Epsilon-factorised differential equations for non-trivial geometries

Stefan Weinzierl

in collaboration with Sebastian Pögel and Xing Wang

February 13, 2023

Section 1

Executive summary

Executive summary

Consider a sequence which starts as

$$\begin{aligned} l &= 0: & 1 \\ l &= 1: & \theta \\ l &= 2: & \theta \cdot \theta \\ l &= 3: & \theta \cdot \theta \cdot \theta \end{aligned}$$

We would like to understand the general term at l loops.

Executive summary

We first compute the $(l = 4)$-term:

$$\begin{aligned} l &= 0: & & 1 \\ l &= 1: & & \theta \\ l &= 2: & & \theta \cdot \theta \\ l &= 3: & & \theta \cdot \theta \cdot \theta \\ l &= 4: & & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \end{aligned}$$

Executive summary

The general term at l loops is given by

$$\theta \cdot \frac{1}{Y_{l-1}} \cdot \theta \cdot \frac{1}{Y_{l-2}} \cdot \theta \cdot \frac{1}{Y_{l-3}} \cdot \ldots \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_1} \cdot \theta$$

and we have

$$Y_1 = 1$$

and the duality

$$Y_j = Y_{l-j}.$$

Executive summary

Up to seven loops we therefore have

$$
\begin{aligned}
l = 0: &\quad 1 \\
l = 1: &\quad \theta \\
l = 2: &\quad \theta \cdot \theta \\
l = 3: &\quad \theta \cdot \theta \cdot \theta \\
l = 4: &\quad \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \\
l = 5: &\quad \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \\
l = 6: &\quad \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \\
l = 7: &\quad \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta
\end{aligned}
$$

Executive summary

- θ is the **Euler operator** $\theta = q\frac{d}{dq}$ in the variable q, the functions Y_j are called Y-invariants.
- $N = \theta^2 \frac{1}{Y_2} \theta \frac{1}{Y_3} \dots \frac{1}{Y_3} \theta \frac{1}{Y_2} \theta^2$ is the **special local normal form** of a Calabi-Yau operator.
- Operators like N are related to **Picard-Fuchs operators** of Calabi-Yau Feynman integrals.
- From the factorisation of N we may construct the ε-factorised differential equation.

Section 2

Calabi-Yau Feynman integrals

Calabi-Yau manifolds

Definition

A Calabi-Yau manifold of complex dimension n is a compact Kähler manifold M with vanishing first Chern class.

Theorem (conjectured by Calabi, proven by Yau)

An equivalent condition is that M has a Kähler metric with vanishing Ricci curvature.

Calabi-Yau manifolds

Remark

Sometimes one defines Calabi-Yau manifolds by the stronger condition that M has a Kähler metric with local holonomy $\mathrm{Hol}(p) = \mathrm{SU}(n)$.

- This implies the previous condition, but the converse is not true.
- This requirement excludes for example Abelian surfaces or Enriques surfaces.
- This condition implies for the Hodge numbers

$$h^{n,0} = 1, \quad \text{and} \quad h^{j,0} = 0, \qquad 0 < j < n.$$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

K3 surface

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

Abelian surface

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Enriques surface

Calabi-Yau manifolds

Theorem

Let X be a hypersurface defined by a homogeneous polynomial P of degree $n+2$ in $\mathbb{CP}^{n+1}$. If X is smooth, than X is a Calabi-Yau n-fold.

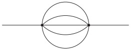
Example

$$P = a_1 a_2 a_3 + (a_1 + a_2 + a_3)(a_1 a_2 + a_2 a_3 + a_3 a_1)\, y$$

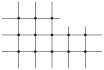
is homogeneous of degree 3 in the variables $[a_1 : a_2 : a_3]$. It defines a hypersurface in $\mathbb{CP}^2$. The hypersurface is smooth for $y \notin \{0, -1, -\frac{1}{9}, \infty\}$. Hence, P defines for generic y a Calabi-Yau one-fold.

