Karlsruher Institut für Technologie

TRR 257 - Particle Physics Phenomenology after the Higgs Discovery

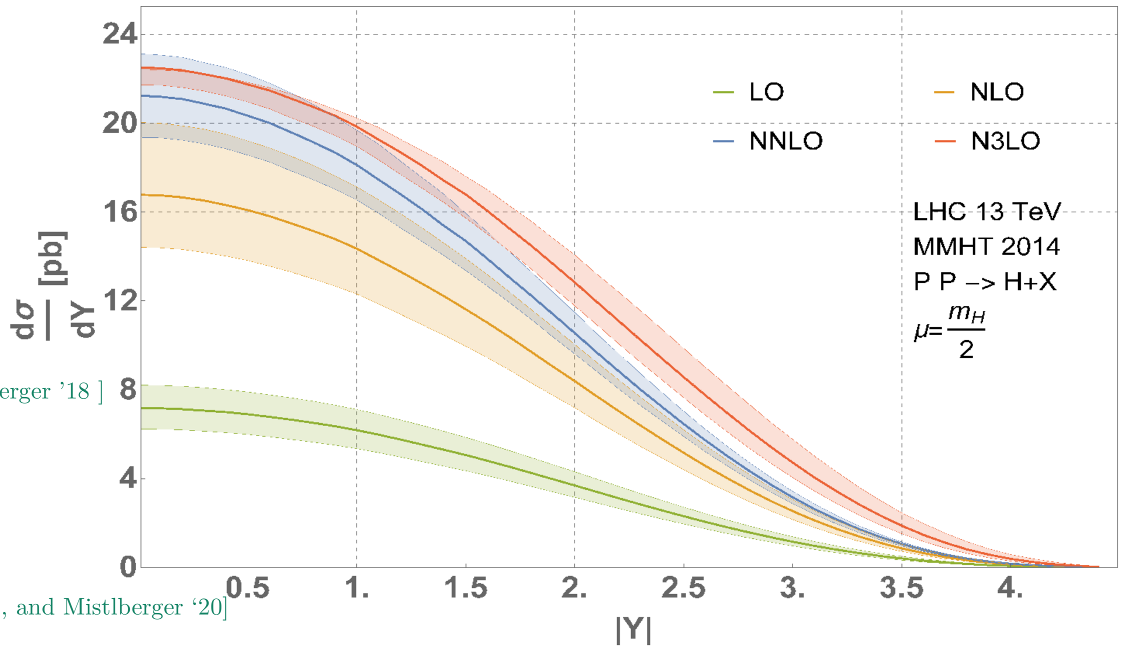
On phase-space integrals with Heaviside functions

Based on 2111.13594 & 2204.094559 in collaboration with Maximilian Delto, Kirill Melnikov and Chen-Yu Wang

Daniel Baranowski; KIT | June 8, 2022
CRC Meeting, Karlsruhe

Motivation

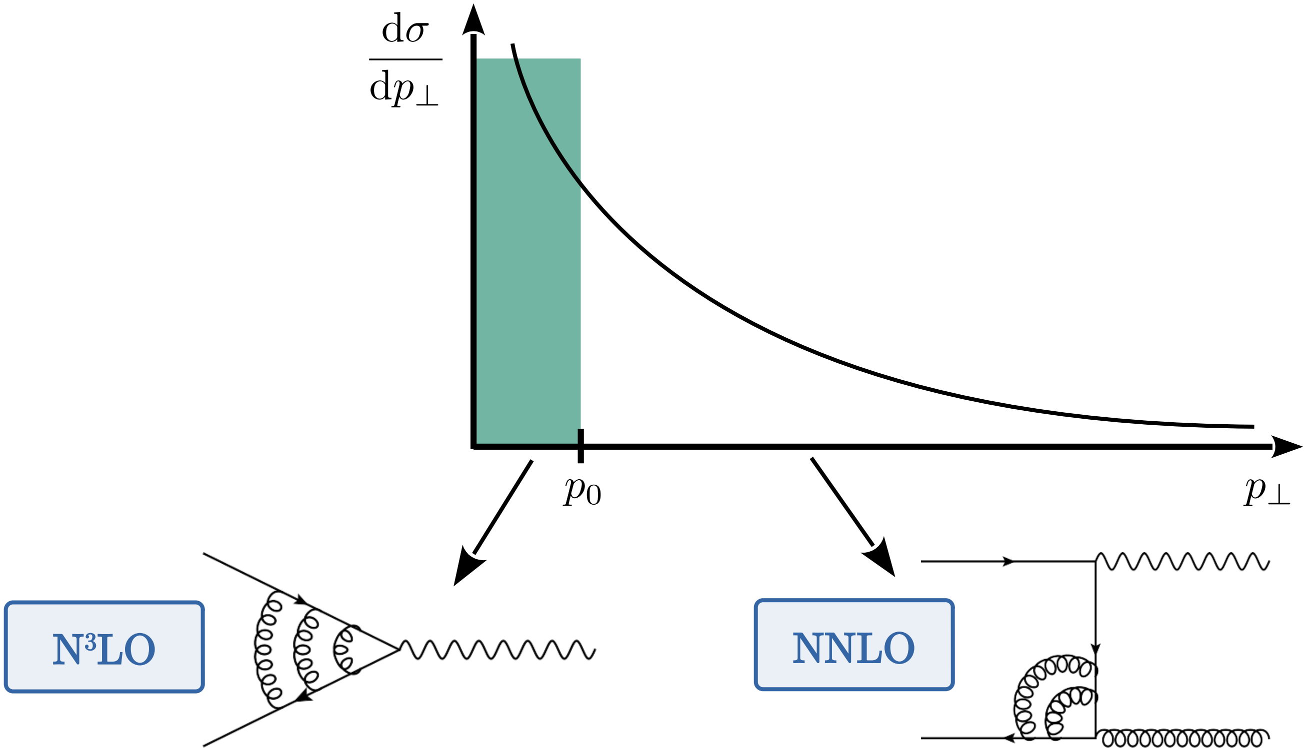
- Continuous push for more precision.
- N^3LO current precision frontier:
  - Higgs rapidity distribution [Dulat, Mistlberger, Pelloni '18]
  - Fully inclusive:
    - **gg → H** [Anastasiou et al. '15.Anastasiou et al`16, Mistelberger '18]
    - **pp → γ*** [Duhr, Dulat, Mistlberger '20]
    - **bb → H** [Duhr, Dulat, Hirschi, Mistlberger '20;Duhr, Dulat, and Mistlberger '20]
    - ...



