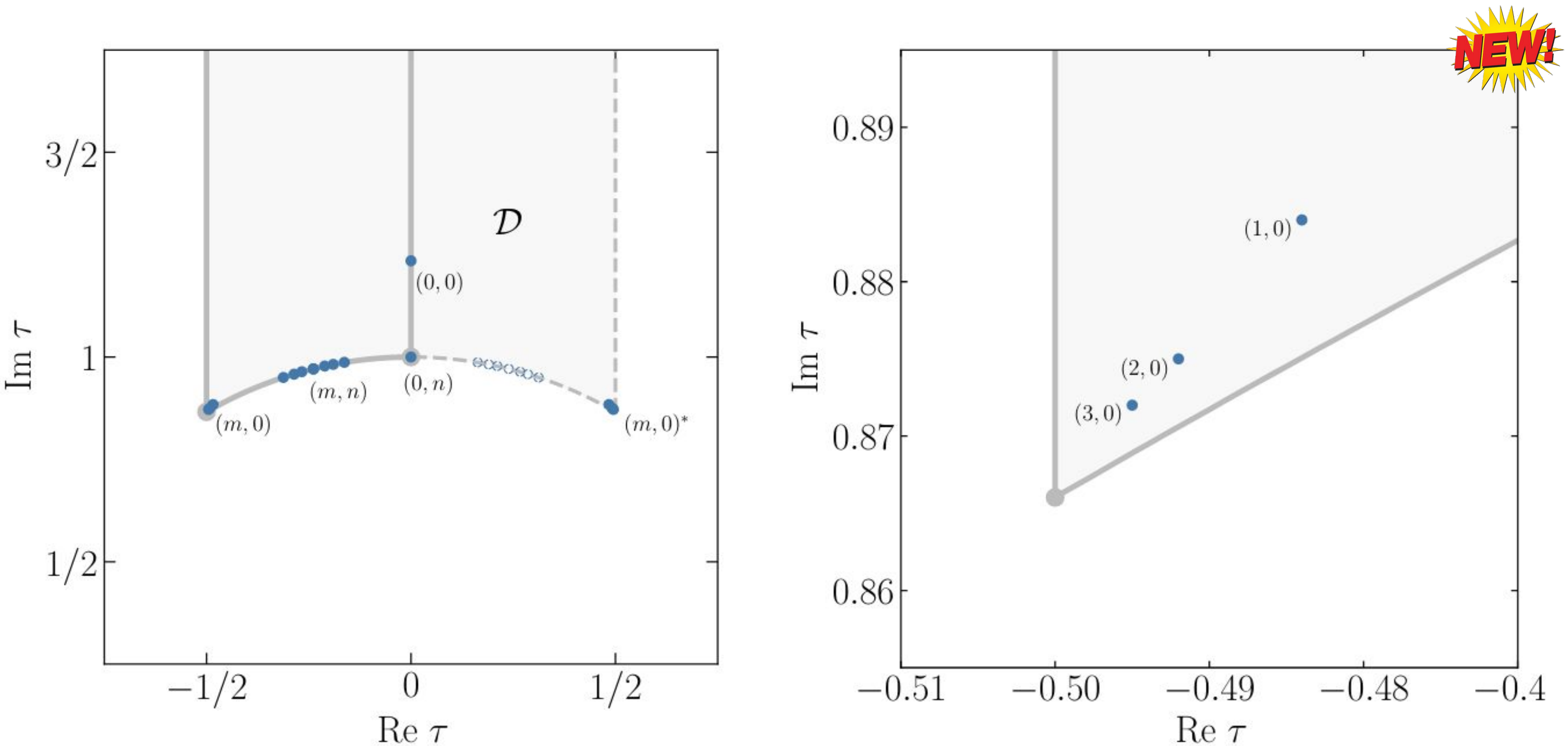
Global minima for (*m*,*n*)-potentials



“(...) we conjecture that all extrema of V entirely lie on [the boundary].” — Cvetič et al.

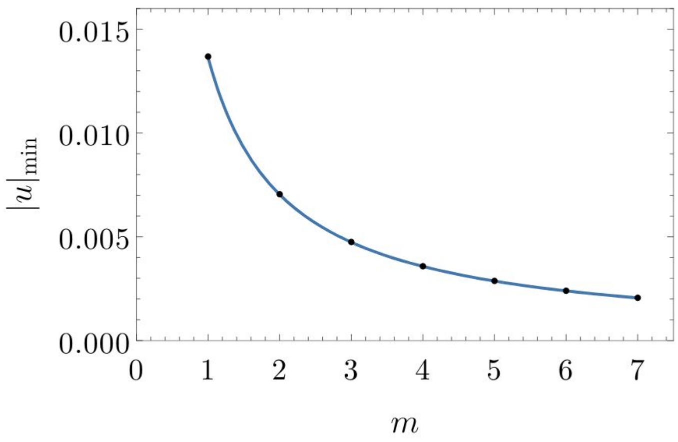
The (*m*,*0*) family of potentials

- *u*-expand (*m*,*0*) potentials to analyse them near the left cusp

$$V_{m,0} = \Lambda_V^4 \frac{1728^m}{\sqrt{3}\,\tilde{\eta}_0^{12}} \left\{-1 - 2\,|u|^2 + \left(A_m^2 - 3\right)|u|^4\right\} + \mathcal{O}(|u|^6)$$

$$A_m \equiv \frac{864\,|\tilde{\eta}_3|^3}{\pi^6\,\tilde{\eta}_0^{27}}\,m + \frac{6\,|\tilde{\eta}_3|}{\tilde{\eta}_0} \simeq 68.78\,m + 4.30$$

- Mexican hat potential
  (cusp is a maximum!)



$$|u|_{\text{min}} \simeq \left(A_m^2 - 3\right)^{-1/2} \simeq A_m^{-1} = \frac{0.0145}{m + 0.0625}$$

The (*m*,0) family of potentials

(phase dependence)

$$u = |u| e^{i\phi}$$

- *u*-expanding to higher order shows dependence on $\phi \in [-\pi/3, 0]$

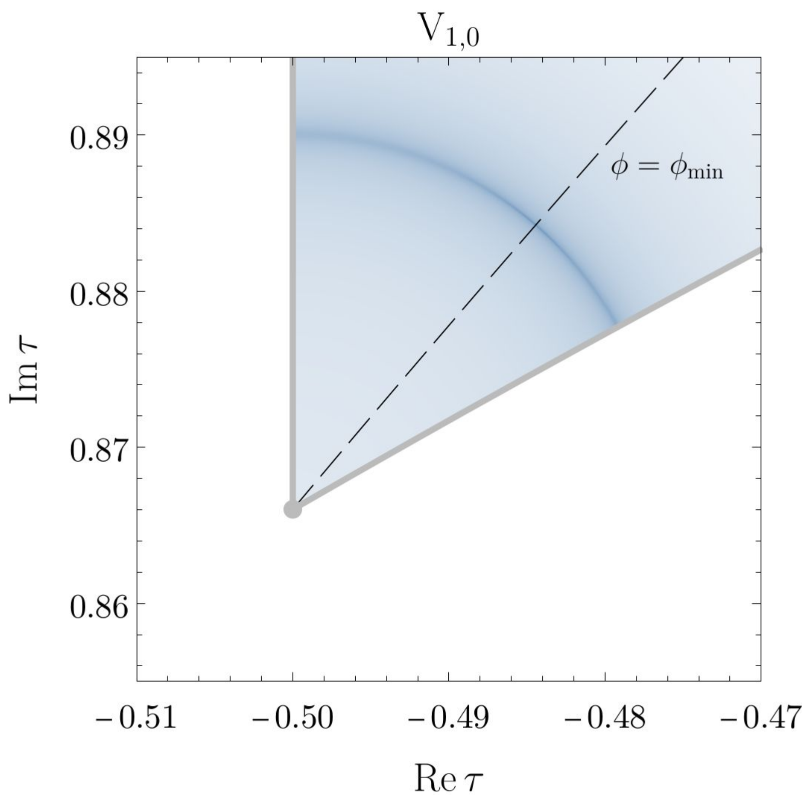
$$V_{m,0} \propto -1 - 2\,|u|^2 + \left(A_m^2 - 3\right)|u|^4 + \left(-4 + 2A_m^2 + B_m^2 \cos 6\phi\right)|u|^6 + 2A_m B_m^2 \cos 3\phi\,|u|^7 + \left(-5 + 3A_m^2 + 2B_m^2 \cos 6\phi\right)|u|^8 + \mathcal{O}(|u|^9)$$