Fantastic Beasts and Where to Find Them

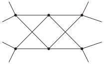
- Bananas



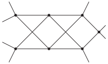
- Fishnets



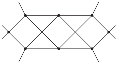
- Amoebas



- Tardigrades



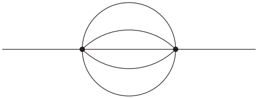
- Paramecia



Aluffi, Marcolli, ’09, Bloch, Kerr, Vanhove, ’14
Bourjaily, McLeod, von Hippel, Wilhelm, ’18
Duhr, Klemm, Loebbert, Nega, Porkert, ’22

Fantastic Beasts and Where to Find Them

The standard example for this talk will be the family of **equal-mass banana integrals**.



These integrals depend on one kinematic variable

$$y = -\frac{m^2}{p^2}.$$

In two space-time dimensions and with unit powers of the propagators the integral is given in the Feynman parameter representation by

$$I_{1...11} = \int d^{l+1}a \, \delta(1 - a_1 - \cdots - a_{l+1}) \, \frac{1}{\mathcal{F}}$$

where $\mathcal{F}$ is the second graph polynomial.

Fantastic Beasts and Where to Find Them

Where is the Calabi-Yau manifold?

For the banana integrals it's very simple: The second graph polynomial $\mathcal{F}$ is homogeneous of degree $(l+1)$ and defines a hypersurface in $\mathbb{CP}^l$. The hypersurface is smooth for generic values of y, hence $\mathcal{F}$ defines a Calabi-Yau $(l-1)$-fold in Feynman parameter space.

Example

The second graph polynomial of the two-loop banana graph (i.e. the sunrise graph) defines for $y \notin \{0, -1, -\frac{1}{9}, \infty\}$ the Calabi-Yau one-fold

$$a_1 a_2 a_3 + (a_1 + a_2 + a_3)(a_1 a_2 + a_2 a_3 + a_3 a_1) y = 0.$$

Section 3

The non-ε-factorised differential equation

Integration-by-parts and differential equations

- The family of equal mass banana integrals has **$(l+1)$ master integrals at l loops**. A possible basis is given by

$$I \quad = \quad \left(I_{1...10}, I_{1...11}, I_{1...12}, \dots, I_{1...1l} \right).$$

  An alternative basis is the derivative basis given by

$$\left(I_{1...10}, \quad I_{1...11}, \quad \frac{d}{dy} I_{1...11}, \quad \dots, \quad \frac{d^{l-1}}{dy^{l-1}} I_{1...11} \right).$$

- We start from a non-ε-factorised differential equation

$$dI \quad = \quad AI.$$

  In principle this differential equation can be obtained by using **only linear algebra**.

- Standard integration-by-parts reduction programs work efficiently for $l \leq 5$.

Integration-by-parts and differential equations

- The l-loop banana integral has $(l+1)$ propagators, i.e. the number grows linearly with l.
- IBP-reduction programs work with an auxiliary graph, where every scalar product involving a loop momentum is expressible as a linear combination of inverse propagators. The auxiliary graph has

$$N_V = \frac{1}{2} l (l+3)$$

propagators, this number **grows quadratically** with l:

l	1	2	3	4	5	6	7
N_V	2	5	9	14	20	27	35

The Bessel representation

- The integral $I_{1\dots 11}$ has the integral representation

$$I_{1\dots 11} \quad = \quad e^{l\varepsilon\gamma_E} 2^{l(1-\varepsilon)} y^{-\frac{\varepsilon}{2}} \int\limits_0^\infty dt\, t^{1+l\varepsilon} J_{-\varepsilon}\left(\frac{t}{\sqrt{y}}\right) \left[K_{-\varepsilon}(t)\right]^{l+1}$$

$J_\nu(z)$: Bessel function of the first kind,
$K_\nu(z)$: modified Bessel function of the second kind.