- Already fully differential at N^3LO:
  - Higgs boson production [Chen, Gehrmann, Glover, Huss, Mistlberger, Pelloni '21]
  - Drell-Yan process [Chen, Gehrmann, Glover, Huss, Yang, Zhu '21;Chen, Gehrmann, Glover, Huss, Yang, et al. '22]

Motivation

- Differential calculation require a good handle on IR divergences
- Two possible methods:
  - Subtraction
  - Slicing → **more feasible @ N3LO**



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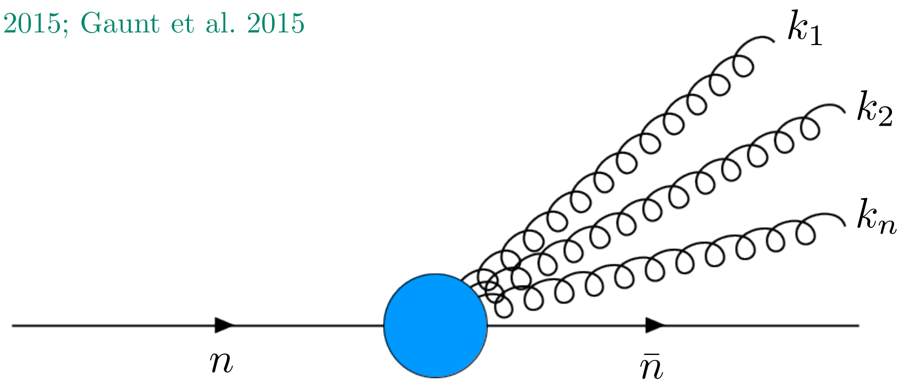
Zero-jettiness slicing
Boughezal, Focke, et al. 2015; Gaunt et al. 2015



- Definition of zero-jettiness

$$\tau = \sum_j \min_{i \in 1,2} \left[\frac{n_i \cdot k_j}{Q_i} \right]$$

- Simplification through factorization theorem derived in SCET
  [Stewart, Tackmann, Waalewijn '10]

$$\lim_{\tau \to 0} \mathrm{d}\sigma(O) = B_\tau \otimes B_\tau \otimes S_\tau \otimes H_\tau \otimes \mathrm{d}\sigma^{LO}$$

Beam function:

- Collinear singularities
- Known through **N³LO**
  [Ebert, Mistlberger, Vita '20]
  [Behring,Melnikov,Rietkerk,Tancredi,Wever'19]

Soft function:

- Soft singularities
- **NOT** known at **N³LO**

Beam function

$$B_{ij}(t,z) \sim \sum_{\{m\}} \int \boxed{\mathrm{dPS}^{(m)}(t,z)}\ \boxed{P_{j\to i^*\{m\}}}$$

Delta functions — Linear and quadratic propagators

General Procedure

1. Amplitude in terms of integral families

   $$B = \sum_j C_j I_j$$

2. Use reverse unitarity to write delta functions as propagators

   $$\delta\left(p^2-m^2\right) = \frac{\mathrm{i}}{2\pi}\left[\frac{1}{p^2-m^2+\mathrm{i}\epsilon} - \frac{1}{p^2-m^2-\mathrm{i}\epsilon}\right]$$

   [Anastasiou,Melnikov'02]

3. Use Integration-by-parts to reduce expression to sum of master integrals (MIs)

   $$\int \mathrm{d}^d k \frac{\partial}{\partial k_\nu}\left[\nu_\mu \frac{1}{\prod_i D_i}\right] = 0$$

   [Chetyrkin,Tkachov'81]

4. Compute MIs

   $$B = \sum_j C'_j I'_j$$

Beam function

$$B_{ij}(t,z) \sim \sum_{\{m\}} \int \boxed{\mathrm{dPS}^{(m)}(t,z)}\ \boxed{P_{j\to i^*\{m\}}}$$

Delta functions | Linear and quadratic propagators

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$$B = \sum_j C_j I_j$$

$$\delta\left(p^2 - m^2\right) = \frac{\mathrm{i}}{2\pi}\left[\frac{\ldots}{p\ldots - \mathrm{i}\epsilon} - \frac{1}{p^2 - m^2 - \mathrm{i}\epsilon}\right]$$

[Anastasiou,Melnikov'02]

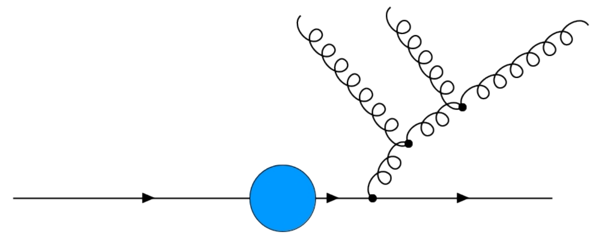
$$\int \mathrm{d}^d k \frac{\partial}{\partial k_\nu}\left[\nu_\mu \frac{1}{\prod_i \ldots}\ldots\right.$$