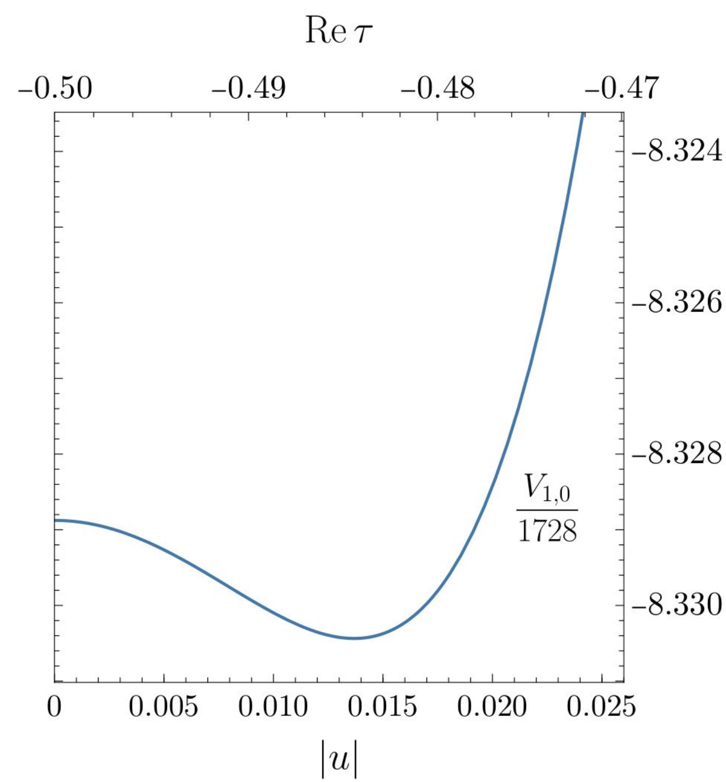
$$B_m^2 \equiv \frac{864\,|\tilde{\eta}_3|^3}{\pi^6\,\tilde{\eta}_0^{27}}\,m\left[\frac{864\,|\tilde{\eta}_3|^3}{\pi^6\,\tilde{\eta}_0^{27}}(m-2) + \frac{3\left(31\,\tilde{\eta}_3^2 - 10\tilde{\eta}_0\tilde{\eta}_6\right)}{\tilde{\eta}_0|\tilde{\eta}_3|}\right] + \frac{6\left(7\tilde{\eta}_3^2 - 2\tilde{\eta}_0\tilde{\eta}_6\right)}{\tilde{\eta}_0^2} \simeq 4730.60\,m^2 - 2069.73\,m + 33.26\,.$$

- Phase of *u* mostly determined by $|u|^6$ and $|u|^7$ terms

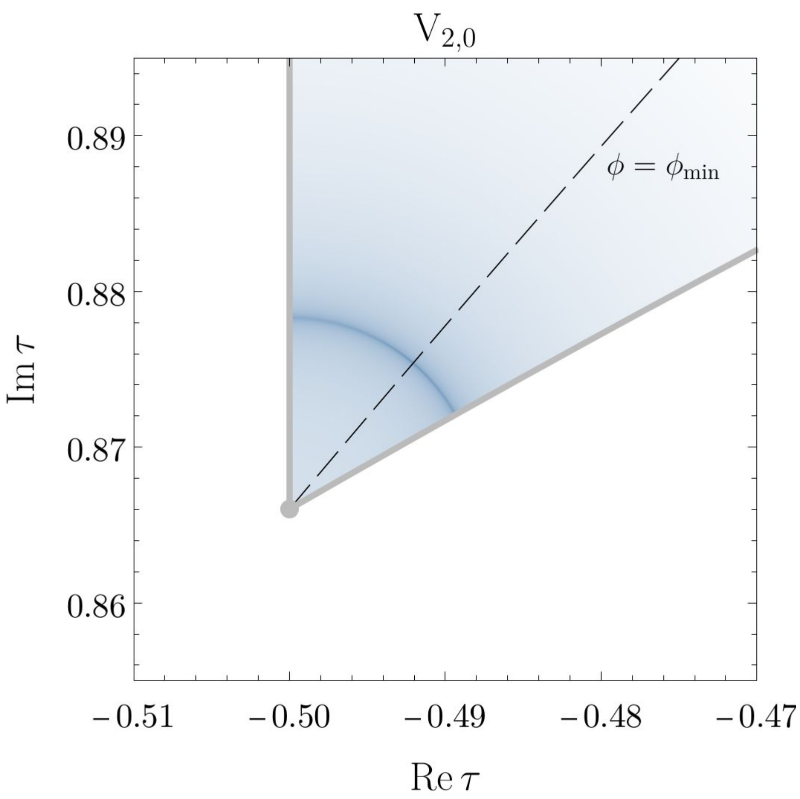
$$\phi_{\min} \simeq -\frac{2\pi}{9} = -40^\circ$$

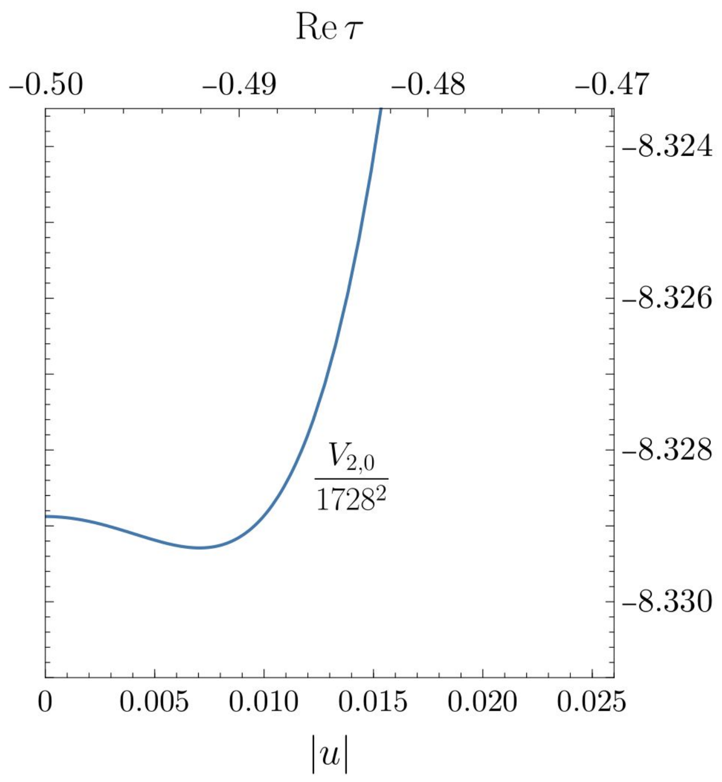
The (*m*,*0*) family of potentials (*m = 1*)