Berends, Buza, Böhm, Scharf, '94

- We may derive the differential equation from this representation.
Vanhove, '14,
Bönisch, Duhr, Fischbach, Klemm, Nega, '21

- This allows us to obtain the differential equation up to l=15 loops **in a few seconds**.

Singularities

Apart from the points 0 and ∞ the possible **singularities of the differential equation** are obtained by considering all sign choices of

$$p^2 = (m \pm m \pm \cdots \pm m)^2$$

with $(l+1)$ summands inside the bracket.

The set $S^{(l)}$ encodes these singularities and is given by squares of odd or even numbers:

l	$S^{(l)}$
0	$\{1\}$
1	$\{4\}$
2	$\{1,9\}$
3	$\{4,16\}$
4	$\{1,9,25\}$
5	$\{4,16,36\}$

The non-ε-factorised differential equation

We therefore obtain alternatively:

- A system of $(l+1)$ first-order differential equations

$$dI = AI.$$

- A homogeneous differential equation of order $(l+1)$ for $I_{1...11}$.
- An inhomogeneous differential equation of order l for $I_{1...11}$.

$$L^{(l)} I_{1...11} = (-1)^l \frac{(l+1)!}{y^{l-1} \prod\limits_{a \in S^{(l)}} (1+ay)} \varepsilon^l I_{1...10}$$

We call $L^{(l)}$ the **Picard-Fuchs operator** of $I_{1...11}$. The ε-dependence of $L^{(l)}$ is polynomial. We denote by $L^{(l,0)}$ the ε^0-part of $L^{(l)}$.

Section 4

Calabi-Yau operators

Essentially self-adjoint operators

- The adjoint operator L^* of an operator L is defined to be

$$L = \sum_{j=0}^{l} r_j(y) \frac{d^j}{dy^j} \quad \Rightarrow \quad L^* = \sum_{j=0}^{l} (-1)^{l-j} \frac{d^j}{dy^j} r_j(y)$$

- An operator L is called self-adjoint, if $L^* = L$.
- An operator L is called **essentially self-adjoint** or **self-dual**, if there exists a function $\alpha(y)$ such that

$$\alpha L^* = L \alpha.$$

Fact

The Picard-Fuchs operator $L^{(l,0)}$ is essentially self-adjoint.

The Frobenius method

- Consider a homogeneous linear differential equation of order l

$$L\psi = 0.$$

- The point $y = y_0$ is called a point of **maximal unipotent monodromy** if the indicial equation for the operator L at this point is of the form $(\rho - \rho_0)^l = 0$.
- If $y = 0$ is a point of maximal unipotent monodromy we may write the l independent solutions $\psi_0, \dots, \psi_{l-1}$ as

$$\psi_k = \frac{1}{(2\pi i)^k} \sum_{j=0}^{k} \frac{\ln^j y}{j!} \sum_{n=0}^{\infty} a_{k-j,n} y^{n+\rho_0}, \qquad a_{0,0} = 1$$

Fact

$y = 0$ is a point of maximal unipotent monodromy for the Picard-Fuchs operator $L^{(l,0)}$ with local exponent $\rho_0 = 1$.

The mirror map

- The holomorphic solution ψ_0 and the single-logarithmic solution ψ_1 are used to define a change of variables from y to τ (or q):

$$\tau = \frac{\psi_1}{\psi_0}, \qquad q = e^{2\pi i \tau}.$$

- In the context of Calabi-Yau manifolds the map from y to τ is called the **mirror map**.

Candelas, De La Ossa, Green, Parkes, '91

- In the special case of $l = 2$ the map corresponds to the transformation from y to the modular parameter τ of an elliptic curve.