[Chetyrk… …81]

$$B = \sum_j C'_j I'_j$$

Number and complexity of integrals

Soft function



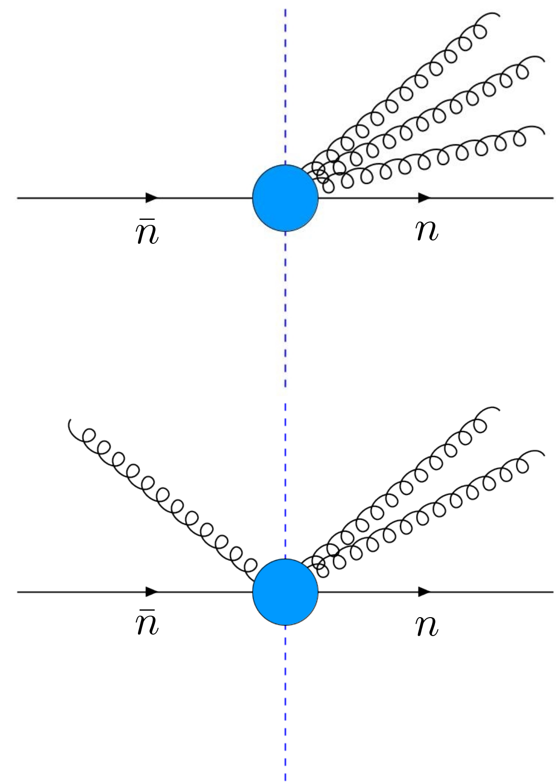
- $S_{ggg}^{nnn} \propto \int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn} \, |\boldsymbol{J}(k_1, k_2, k_3)|^2$

- Triple gluon emission eikonal in simplified form [Catani, Colferai, et al. 2020]

- Measurement function selects closest beam-line to each emission

$$\mathrm{d}\Phi_{\theta_1\theta_2\theta_3}^{nnn} = \left(\prod_{i=1}^{3} [\mathrm{d}k_i] \theta_i(k_i \cdot \overline{n} - k_i \cdot n)\right) \delta\left(\tau - \sum_{i=1}^{3} k_i \cdot n\right)$$

$$\mathrm{d}\Phi_{\theta_1\theta_2\theta_3}^{nn\bar{n}} = \left(\prod_{i=1}^{2} [\mathrm{d}k_i] \theta_i(k_i \cdot \overline{n} - k_i \cdot n)\right)$$
$$\times [\mathrm{d}k_3] \theta_3(k_3 \cdot n - k_3 \cdot \bar{n}) \delta\left(\tau - \sum_{i=1}^{2} k_i \cdot n - k_3 \cdot \bar{n}\right)$$



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Soft function

$$S \sim \sum_{\{m\}} \int \boxed{\mathrm{dPS}^{(m)}(\tau)} \; \boxed{\xi_{\{m\}}}$$

Delta **and theta** functions

Linear and quadratic propagators

General Procedure

1. Amplitude in terms of integral families
2. Use reverse unitarity to write delta functions as propagators
3. Use Integration-by-parts to reduce expression to sum of master integrals (MIs)
4. Compute MIs

$$\delta\left(p^2 - m^2\right) = \frac{\mathrm{i}}{2\pi}\left[\frac{1}{p^2 - m^2 + \mathrm{i}\epsilon} - \frac{1}{p^2 - m^2 - \mathrm{i}\epsilon}\right]$$

[Anastasiou,Melnikov'02]

$$\theta\left(k_i \cdot \bar{n} - k_i \cdot n\right) = ?$$

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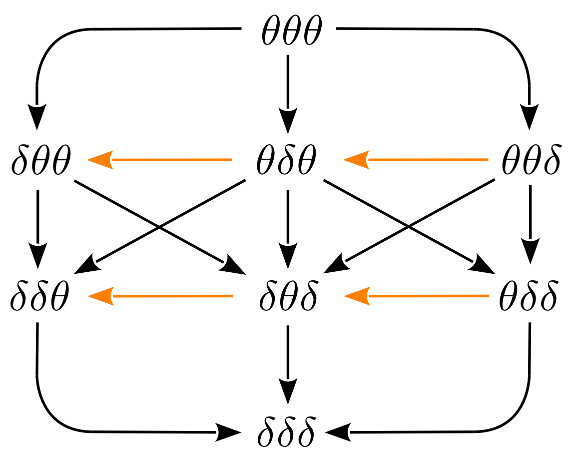
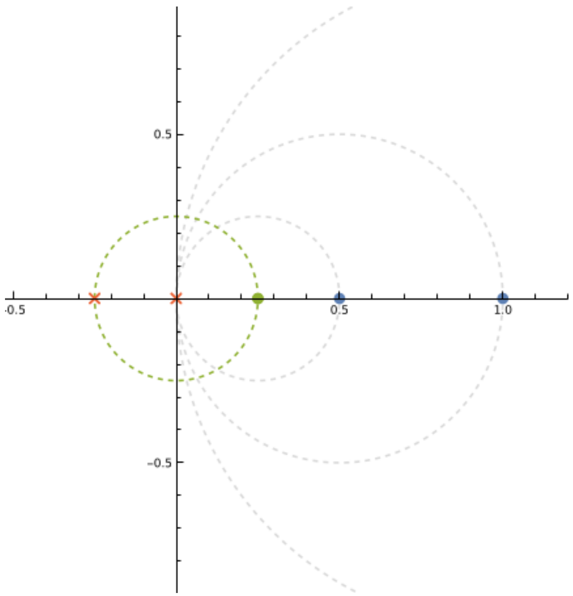
Outline