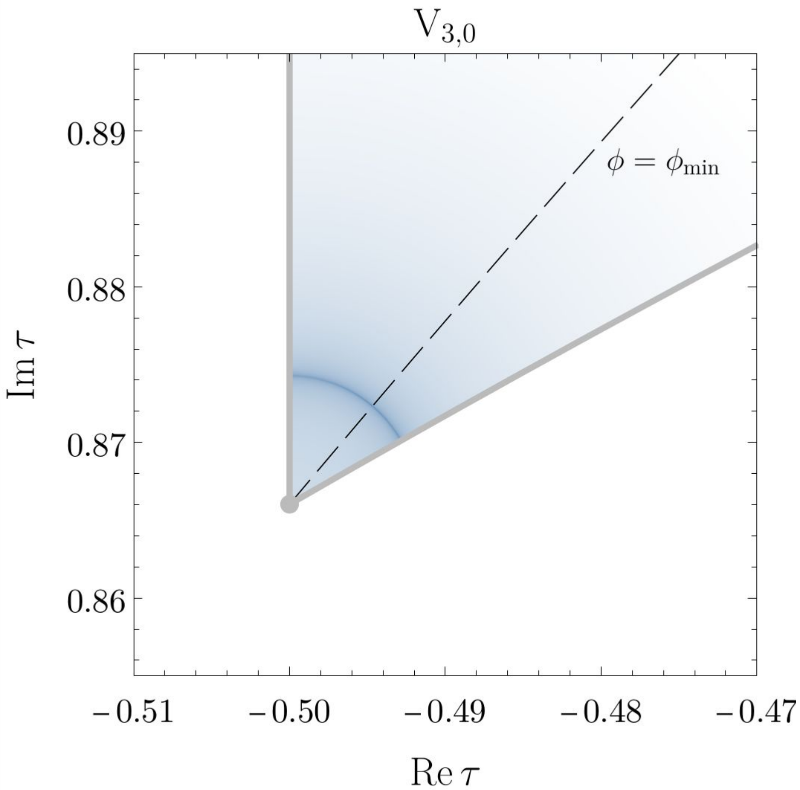


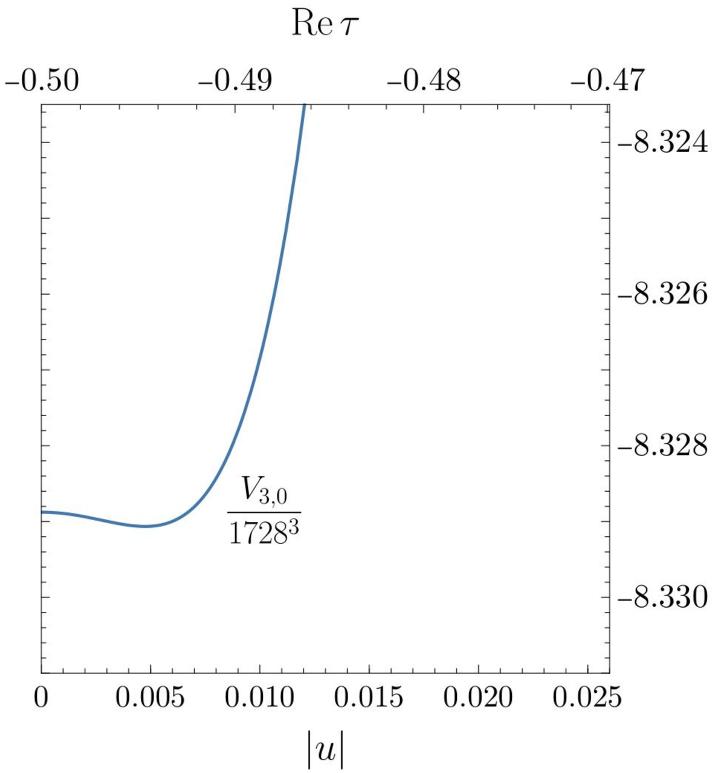
The $(m,0)$ family of potentials $(m = 2)$



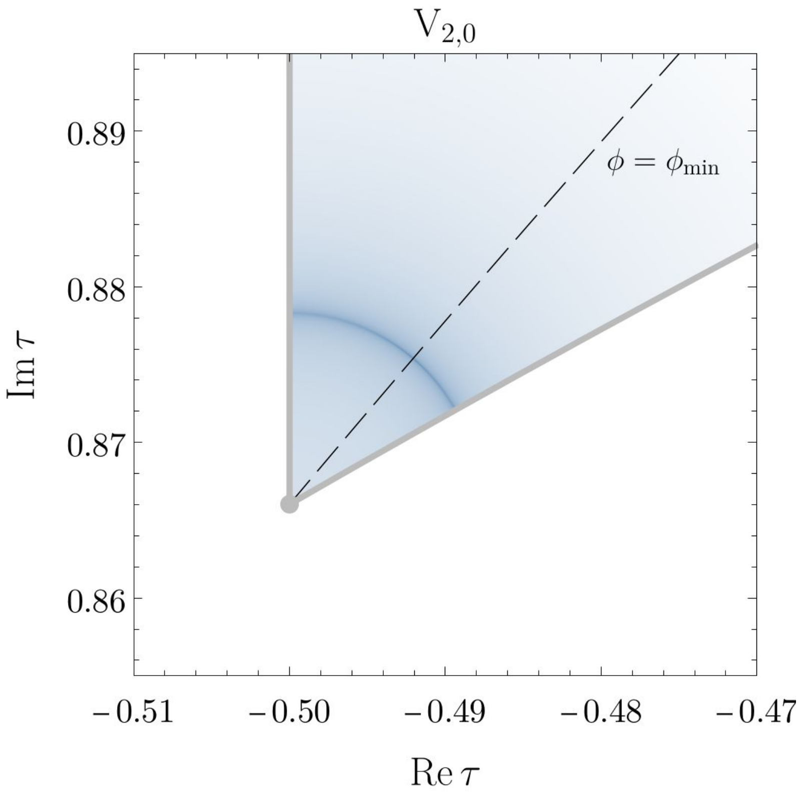


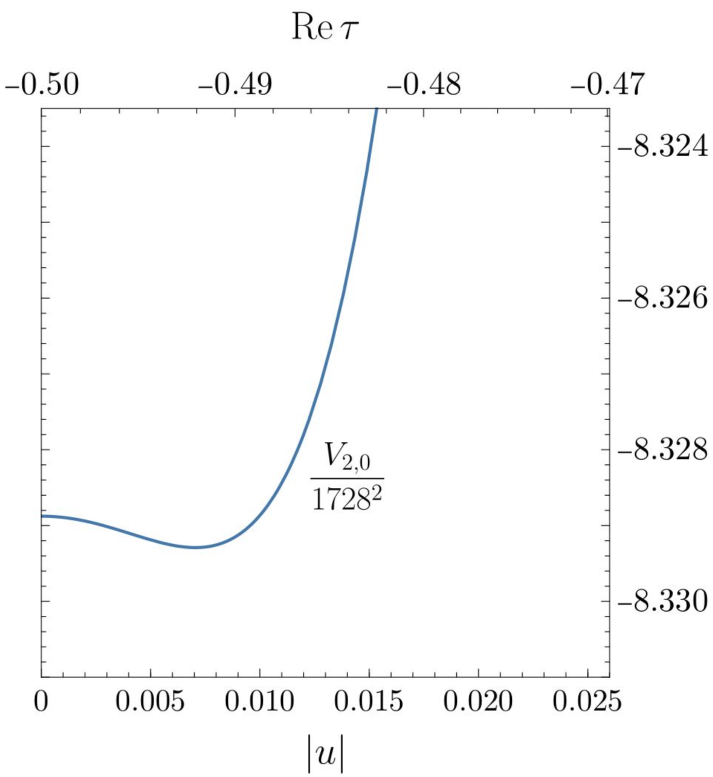
The (m,0) family of potentials (m = 3)