The Y-invariants

- Define recursively **operators N_j** by

$$N_0 = 1, \qquad N_{j+1} = y \frac{d}{dy} \frac{1}{(2\pi i)^j N_j(\psi_j)} N_j$$

The operators N_j have the property that

$$N_j(\psi_i) = 0 \text{ for } i < j.$$

- **Structure series** $(\alpha_1, \alpha_2, \dots, \alpha_{l-1})$:

$$\alpha_j = \frac{1}{(2\pi i)^j} \frac{1}{N_j(\psi_j)}.$$

- ***Y*-invariants**:

$$Y_j = \frac{\alpha_1}{\alpha_j}, \qquad j \in \{1, \dots, l-1\}$$

Remark: From the definition it follows immediately that $Y_1 = 1$.

Integral power series

Definition

A power series

$$\sum_{n=0}^{\infty} a_n y^n$$

is called N-integral, if the substitution $y = N y'$ leads to a power series in the new variable y' with integer coefficients.

Calabi-Yau operators

A differential operator is called a Calabi-Yau operator if

1. L is self-dual.
2. The point $y = 0$ is a point of maximal unipotent monodromy and the local exponent at y is an integer.
3. The holomorphic solution ψ_0 as a power series in y is N-integral.
4. The variable q as a power series in y is N-integral.
5. All functions $(\alpha_1, \alpha_2, \dots, \alpha_{l-1})$ as power series in y are N-integral.

M. Bogner, '13

Fact

The Picard-Fuchs operator $L^{(l,0)}$ is a Calabi-Yau operator.

The special local normal form

- The differential operator L can be written in the q-coordinate as

$$L = \beta\theta \frac{1}{Y_{l-1}} \theta \frac{1}{Y_{l-2}} \theta \frac{1}{Y_{l-3}} \dots \frac{1}{Y_3} \theta \frac{1}{Y_2} \theta \frac{1}{Y_1} \theta \frac{1}{\psi_0}$$

  where β is a function of q.

- With $Y_1 = 1$ and $Y_j = Y_{l-j}$ this simplifies to

$$L = \beta\theta^2 \frac{1}{Y_2} \theta \frac{1}{Y_3} \dots \frac{1}{Y_3} \theta \frac{1}{Y_2} \theta^2 \frac{1}{\psi_0}.$$

- The operator

$$N(L) = \theta^2 \frac{1}{Y_2} \theta \frac{1}{Y_3} \dots \frac{1}{Y_3} \theta \frac{1}{Y_2} \theta^2$$

  is called the **special local normal form** of the operator L.

M. Bogner, '13

Section 5

The ansatz

Iterated integrals

Recall that $q = \exp(2\pi i \tau)$. We define **iterated integrals** by

$$I(f_1, f_2, ..., f_n; \tau) = \lim_{q_0 \to 0} R \left[\int\limits_{q_0}^{q} \frac{dq_1}{q_1} \int\limits_{q_0}^{q_1} \frac{dq_2}{q_2} ... \int\limits_{q_0}^{q_{n-1}} \frac{dq_n}{q_n} \, f_1(\tau_1) f_2(\tau_2) ... f_n(\tau_n) \right]$$

R regularises trailing zeros (removes all $\ln(q_0)$-terms).

Differentiation chops off the first function:

$$\frac{1}{2\pi i} \frac{d}{d\tau} I(f_1, f_2, ..., f_n; \tau) \;=\; q \frac{d}{dq} I(f_1, f_2, ..., f_n; \tau) \;=\; f_1(\tau) \, I(f_2, ..., f_n; \tau)$$

The ansatz

- We set $D = 2 - 2\varepsilon$.
- Instead of y we work with the variable τ (or q).
- We now **construct master integrals**

$$M = \left(M_0, M_1, \ldots, M_l\right)^T,$$

  which put the differential equation into an ε-factorised form.
- M_0 is proportional to the l-loop tadpole integral:

$$M_0 = \varepsilon^l I_{1 \ldots 10}.$$

The ansatz

- $I_{1\dots11}$ has Picard-Fuchs operator $L^{(l)}$, the ε^0-part $L^{(l,0)}$ is of the form

$$L^{(l,0)} = \beta\theta^2 \frac{1}{Y_{l-2}} \theta \frac{1}{Y_{l-3}} \dots \frac{1}{Y_3} \theta \frac{1}{Y_2} \theta^2 \frac{1}{\psi_0}$$