1) Integral Reduction



2) Integral Evaluation



Integral Reduction

- Replace theta functions with an integral

$$\theta(k\cdot\bar{n} - k\cdot n) = \int_0^1 \mathrm{d}z\ \delta\left(zk\cdot\bar{n} - k\cdot n\right)\ k\cdot\bar{n}$$

- **Standard IBP programs can be used**
- Need to integrate over 3 auxillary parameters

$$S = \int \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \sum_j C_j'\left(z_i\right) I_j'\left(z_i\right)$$

- tested at NNLO [DB '20]
- see also:
  - [Angeles-Martinez,Czarkon,Sapeta'18]
  - [Caola,Delto,Frellesvig,Melnikov'18]
  - [Chen'20]

- Implement IBPs for theta functions

$$\int \mathrm{d}^d k \frac{\partial}{\partial k_\nu}\left[\nu_\mu \frac{1}{\prod_i D_i}\right] = 0$$

$$\frac{\partial}{\partial k\cdot\bar{n}}\theta(k\cdot\bar{n} - k\cdot n) = \delta(k\cdot\bar{n} - k\cdot n)$$

- Generate IBP identities manually
- Solve the system using KIRA
- **Simpler master integrals**

$$S = \sum_j C_j' I_j'$$

- see also:
  - [Luo,Yang,Zhu,Zhu'19]

Integral Reduction

- Derive IBP for $I = \int \mathrm{d}^d k \; g(k \cdot \bar{n}) \; \theta(\underbrace{k \cdot \bar{n} - k \cdot n}_{f})$
- IBP splits into two pieces

$$\frac{\partial}{\partial k \cdot \bar{n}} I = \underbrace{\theta(f) \frac{\partial}{\partial k \cdot \bar{n}} g}_{\text{Homogenous}} + \overbrace{g \;\delta(f)}^{\text{Inhomogenous}}$$

- Homogenous: "Normal" piece unaffected by θ function
  - IBP fast index shift operation: $\frac{\partial}{\partial k \cdot \bar{n}} \mathcal{T}_{a_1,a_2,a_3} = \left[-a_3 \hat{3}\right] \mathcal{T}_{a_1,a_2,a_3}$
  - Form closed system on their own

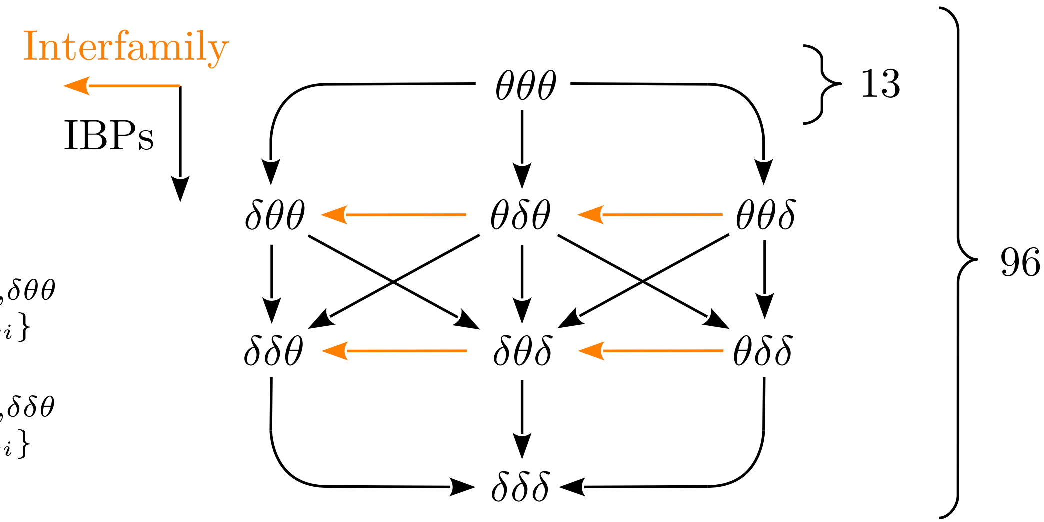
- Inhomogenous: New term, requires partial fractioning → slow

$$\cdots \frac{\theta\left(k \cdot \bar{n} - k \cdot n\right)}{(k \cdot n)^{a_5} (k \cdot \bar{n})^{a_6}} \cdots \to \cdots \frac{\delta\left(k \cdot \bar{n} - k \cdot n\right)}{(k \cdot n)^{a_5} (k \cdot \bar{n})^{a_6}} \cdots$$

  - Can't be written back into original topology

$$\frac{\partial}{\partial k \cdot \bar{n}} I^{\theta} = I'^{\theta} + I^{\delta}$$

Integral Reduction



- Hierarchical structure of Topologies

$$0 = \sum_i b_i^{\text{hom}}(\epsilon)\mathcal{T}^{k,\theta\theta\theta}_{\{a_i\}} + \sum_i \sum_k b_{i,k}^{\text{inhom}}(\epsilon)\mathcal{T}^{k,\delta\theta\theta}_{\{a_i\}}$$

$$0 = \sum_i c_i^{\text{hom}}(\epsilon)\mathcal{T}^{k,\delta\theta\theta}_{\{a_i\}} + \sum_i \sum_k c_{i,k}^{\text{inhom}}(\epsilon)\mathcal{T}^{k,\delta\delta\theta}_{\{a_i\}}$$

$$\ldots$$

- Create list of seeds integrals to apply relations on
  - As small as possible, time consuming due to partial fractioning
- Create complete set of interfamily relations (crucial)
- Create relations