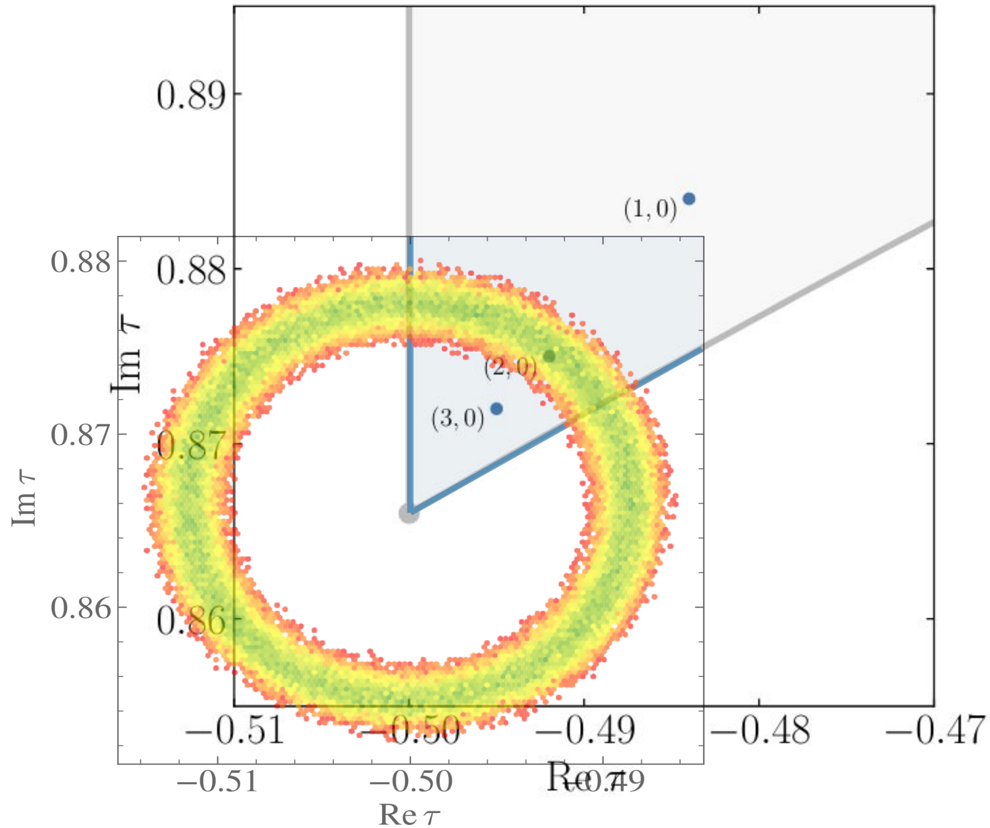
The (m,0) family of potentials (m = 2)





$$|\epsilon| \simeq 0.02 \Leftrightarrow |u| \simeq 0.007$$

Matching puzzle pieces?



$$|\epsilon| \simeq 0.02 \Leftrightarrow |u| \simeq 0.007$$

Summary (3/3)

Summary (3/3)

- There are simple potentials for modulus stabilisation, which are independent of the level N
- Novel **CP-breaking minima** are found, located **in the vicinity** of (but not directly on) the cusps
- The found deviation $|u|$ matches the BU requirement

Pieces of a puzzle / future
(a personal view)

- use TD to fix irreps, weights? (Andreas' talk, in 5 mins)
- hints of universality? (Feruglio 2211.00659)
- phenomenology beyond masses and mixing?
- modular symmetry breaking as the only source of CPV?
- do away with SUSY?

Vielen Dank!

Backup slides

Modular-invariant SUSY actions

Ferrara et al, '89

$$W(\psi;\tau) = \sum_{n} \sum_{\{i_1,\ldots,i_n\}} \sum_{s} g_{i_1\ldots i_n,s} \left(Y_{i_1\ldots i_n,s}(\tau)\, \psi_{i_1}\cdots\psi_{i_n}\right)_{\mathbf{1},s}$$

$$\mathcal{S} = \int d^4x\, d^2\theta\, d^2\overline{\theta}\; K(\psi,\overline{\psi};\tau,\overline{\tau}) + \int d^4x\, d^2\theta\; W(\psi;\tau) + \text{h.c.}$$

τ is a dimensionless spurion: once its value is fixed, it **parameterises all** modular sym. breaking

One may argue that *Y*'s play the role of flavons, but structures are **completely fixed** given the modulus VEV

Modular-invariant SUSY actions

$$W(\psi;\tau) = \sum_n \sum_{\{i_1,\ldots,i_n\}} \sum_s g_{i_1 \ldots i_n,s} \left(Y_{i_1 \ldots i_n,s}(\tau)\, \psi_{i_1} \ldots \psi_{i_n} \right)_{\mathbf{1},s}$$

$$\begin{cases} \tau \to \gamma\tau = \dfrac{a\tau + b}{c\tau + d} \\ \psi_i \to (c\tau + d)^{-k_i} \rho_i(\gamma)\, \psi_i \\ Y(\tau) \to Y(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma)\, Y(\tau) \end{cases} \quad \text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

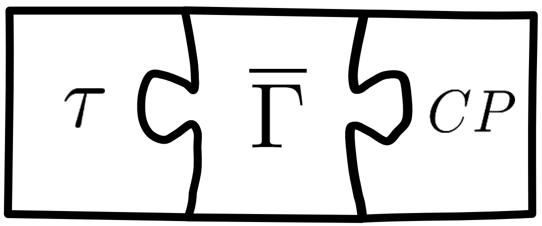
weights (k_i, k_Y)

k_Y positive & even, for PSL(2,Z)

$Y(\tau)$ are **modular forms** obeying $\begin{cases} k_Y = k_{i_1} + \ldots + k_{i_n} \\ \rho_Y \otimes \rho_{i_1} \otimes \ldots \otimes \rho_{i_1} \supset \mathbf{1} \end{cases}$

Live in linear spaces of finite dimension

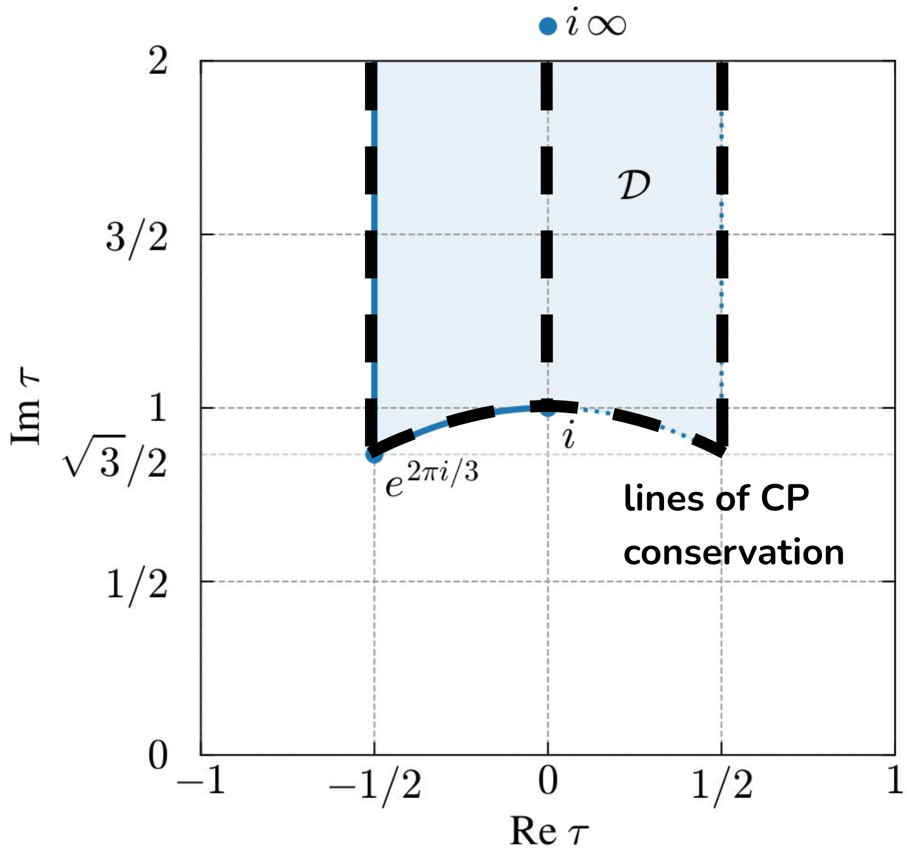
Combining modular and CP symmetries



$$\tau \xrightarrow{\text{CP}} -\tau^*$$

$$\psi(x) \xrightarrow{\text{CP}} X_{\mathbf{r}}^{\text{CP}}\,\overline{\psi}(x_{\text{P}})$$

$$Y(\tau) \xrightarrow{\text{CP}} X_{\mathbf{r}}^{\text{CP}}\,Y^*(\tau)$$



Constraints on the Kähler potential?