- M_1 should start at order ε^l.
- $L^{(l,0)}$ **annihilates** $I_{1\dots11}$ modulo ε and modulo tadpoles.
- This suggests

$$M_1 = \frac{\varepsilon^l}{\psi_0} I_{1\dots11}.$$

The ansatz

- We construct a derivative basis. The **factorisation** of $L^{(l,0)}$ in the variable q suggests for the master integrals $M_2 - M_l$

$$M_j = \frac{\mathbf{1}}{\mathbf{Y_{j-1}}} \left[\frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} M_{j-1} + \text{junk} \right],$$

- **Griffiths transversality**:

$$M_j = \frac{1}{Y_{j-1}} \left[\frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} M_{j-1} - \sum_{k=1}^{j-1} \mathbf{F_{(j-1)k} M_k} \right],$$

with a priori unkown but ε-independent functions $F_{ij}(\tau)$.

Summary of the ansatz

$$
\begin{aligned}
M_0 &= \varepsilon^l I_{1\dots10} \\
M_1 &= \frac{\varepsilon^l}{\psi_0} I_{1\dots11} \\
M_j &= \frac{1}{Y_{j-1}} \left[\frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} M_{j-1} - \sum_{k=1}^{j-1} F_{(j-1)k} M_k \right] \quad \text{for } j \geq 2
\end{aligned}
$$

The differential equation

The ansatz leads to the differential equation

$$\frac{1}{2\pi i\varepsilon}\frac{d}{d\tau}M = \varepsilon\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & F_{11} & 1 & 0 & 0 & & 0 & 0 \\ 0 & F_{21} & F_{22} & Y_2 & 0 & & 0 & 0 \\ 0 & F_{31} & F_{32} & F_{33} & Y_3 & & 0 & 0 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & F_{(l-2)1} & F_{(l-2)2} & F_{(l-2)3} & F_{(l-2)4} & \dots & Y_{l-2} & 0 \\ 0 & F_{(l-1)1} & F_{(l-1)2} & F_{(l-1)3} & F_{(l-1)4} & \dots & F_{(l-1)(l-1)} & 1 \\ * & * & * & * & * & \dots & * & * \end{pmatrix} M.$$

- The first l rows are in an ε-factorised form.
- Determine the functions F_{ij} such that the $(l+1)$-th row is in ε-factorised form.

The differential equation

The condition that in the $(l+1)$-th row only terms of order ε^1 are present leads to

- differential equations
- **algebraic equations** from self-duality

$$\frac{1}{2\pi i \varepsilon}\frac{d}{d\tau}M = \varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & F_{11} & 1 & 0 & 0 & & 0 & 0 \\ 0 & F_{21} & F_{22} & Y_2 & 0 & & 0 & 0 \\ 0 & F_{31} & F_{32} & F_{33} & Y_3 & & 0 & 0 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & F_{(l-2)1} & F_{(l-2)2} & F_{(l-2)3} & F_{(l-2)4} & \dots & Y_{l-2} & 0 \\ 0 & F_{(l-1)1} & F_{(l-1)2} & F_{(l-1)3} & F_{(l-1)4} & \dots & F_{(l-1)(l-1)} & 1 \\ * & * & * & * & * & \dots & * & * \end{pmatrix} M$$

The differential equation

- The equations for F_{ij}'s have a natural triangular structure and can be solved systematically.
- We arrive at the **differential equation in ε-factorised form**:

$$dM = \varepsilon A M$$

Section 6

Boundary values

Boundary values

- In addition to the differential equation in ε-factorised form we need boundary values at a specific kinematic point.
- As boundary point we choose $y = 0$ corresponding to $\tau = i\infty$ or $q = 0$.
- With the help of **Mellin-Barnes** one obtains

$$M_1|_{y \to 0} \quad = \quad e^{l \varepsilon \gamma_E} \left(l+1\right) \sum_{j=0}^{l} \binom{l}{j} \left(-1\right)^j y^{j\varepsilon} \frac{\Gamma\left(1+\varepsilon\right)^{l-j} \Gamma\left(1-\varepsilon\right)^{1+j} \Gamma\left(1+j\varepsilon\right)}{\Gamma\left(1-\left(j+1\right)\varepsilon\right)}$$

- The boundary values are multiple zeta values and of weight n at order ε^n.