} Mathematica

- Userdefined system of equations (udeq) in [Kira] to solve for MIs
- Check homogeneous solutions

} Kira

Analytic Regulator

- Soft function regulated by dimensional regularization
- All integrals **before and after reduction regularized**

$$S_{ggg}^{nnn} = \sum_{\alpha} c_\alpha I_\alpha$$

- ONLY @ N³LO: Integrals that appear only at IBP level, unregularized **collinear divergence**

$$k_i^\mu = \alpha_i \left(\frac{n^\mu}{2} + z_i \frac{\bar{n}^\mu}{2} + \sqrt{z_i} e_{i,\perp}^\mu \right)$$

$$I^{\text{ibp}} \sim \int_0^1 dz_1 \, \frac{1}{z_1} (z_1)^\epsilon \, (z_1)^{-\epsilon} \dots$$

- Regularized by **additional analytic regulator**

$$\mathrm{d}\Phi_{f_1 f_2 f_3}^{nnn} \to \mathrm{d}\Phi_{f_1 f_2 f_3}^{nnn} (k_1 \cdot n)^\nu (k_2 \cdot n)^\nu (k_3 \cdot n)^\nu$$

Analytic Regulator

- Soft function regulated by dimensional regularization
- All integrals **before and after reduction regularized**

$$S^{nnn}_{ggg} = \sum_\alpha c_\alpha(\nu) I^\nu_\alpha + \nu \sum_\alpha \widetilde{c}_\alpha(\nu) \overline{I}^\nu_\alpha,$$

- ONLY @ N^{3}LO: Integrals that appear only at IBP level, unregularized **collinear divergence**

$$k_i^\mu = \alpha_i \left(\frac{n^\mu}{2} + z_i \frac{\bar{n}^\mu}{2} + \sqrt{z_i} e^\mu_{i,\perp} \right)$$

$$I^{\text{ibp}} \sim \int_0^1 dz_1 \, \frac{1}{z_1^{1+\nu}} \dots$$

- Regularized by **additional analytic regulator**

$$\mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} \to \mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} (k_1 \cdot n)^\nu (k_2 \cdot n)^\nu (k_3 \cdot n)^\nu$$

Integral Evaluation

- Soft function in terms of MI $S_{ggg}^{nnn} = \sum_\alpha c_\alpha(\nu) I_\alpha^\nu + \nu \sum_\alpha \widetilde{c}_\alpha(\nu) \overline{I}_\alpha^\nu,$
- No $\theta\theta\theta$ MI
- 40 MI without $1/(k_1+k_2+k_3)^2$ propagator
  - Calculate analytically using HypExp and HyperInt
- 48 MI with $1/(k_1+k_2+k_3)^2$ propagator
  - hard to calculate directly
  - add m^2 to the propagator:

$$J = \int \mathrm{d}\Phi_{f_1 f_2 f_3}^{nnn} \, \frac{1}{k_{123}^2 + m^2} \frac{\cdots}{(k_1 \cdot k_2)(k_1 \cdot n)\cdots}$$

  - Construct DEQ in m^2:

$$\frac{\partial}{\partial m^2} \boldsymbol{J} = \boldsymbol{M} \boldsymbol{J}$$

  - Fix boundary at $m \to \infty$, bad propagator “disappears”
  - Solve DEQ numerically and recover Taylor expansion in m^2 at $m = 0$.

Integral Evaluation

- **Straightforward?**
- Finding a good basis: select $\nu = 0$ MI as preferred basis
- Growth in basis vector of DEQ $48 \underset{m \neq 0}{\rightarrow} 172 \underset{\nu \neq 0}{\rightarrow} 265$
  - How to predict $1/\nu$ behavior of integrals?
  - Set $\nu = 0$ where possible before hand $\underset{\nu \to 0}{\rightarrow} 173$
  - Do m and ν limits commute? (yes but not trivially)
- What regions contribute to boundary condition?

$$\frac{\delta\delta\delta}{(k_1 + k_2 + k_3) + m^2} \approx \frac{\delta\delta\delta}{m^2}$$

$$\frac{\theta\delta\delta}{(k_1 + k_2 + k_3) + m^2} \approx \frac{1}{m^{2-2\epsilon}} \frac{\theta\delta\delta}{k_1 \cdot \bar{n}\ k_{23} \cdot n}$$

$$\frac{\theta\theta\delta}{(k_1 + k_2 + k_3) + m^2} \approx \frac{1}{m^{2-4\epsilon}} \frac{\theta\theta\delta}{2k_1 \cdot k_2 + k_3 \cdot n\ k_{23} \cdot \bar{n}}$$

Results

- S_3^{nnn} the same-hemisphere triple-gluon emission contribution to the N3LO zero-jettiness soft function.
- Reconstructed from the numerical result and verified numerically with more than 2000 digits.
- Singular terms only contain zeta values.
- Regular terms contain multiple polylogarithms G with sixth root of unity $\exp(ik\pi/3)$ letters.