- **Kähler** not constrained by the symmetry.
- Under a modular transformation, invariant up to:

$$K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) \rightarrow K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) + f(\chi_i; \tau) + f(\overline{\chi}_i; \overline{\tau})$$

- Minimal choice:

$$K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) = -h\,\Lambda_0^2 \log(-i(\tau - \overline{\tau})) + \sum_i \frac{|\chi_i|^2}{(-i(\tau - \overline{\tau}))^{k_i}}$$

should be justified from the top-down

Chen, Ramos-Sánchez and Ratz, 1909.06910

- Further constraints may arise from combining modular group + traditional finite flavour symmetry

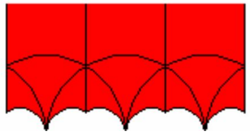
Nilles, Ramos-Sanchez, Vaudrevange, 2004.05200

SUSY breaking effects?

- **RGEs & threshold corrections** need to be considered, depend on tan β and unknown SUSY spectrum
- **SUSY-breaking** corrections can be made negligible via separation of scales (power counting argument)
- Under reasonable conditions, predictions may be unaffected

Feruglio and Criado, 1807.01125

Larger fundamental domains?



- Despite working with representations of the quotients, our theories are **fully modular invariant**
- To have canonical kinetic terms,

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \Rightarrow \quad g_i \to (c\tau + d)^{-k_{Y_i}} g_i$$

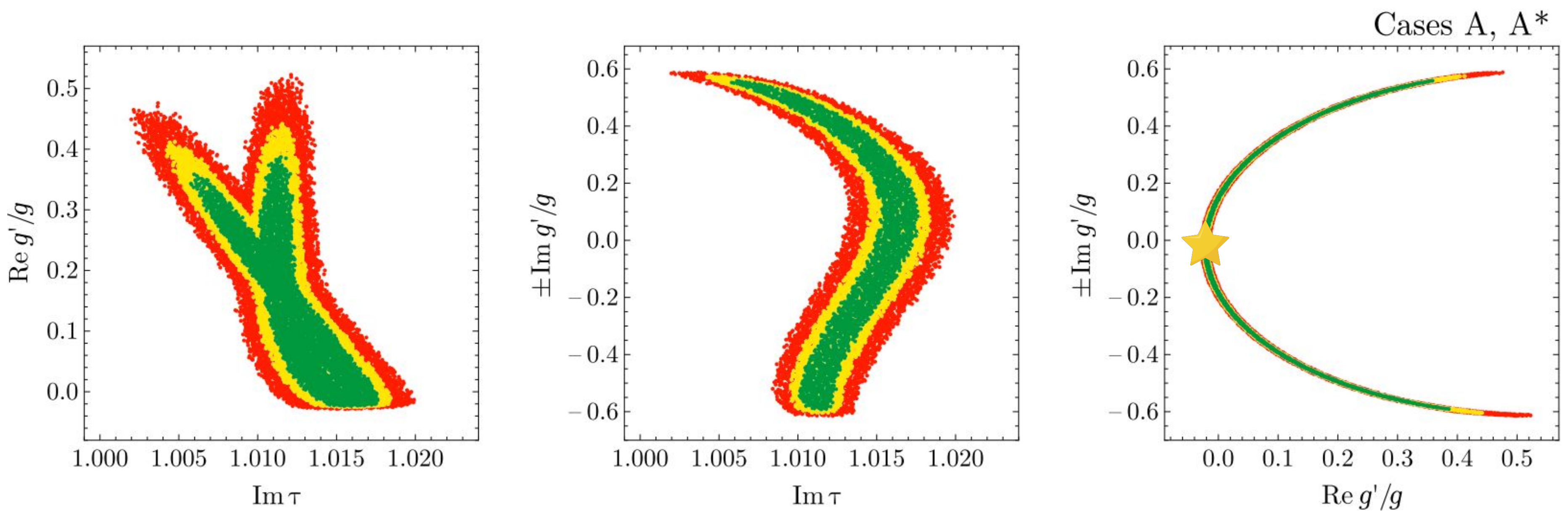
- e.g. in a particular model, see sec. 4 of Novichkov, JP, Petcov, Titov, 1811.04933

$$(\tau,\, \beta/\alpha,\, \gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots) \to$$
$$\left(\frac{a\tau + b}{c\tau + d},\, (c\tau + d)^{-2}\,\beta/\alpha,\, (c\tau + d)^{-2}\,\gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots\right)$$

these different parameter sets lead to the same observables

- Things may be different if **flavons** are present!

Correlations between parameters in the first S4 example model



see Novichkov, JP, Petcov, Titov, 1811.04933

Decompositions under residual groups: S3, A4’

$\mathbf{r}$	$\mathbb{Z}_4^S\,(\tau=i)$	$\mathbb{Z}_3^{ST}\times\mathbb{Z}_2^R\,(\tau=\omega)$	$\mathbb{Z}_2^T\times\mathbb{Z}_2^R\,(\tau=i\infty)$
$\mathbf{1}$	$\mathbf{1}_k$	$\mathbf{1}_k^\pm$	$1_0^\pm$
$\mathbf{1}'$	$\mathbf{1}_{k+2}$	$\mathbf{1}_k^\pm$	$1_1^\pm$
$\mathbf{2}$	$\mathbf{1}_k\oplus\mathbf{1}_{k+2}$	$\mathbf{1}_{k-1}^\pm\oplus\mathbf{1}_{k+1}^\pm$	$1_0^\pm\oplus 1_1^\pm$

$\mathbf{r}$	$\mathbb{Z}_4^S\,(\tau=i)$	$\mathbb{Z}_3^{ST}\times\mathbb{Z}_2^R\,(\tau=\omega)$	$\mathbb{Z}_3^T\times\mathbb{Z}_2^R\,(\tau=i\infty)$
$\mathbf{1}$	$\mathbf{1}_k$	$\mathbf{1}_k^\pm$	$1_0^\pm$
$\mathbf{1}'$	$\mathbf{1}_k$	$\mathbf{1}_{k+1}^\pm$	$\mathbf{1}_1^\pm$
$\mathbf{1}''$	$\mathbf{1}_k$	$\mathbf{1}_{k+2}^\pm$	$\mathbf{1}_2^\pm$
$\hat{\mathbf{2}}$	$\mathbf{1}_{k+1}\oplus\mathbf{1}_{k+3}$	$\mathbf{1}_k^\mp\oplus\mathbf{1}_{k+1}^\mp$	$1_0^\mp\oplus\mathbf{1}_1^\mp$
$\hat{\mathbf{2}}'$	$\mathbf{1}_{k+1}\oplus\mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^\mp\oplus\mathbf{1}_{k+2}^\mp$	$\mathbf{1}_1^\mp\oplus\mathbf{1}_2^\mp$
$\hat{\mathbf{2}}''$	$\mathbf{1}_{k+1}\oplus\mathbf{1}_{k+3}$	$\mathbf{1}_k^\mp\oplus\mathbf{1}_{k+2}^\mp$	$1_0^\mp\oplus\mathbf{1}_2^\mp$
$\mathbf{3}$	$\mathbf{1}_k\oplus\mathbf{1}_{k+2}\oplus\mathbf{1}_{k+2}$	$\mathbf{1}_k^\pm\oplus\mathbf{1}_{k+1}^\pm\oplus\mathbf{1}_{k+2}^\pm$	$1_0^\pm\oplus\mathbf{1}_1^\pm\oplus\mathbf{1}_2^\pm$