Remark

- For Feynman integrals evaluating to multiple polylogarithms we may define a transcendental weight and have the notion of master integrals of **uniform weight**.
- If we try to extend this concept beyond multiple polylogarithms, a reasonable minimal assumption is that the definition of transcendental weight in the more general case should be compatible with the restriction of the kinematic space to a sub-space.
- Having an ε-factorised differential alone is not enough to guarantee uniform weight, the boundary constants have to be of uniform weight as well.

Sideremark

- Specialise to $l = 2$: We know about two bases, which put the differential equation into an ε-factorised form:
  - The basis (M_0, M_1, M_2) constructed in this talk.
  - A second basis constructed from the requirement that the period matrix on the maximal cut is proportional to the identity matrix.
- The latter basis does not have boundary constants of uniform weight.

Frellesvig, S.W., '23

Section 7

Results

Results: Six loops

The ε-expansion of the master integral M_1 starts at order ε^6:

$$M_1 = \varepsilon^6 M_1^{(6,6)} + \varepsilon^7 M_1^{(6,7)} + \mathcal{O}\left(\varepsilon^8\right)$$

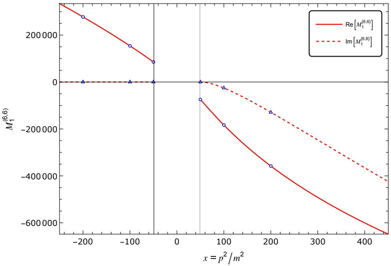
The first term in the ε-expansion is given by

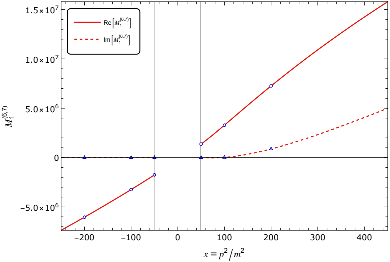
$$M_1^{(6,6)} = 1120\zeta_3^2 - 2016\zeta_5 L_q - 3360\zeta_3 I(1, Y_2, Y_3) + I(1, Y_2, Y_3, Y_2, 1, f_{7,0})$$

The first few terms of the q-expansion of $M_1^{(6,6)}$ read ($L_q = ln(q)$)

$$\begin{aligned} M_1^{(6,6)} = \; & 1120\zeta_3^2 - 560\zeta_3 L_q^3 - 2016\zeta_5 L_q + 7L_q^6 + 210q(-32\zeta_3 + 48\zeta_3 L_q - 3L_q^4 + 8L_q^3) \\ & + \frac{105}{2} q^2 \left(208\zeta_3 - 1392\zeta_3 L_q + 87L_q^4 - 52L_q^3 - 180L_q^2 - 72L_q + 192\right) \\ & + \mathcal{O}\left(q^3\right) \end{aligned}$$

Results: Six loops





Expansion around $y = 0$ converges at six loops for $|p^2| > 49m^2$.
Agrees with results from `pySecDec`.

The geometry of this Feynman integral is a **Calabi-Yau five-fold**.

Conclusions

- There are (many) Feynman integrals related to Calabi-Yau geometries.
- The l-loop equal-mass banana integral corresponds to a Calabi-Yau $(l-1)$-fold.
- We have shown that an ε-factorised differential equation with boundary values of uniform weight exists for the family of equal-mass banana integrals.
- We expect this to hold for other Calabi-Yau Feynman integrals as well.
- We profited from research in mathematics on Calabi-Yau operators.