$$
\begin{aligned}
S_3^{nnn} = {} & \frac{24}{\epsilon^5} + \frac{308}{3\epsilon^4} + \frac{1}{\epsilon^3}\left(-12\pi^2 + \frac{3380}{9}\right) + \frac{1}{\epsilon^2}\left(-1000\zeta_3 + \frac{440\pi^2}{9} + \frac{10048}{9}\right) + \frac{1}{\epsilon}\left(-\frac{2377\pi^4}{45} + \frac{440\zeta_3}{3} + \frac{7192\pi^2}{27} + \frac{253252}{81}\right) \\
& + \left(-28064\zeta_5 + \frac{1972\zeta_3\pi^2}{3} - \frac{638\pi^4}{15} + 4224\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 3696\zeta_3\ln(2) - 176\pi^2\ln^2(2) + 176\ln^4(2) + \frac{13208\zeta_3}{3} + \frac{78848\pi^2}{81} + 96\ln(2) + \frac{1925074}{243}\right) \\
& + \epsilon\Bigg(2304\ \zeta_{-5,-1} - 4464\zeta_5\ln(2) + 25784\zeta_3^2 - \frac{67351\pi^6}{567} - 6336G_R(0,0,r_2,1,-1) \\
& - 6336G_R(0,0,1,r_2,-1) - 3168G_R(0,0,1,r_2,r_4) - 6336G_R(0,0,r_2,-1)\ln(2) + \frac{268895\zeta_5}{3} \\
& - 45056\,\mathrm{Li}_5\left(\frac{1}{2}\right) - 45056\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2) + 176\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\pi \\
& + 176\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\pi - 1056\zeta_3\,\mathrm{Li}_2\left(\frac{1}{4}\right) - 3982\zeta_3\pi^2 - 21824\zeta_3\ln^2(2) + 2112\zeta_3\ln(2)\ln(3) - 1584\,\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\ln(3) - \frac{4400\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\pi^3}{27} + \frac{88\pi^4\ln(2)}{45} \\
& - \frac{616\pi^4\ln(3)}{27} + \frac{11264\pi^2\ln^3(2)}{9} - \frac{22528\ln^5(2)}{15} + 8576\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 7504\zeta_3\ln(2) + \frac{4174\pi^4}{27} \\
& - \frac{1072\pi^2\ln^2(2)}{3} + \frac{1072\ln^4(2)}{3} + \frac{554032\zeta_3}{27} - 32\pi^2\ln(2) + \frac{730378\pi^2}{243} - 384\ln^2(2) + 832\ln(2) \\
& + \frac{1408681}{81} + \sqrt{3}\left(192\,\Im\left\{\mathrm{Li}_3\left(\frac{\exp(i\pi/3)}{2}\right)\right\} + 160\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln(2) - 16\pi\ln^2(2) - \frac{560\pi^3}{81}\right)\Bigg) + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

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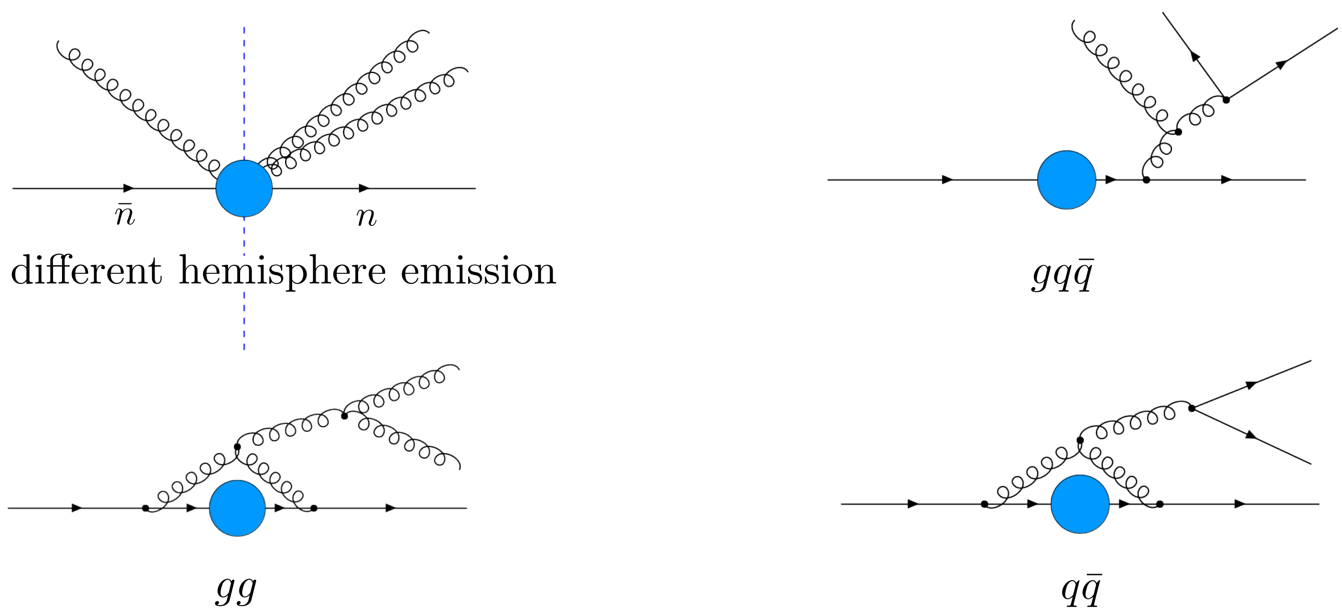
Conclusion

- Custom IBP relations enables reduction for integrals containing Heaviside functions.
- We computed the same-hemisphere triple-gluon zero-jettiness soft function at N3LO.
- Adding an auxiliary mass parameter overcomes the technical difficulty of computing master integrals.
- Outlook: Apply modified IBP to