Decompositions under residual groups: S4’

$\mathbf{r}$	$\mathbb{Z}_4^S\ (\tau = i)$	$\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R\ (\tau = \omega)$	$\mathbb{Z}_4^T \times \mathbb{Z}_2^R\ (\tau = i\infty)$
$\mathbf{1}$	$\mathbf{1}_k$	$\mathbf{1}_k^{\pm}$	$1_0^{\pm}$
$\hat{\mathbf{1}}$	$\mathbf{1}_{k+1}$	$\mathbf{1}_k^{\mp}$	$\mathbf{1}_3^{\mp}$
$\mathbf{1}'$	$\mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm}$	$1_2^{\pm}$
$\hat{\mathbf{1}}'$	$\mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp}$	$\mathbf{1}_1^{\mp}$
$\mathbf{2}$	$\mathbf{1}_{k+2} \oplus \mathbf{1}_k$	$\mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus 1_2^{\pm}$
$\hat{\mathbf{2}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_1^{\mp} \oplus \mathbf{1}_3^{\mp}$
$\mathbf{3}$	$\mathbf{1}_{k+2} \oplus \mathbf{1}_k \oplus \mathbf{1}_k$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$\mathbf{1}_1^{\pm} \oplus 1_2^{\pm} \oplus \mathbf{1}_3^{\pm}$
$\hat{\mathbf{3}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$1_0^{\mp} \oplus \mathbf{1}_1^{\mp} \oplus 1_2^{\mp}$
$\mathbf{3}'$	$\mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_k$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_1^{\pm} \oplus \mathbf{1}_3^{\pm}$
$\hat{\mathbf{3}}'$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$1_0^{\mp} \oplus 1_2^{\mp} \oplus \mathbf{1}_3^{\mp}$

Decompositions under residual groups: A5’

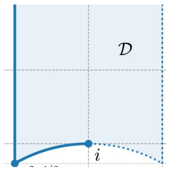
$\mathbf{r}$	$\mathbb{Z}_4^S\,(\tau = i)$	$\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R\,(\tau = \omega)$	$\mathbb{Z}_5^T \times \mathbb{Z}_2^R\,(\tau = i\infty)$
$\mathbf{1}$	$\mathbf{1}_k$	$\mathbf{1}_k^{\pm}$	$1_0^{\pm}$
$\hat{\mathbf{2}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_2^{\mp} \oplus \mathbf{1}_3^{\mp}$
$\hat{\mathbf{2}}'$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_1^{\mp} \oplus \mathbf{1}_4^{\mp}$
$\mathbf{3}$	$\mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_1^{\pm} \oplus \mathbf{1}_4^{\pm}$
$\mathbf{3}'$	$\mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_2^{\pm} \oplus \mathbf{1}_3^{\pm}$
$\mathbf{4}$	$\mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$\mathbf{1}_1^{\pm} \oplus \mathbf{1}_2^{\pm} \oplus \mathbf{1}_3^{\pm} \oplus \mathbf{1}_4^{\pm}$
$\hat{\mathbf{4}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_1^{\mp} \oplus \mathbf{1}_2^{\mp} \oplus \mathbf{1}_3^{\mp} \oplus \mathbf{1}_4^{\mp}$
$\mathbf{5}$	$\mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_1^{\pm} \oplus \mathbf{1}_2^{\pm} \oplus \mathbf{1}_3^{\pm} \oplus \mathbf{1}_4^{\pm}$
$\hat{\mathbf{6}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$1_0^{\mp} \oplus 1_0^{\mp} \oplus \mathbf{1}_1^{\mp} \oplus \mathbf{1}_2^{\mp} \oplus \mathbf{1}_3^{\mp} \oplus \mathbf{1}_4^{\mp}$

Details of the model fit

Model	Section 4.2 (S_4')
$\mathrm{Re}\,\tau$	$-0.496^{+0.009}_{-0.016}$
$\mathrm{Im}\,\tau$	$0.877^{+0.0023}_{-0.024}$
α_2/α_1	—
α_3/α_1	$2.45^{+0.44}_{-0.42}$
α_4/α_1	$-2.37^{+0.36}_{-0.3}$
α_5/α_1	$1.01^{+0.06}_{-0.06}$
g_2/g_1	$1.5^{+0.15}_{-0.14}$
g_3/g_1	$2.22^{+0.17}_{-0.15}$
$v_d\,\alpha_1$, GeV	$4.61^{+1.32}_{-1.33}$
$v_u^2\,g_1/\Lambda$, eV	$0.268^{+0.057}_{-0.063}$
$\epsilon(\tau)$	$0.0186^{+0.0028}_{-0.0023}$
CL mass pattern	$(1, \epsilon, \epsilon^2)$
max(BG)	0.848

m_e/m_μ	$0.00475^{+0.00061}_{-0.00052}$
m_μ/m_τ	$0.0556^{+0.0136}_{-0.0116}$
r	$0.0298^{+0.00196}_{-0.0023}$
δm^2, 10^{-5} eV^2	$7.38^{+0.35}_{-0.44}$
$\lvert\Delta m^2\rvert$, 10^{-3} eV^2	$2.48^{+0.05}_{-0.04}$
$\sin^2\theta_{12}$	$0.304^{+0.039}_{-0.036}$
$\sin^2\theta_{13}$	$0.0221^{+0.0019}_{-0.002}$
$\sin^2\theta_{23}$	$0.539^{+0.0522}_{-0.099}$
m_1, eV	0
m_2, eV	$0.0086^{+0.0002}_{-0.00026}$
m_3, eV	$0.0502^{+0.00046}_{-0.00043}$
$\Sigma_i m_i$, eV	$0.0588^{+0.0002}_{-0.0002}$
$\lvert\langle m\rangle\rvert$, eV	$0.00144^{+0.00035}_{-0.00033}$
δ/π	$1 \pm \mathcal{O}(10^{-6})$
α_{21}/π	0
α_{31}/π	$1 \pm \mathcal{O}(10^{-5})$
$N\sigma$	0.563

q- and u-expansions of η



$|q| \leq e^{-\sqrt{3}\pi} \simeq 0.004$

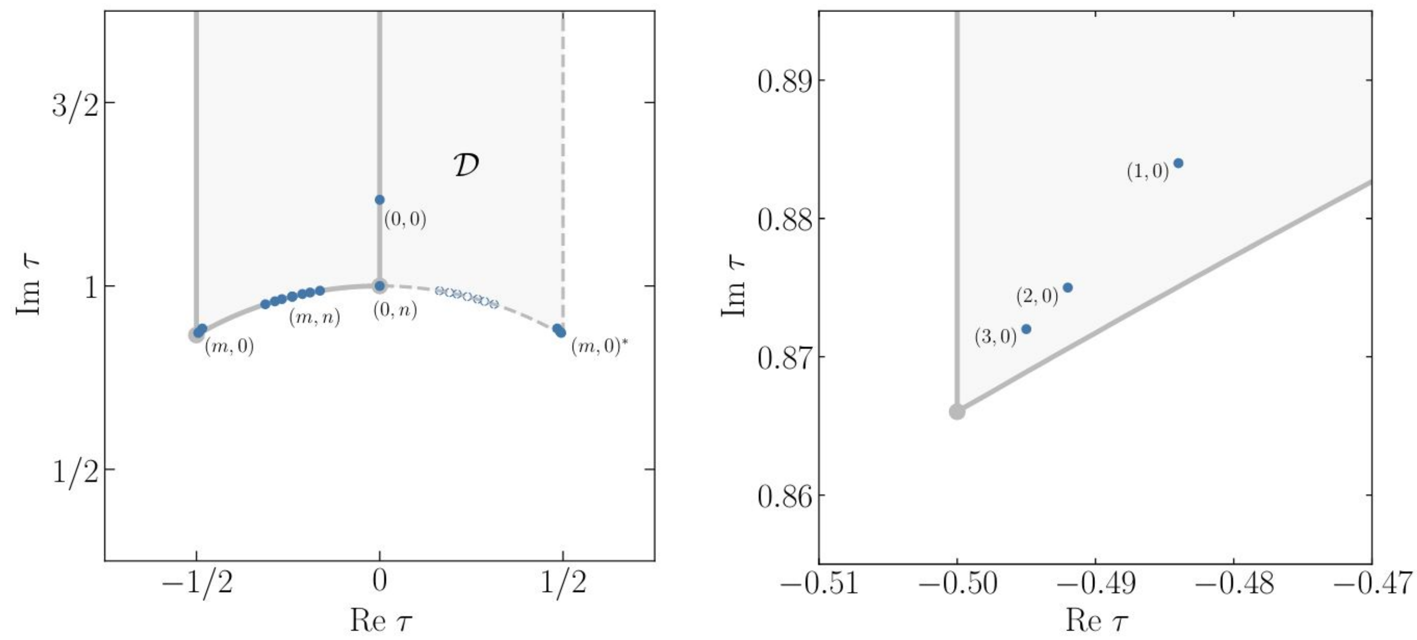
$(q \equiv e^{2\pi i\tau})$

$$\eta = q^{1/24} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2-n}{2}} = q^{1/24} \left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \mathcal{O}(q^{22})\right)$$

$$u \equiv \frac{\tau - \omega}{\tau - \omega^2} \qquad \tilde{\eta}(u) \equiv \frac{\eta(u)}{\sqrt{1-u}}$$

$$u \xrightarrow{ST} \omega^2 u \qquad \tilde{\eta}(u) \xrightarrow{ST} \tilde{\eta}(u)$$

$$\begin{aligned} \tilde{\eta}(u) &\simeq e^{-i\pi/24} \left(0.800579 - 0.573569u^3 - 0.780766u^6 - 0.150007u^9\right) + \mathcal{O}(u^{12}) \\ &\equiv e^{-i\pi/24} \left(\tilde{\eta}_0 + \tilde{\eta}_3 u^3 + \tilde{\eta}_6 u^6 + \tilde{\eta}_9 u^9\right) + \mathcal{O}(u^{12}) \,, \end{aligned}$$



(0, 0) is a single minimum at $\tau \simeq 1.2i$ on the imaginary axis, corresponding to the case $m = n = 0$;

(0, *n*) is a single minimum at the symmetric point $\tau = i$ attained when $m = 0$, $n \neq 0$;

(*m*, 0) and (*m*, 0)* are a pair of degenerate minima for each $m \neq 0$ and $n = 0$: $(m, 0)$ is located in the vicinity of the left cusp $\tau = \omega$, approaching this symmetric point as m increases, while $(m, 0)^*$ is its CP-conjugate;

(*m*, *n*) is a series of minima on the unit arc, corresponding to $m \neq 0$, $n \neq 0$; these minima shift towards $\tau = \omega$ ($\tau = i$) along the arc as m (n) grows.

Extrema at $\tau = i, \omega$

Gonzalo, Ibáñez and Uranga, 1812.06520

	$V\,(T=1)$	Type of Extrema	H	$\frac{dH}{dT}$	SUSY
$m>1$	$V=0$	Min	0	0	Yes
$m=1$	$\frac{1}{T_I^3\lvert\eta\rvert^{12}}\left\{\lvert a\rvert^2\lvert C\rvert^2\right\}>0$	Max $-2.57<\frac{H'''}{H'}<-1.57$ SP $\frac{H'''}{H'}<-2.57$ or $\frac{H'''}{H'}>-1.57$	0	$\neq 0$	No
$m=0$	$\propto \frac{\lvert P(0)\rvert^2}{T_I^3\lvert\eta\rvert^{12}}\{-3\}<0$	Min $\left\lvert\frac{H''}{H}+1.19\right\rvert>\frac{3}{2}$ Max $-\frac{3}{4}<\frac{H''}{H}+1.19<\frac{3}{4}$ SP (Saddle Point) if else	$\neq 0$	0	Yes

Table 2. Classification of the extrema found at $T=i$.

	$V\,(T=\rho)$	Type of Extrema	H	$\frac{dH}{dT}$	SUSY
$n>1$	$V=0$	Minimum	0	0	Yes
$n=1$	$\frac{1}{\lvert\eta\rvert^{12}}\left\{\frac{4}{3}\lvert\mathcal{P}\,(1728)\rvert^2\lvert D\rvert^2\right\}>0$	Maximum	0	$\neq 0$	No
$n=0$	$\propto \frac{1728^m\lvert\mathcal{P}(1728)\rvert^2}{T_I^3\lvert\eta\rvert^{12}}\{-3\}<0$	Maximum	$\neq 0$	0	Yes

Table 3. Classification of the extrema found at $T=\rho$.

No, there is no tuning in choosing this form of the superpotential (arguably)

$$H(\tau) \propto (J(\tau) - 1)^{m/2}$$

Subset of all possible $H(\tau)$ which vanish only at the symmetric point $\tau = i$ (itself distinguished by modular symmetry)

$$J(\tau) \equiv j(\tau)/1728$$

The global SUSY limit (a comment)