$\rightarrow$ Other objects containing Heaviside functions

Backup

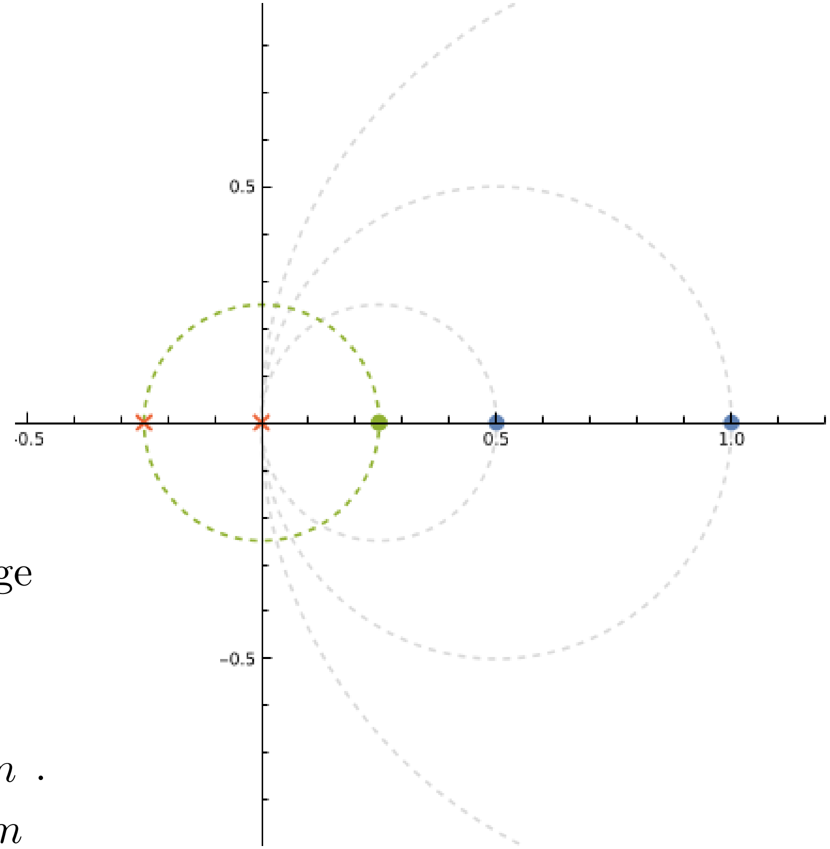
Integral Evaluation

- Expand $\boldsymbol{J}$ around boundary $w^2 = m^{-2} = 0$

$$\boldsymbol{J} = \sum_{i,j,k} \boldsymbol{c}_{ijk}(\epsilon) w^{i+j\epsilon} \ln^k w.$$

- Boundary condition $m \to \infty$ involves several regions as the Heaviside function allows $k_i \cdot \overline{n} = \alpha_i$ to be large

$$\boldsymbol{J}|_{m\to\infty} = \begin{cases} m^0 & \text{with } \alpha_1, \alpha_2, \alpha_3 \ll m \\ m^{-2\epsilon} & \text{with } \alpha_1, \alpha_i \ll m, \text{ while } \alpha_j \sim m \\ m^{-4\epsilon} & \text{with } \alpha_1 \ll m, \text{ while } \alpha_2, \alpha_3 \sim m \end{cases}.$$



- Matching at physical point $m = 0$:

$$\boldsymbol{J} = \sum_{i,j,k} \boldsymbol{c}_{ijk}(\epsilon) m^{i+j\epsilon} \ln^k m$$

- Physical solution $\lim_{m\to 0} \boldsymbol{J}(\epsilon, m) = c_{000}(\epsilon)$

Integral Evaluation

- Expand $\boldsymbol{J}$ around boundary $w^2 = m^{-2} = 0$

$$\boldsymbol{J} = \sum_{i,j,k} \boldsymbol{c}_{ijk}(\epsilon) w^{i+j\epsilon} \ln^k w.$$

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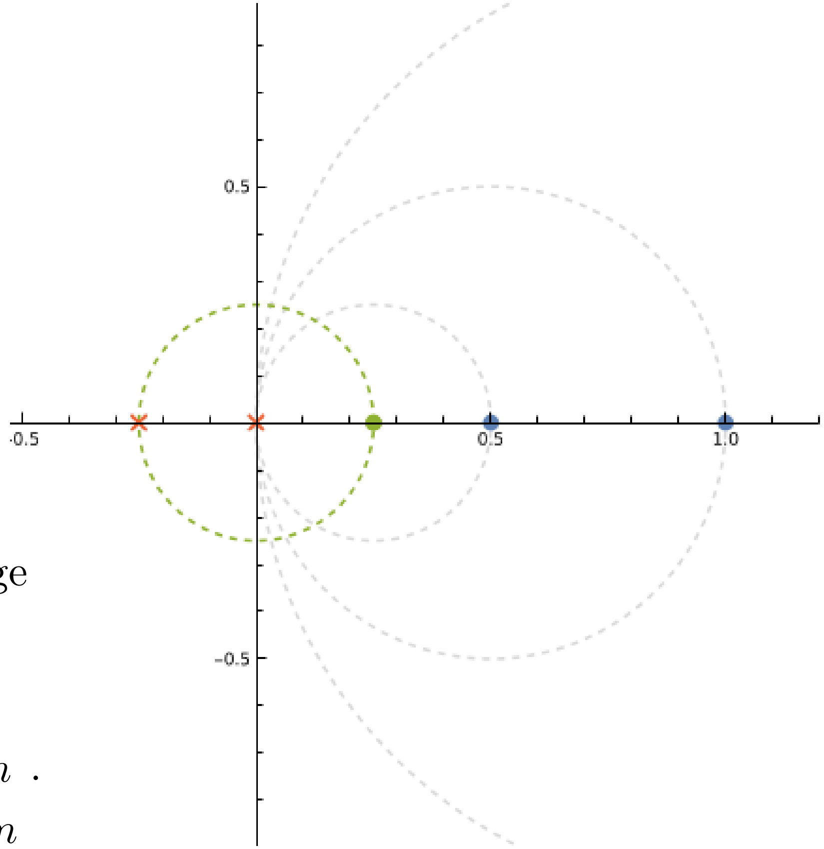
- Expand and evaluate at regular point $J(m' = 1/w_0) = J|_{w=w_0}$

$$\boldsymbol{J} = \sum_i \boldsymbol{c}_i(\epsilon) m'^i.$$

- Repeat this procedure until we move into the radius of convergence around the physical point $m = 0$