$$\mathfrak{n} = \kappa^2 \Lambda_K^2 \to 0$$

$$K(\tau, \bar{\tau}) = -\Lambda_K^2 \log(2 \operatorname{Im} \tau)$$

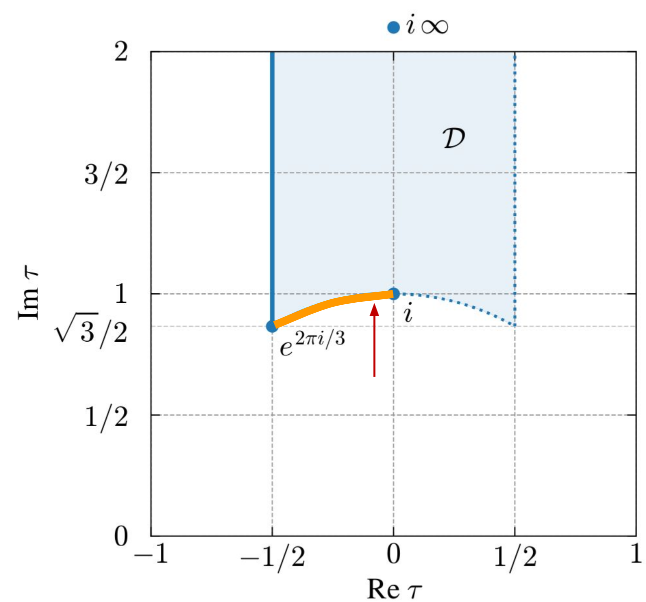
$$\kappa^2 = 8\pi / M_P^2$$

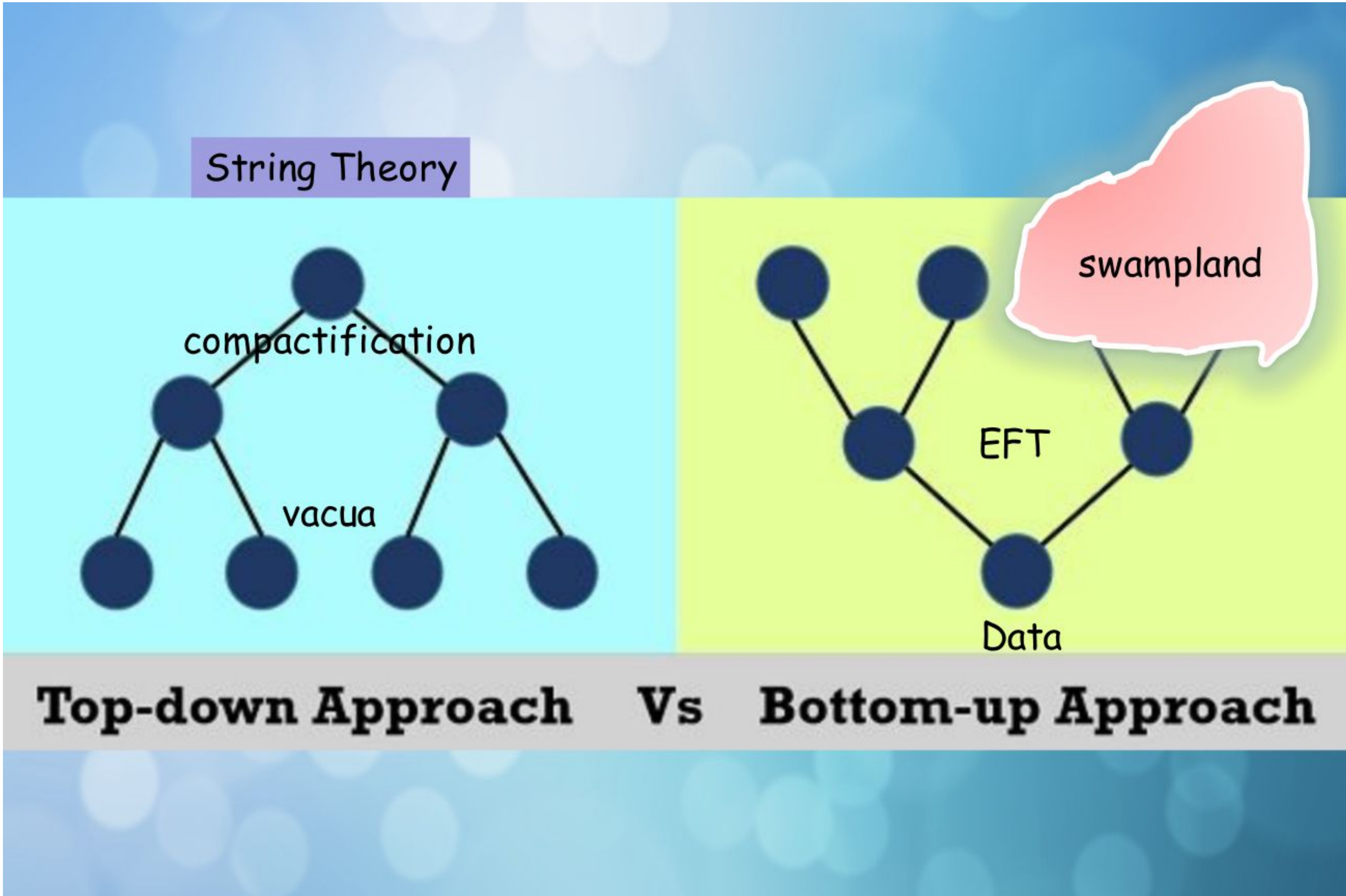
$$W(\tau) = \Lambda_W^3 H(\tau)$$

$$H(\tau) = (j(\tau) - 1728)^{m/2} j(\tau)^{n/3} \mathcal{P}(j(\tau))$$

$$V(\tau, \bar{\tau}) = \frac{4\Lambda_W^6}{\Lambda_K^2} (\operatorname{Im} \tau)^2 \left| H'(\tau) \right|^2$$

- Global minima are zeros of *H*’
- non-trivial $\mathcal{P}(j)$ can be engineered to produce minima at arbitrary points in the fundamental domain





from Feruglio's slides at Mod. Symmetry Bethe Workshop

tests of modulus couplings

G-J. Ding, FF, 2003.13448

non standard neutrino interactions

$$\mathcal{L} = i \sum_{f=e,e^c,\nu} \overline{f}\,\overline{\sigma}^\mu \partial_\mu f + \frac{1}{2}\partial_\mu\varphi_\alpha\partial^\mu\varphi_\alpha - \frac{1}{2}M_\alpha^2\varphi_\alpha^2$$
$$- (m_e + \mathcal{Z}_\alpha^e\varphi_\alpha)e^c e - \frac{1}{2}\nu(m_\nu + \mathcal{Z}_\alpha^\nu\varphi_\alpha)\nu + h.c. + \dots .$$

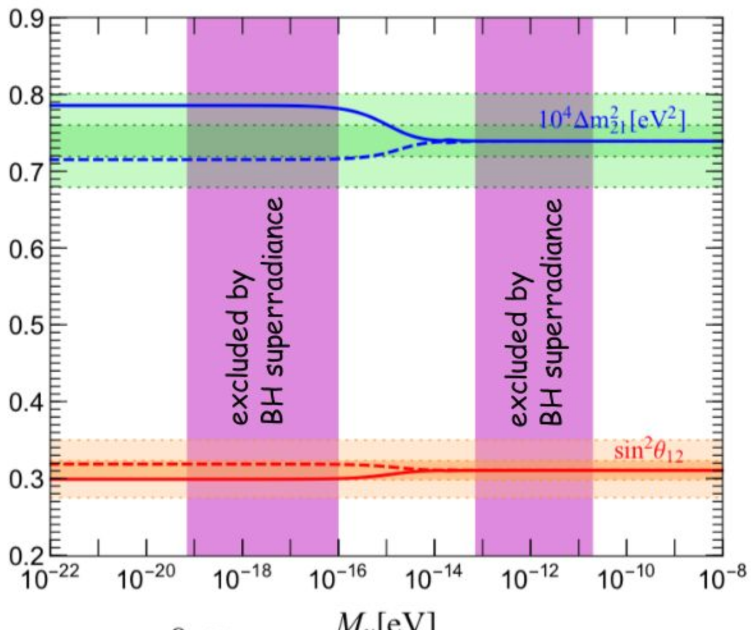
$$\tau = \langle\tau\rangle + \frac{\varphi_u + i\,\varphi_v}{\sqrt{2}}$$

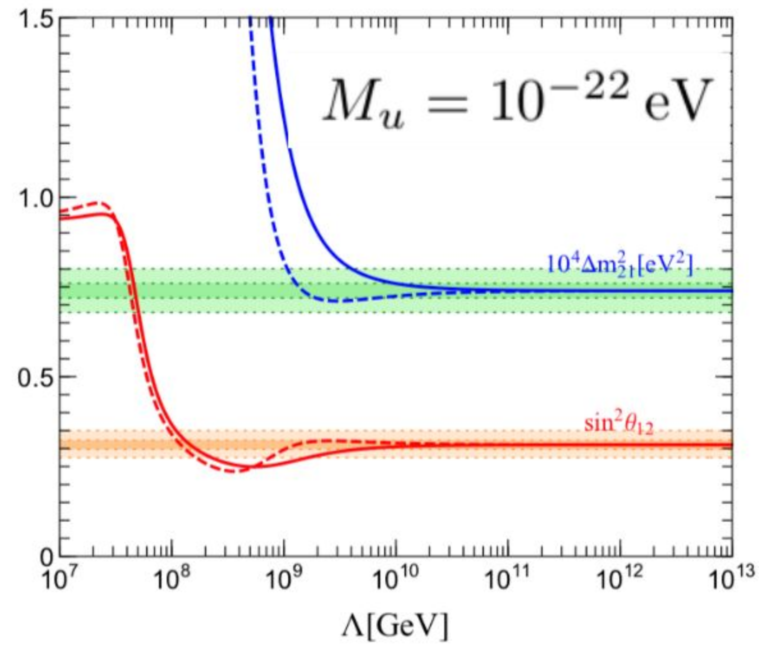
in medium with non-zero electron number density

small, unless the modulus is very light

$$\delta m_\nu(0) = -n_e^0 \frac{\mathrm{Re}(\mathcal{Z}^e)\mathcal{Z}^\nu}{M^2(R)} ,$$

in the sun:





$\Lambda = 5 \times 10^9$ GeV

[modulus VEV]

from Feruglio's slides at Mod. Symmetry Bethe Workshop