Integral Reduction

$$\int \mathrm{d}^d k \frac{\partial}{\partial k_\mu} \left[p_\mu f(k) \right] = 0 \Longrightarrow \int \mathrm{d}^d k \left[\left(\frac{\partial}{\partial k_\mu} p_\mu \right) + p \cdot k \frac{\partial}{\partial k^2} + p \cdot n \frac{\partial}{\partial k \cdot n} + p \cdot \overline{n} \frac{\partial}{\partial k \cdot \overline{n}} \right] f(k) = 0$$

Integral Reduction

$$\int \mathrm{d}^d k \frac{\partial}{\partial k_\mu} \left[p_\mu f(k) \right] = 0 \sim \frac{\partial}{\partial k \cdot \bar{n}} f(k)$$

Integral Reduction

$$\int \mathrm{d}^d k \frac{\partial}{\partial k_\mu} \left[p_\mu f(k) \right] = 0 \sim \frac{\partial}{\partial k \cdot \bar{n}} f(k)$$

- Example: $I\left[a_1, a_2, a_3, \theta\right] = \dfrac{\theta(k \cdot \overline{n} - k \cdot n)}{[k^2]_c^{a_1} [1 - k \cdot n]_c^{a_2} (k \cdot \overline{n})^{a_3}}$

$$\frac{\partial}{\partial k \cdot \overline{n}} I\left[a_1, a_2, a_3, \theta\right] = \underbrace{-a_3 \frac{\theta(k \cdot \overline{n} - k \cdot n)}{[k^2]_c^{a_1} [1 - k \cdot n]_c^{a_2} (k \cdot \overline{n})^{a_3+1}}}_{\text{homogeneous}} + \underbrace{\frac{\delta(k \cdot \overline{n} - k \cdot n)}{[k^2]_c^{a_1} [1 - k \cdot n]_c^{a_2} (k \cdot \overline{n})^{a_3}}.}_{\text{inhomogeneous}}$$

- Homogeneous:

$$\frac{\partial}{\partial k \cdot \overline{n}} I\left[a_1, a_2, a_3, \theta\right] = -a_3 I\left[a_1, a_2, a_3 + 1, \theta\right]$$

Integral Reduction

$$\int \mathrm{d}^d k \frac{\partial}{\partial k_\mu} \left[p_\mu f(k) \right] = 0 \sim \frac{\partial}{\partial k \cdot \overline{n}} f(k)$$

- Example: $I\left[a_1, a_2, a_3, \theta\right] = \dfrac{\theta(k \cdot \overline{n} - k \cdot n)}{[k^2]_c^{a_1} [1 - k \cdot n]_c^{a_2} (k \cdot \overline{n})^{a_3}}$

3 linear dependent Propagators → Partial fraction

$$\frac{\partial}{\partial k \cdot \overline{n}} I\left[a_1, a_2, a_3, \theta\right] = \underbrace{-a_3 \frac{\theta(k \cdot \overline{n} - k \cdot n)}{[k^2]_c^{a_1} [1 - k \cdot n]_c^{a_2} (k \cdot \overline{n})^{a_3+1}}}_{\text{homogeneous}} + \underbrace{\frac{\delta(k \cdot \overline{n} - k \cdot n)}{[k^2]_c^{a_1} [1 - k \cdot n]_c^{a_2} (k \cdot \overline{n})^{a_3}}}_{\text{inhomogeneous}}.$$

- Homogeneous:

$$\frac{\partial}{\partial k \cdot \overline{n}} I\left[a_1, a_2, a_3, \theta\right] = -a_3 I\left[a_1, a_2, a_3 + 1, \theta\right]$$

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Integral Reduction

$$\int \mathrm{d}^d k \frac{\partial}{\partial k_\mu} \left[p_\mu f(k) \right] = 0 \sim \frac{\partial}{\partial k \cdot \overline{n}} f(k)$$

- Example: $I\left[a_1, a_2, a_3, \theta\right] = \dfrac{\theta(k \cdot \overline{n} - k \cdot n)}{[k^2]_c^{a_1} [1 - k \cdot n]_c^{a_2} (k \cdot \overline{n})^{a_3}}$

3 linear dependent Propagators → Partial fraction

$$\frac{\partial}{\partial k \cdot \overline{n}} I\left[a_1, a_2, a_3, \theta\right] = \underbrace{-a_3 \frac{\theta(k \cdot \overline{n} - k \cdot n)}{[k^2]_c^{a_1} [1 - k \cdot n]_c^{a_2} (k \cdot \overline{n})^{a_3+1}}}_{\text{homogeneous}} + \underbrace{\frac{\delta(k \cdot \overline{n} - k \cdot n)}{[k^2]_c^{a_1} [1 - k \cdot n]_c^{a_2} (k \cdot \overline{n})^{a_3}}}_{\text{inhomogeneous}}.$$

- Homogeneous:

$$\frac{\partial}{\partial k \cdot \overline{n}} I\left[a_1, a_2, a_3, \theta\right] = -a_3 I\left[a_1, a_2, a_3 + 1, \theta\right]$$

- Inhomogeneous:

$$\frac{\partial}{\partial k \cdot \overline{n}} I\left[a_1, a_2, a_3, \theta\right] = I\left[a_1, a_2, 1, \delta\right]$$

→ closed form for arbitrary indices **in general not possible** due to partial fractioning **→ slow**

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