Integration-By-Parts reduction using syzygies

Mathematical Structures in Feynman Integrals (2023)

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Introduction

- Method of syzygies has become an important tool for Feynman integral reduction for multi-loop multi-leg scattering amplitudes.
- Facilitated calculation of many challenging processes
- Will try to answer the following:

1. What are syzygies?
2. Why do we need syzygies?
3. How to construct them? Examples.

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Integration-By-Parts Reduction

General scalar Feynman integral with L-loops and N-edges (propagators in integral family) :

$$I(\nu_1, \ldots, \nu_N) = \int d^D k_1 \ldots d^D k_L \, \Pi_{i=1}^N \frac{1}{\left(q_i^2 - m_i^2 + i\epsilon\right)^{\nu_i}}$$

$k_1, \ldots k_L$: Loop momenta

q_i : Momentum of edge i

m_i : Mass of edge i

ν_i : Exponent of edge i

Integration-by-parts identity [Chetyrkin & Tkachov (1981)] :

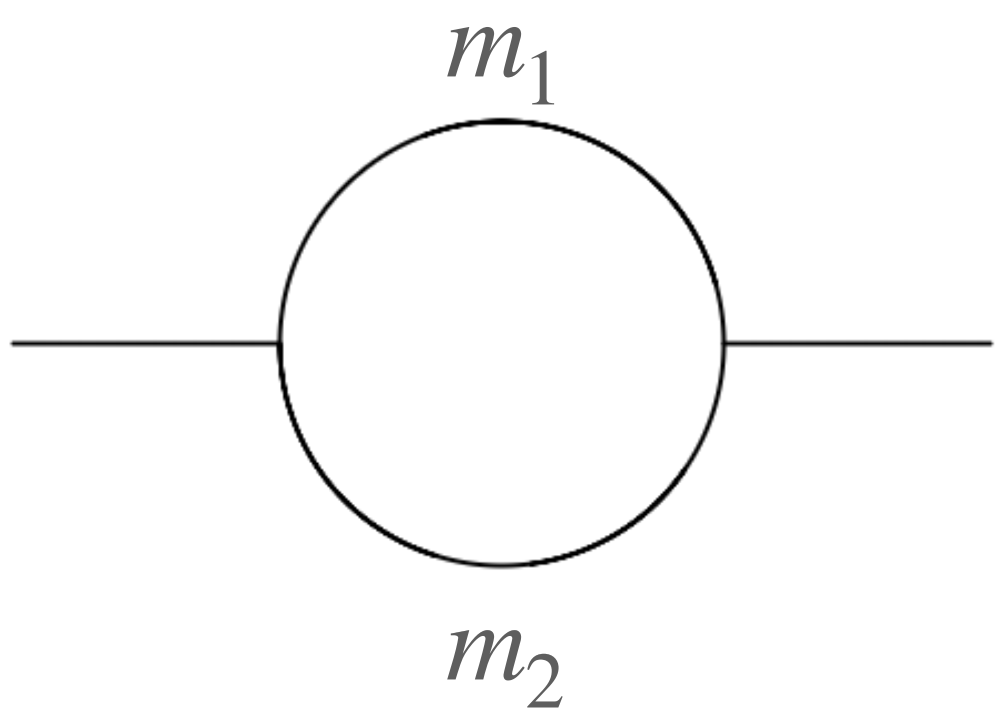
$$0 = \int d^D k_1 \ldots d^D k_L \, \frac{\partial}{\partial k_\mu} v_\mu \left(\Pi_{i=1}^N \frac{1}{\left(q_i^2 - m_i^2 + i\epsilon\right)^{\nu_i}} \right)$$

v_μ is in general a linear combination of loop and external momenta.

Generate a linear system of equations and systematically reduce using Laporta's algorithm [Laporta (2000)] to arrive at a set of basis integrals.

Integration-By-Parts Reduction

Example : 1-loop bubble



$$= \int d^D k \, \frac{1}{\left(k^2 - m_1^2 + i\epsilon\right)\left((k+p)^2 - m_2^2 + i\epsilon\right)} = I(1,1)$$

IBP identities:

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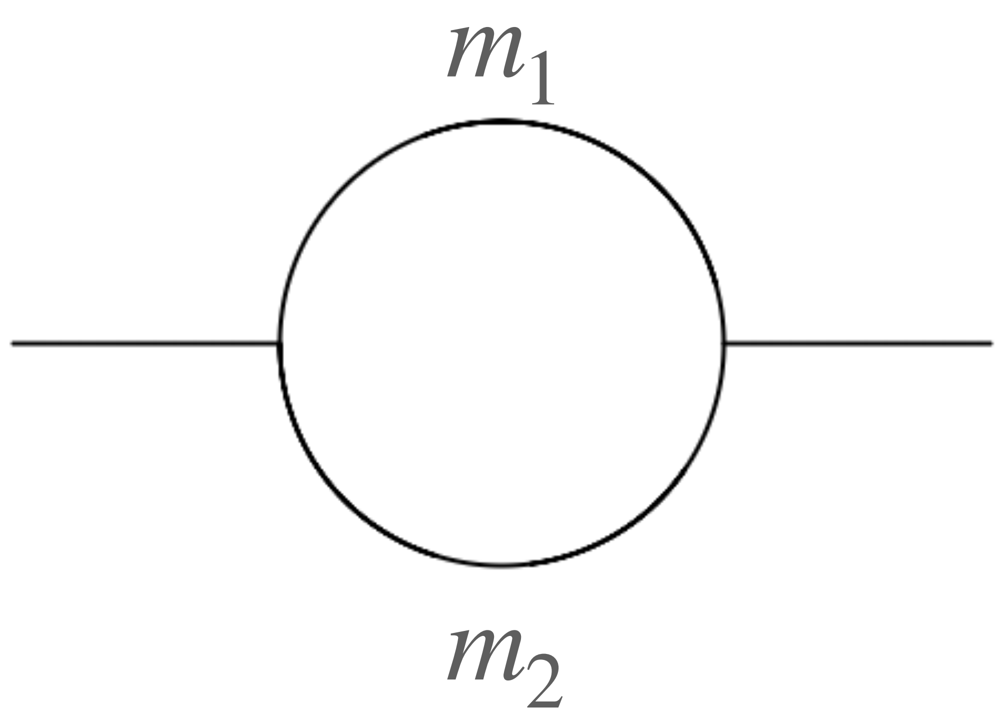

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What to choose for v_μ ?

Integration-By-Parts Reduction

Traditionally, two simple choices for $v_\mu : k_\mu \, , p_\mu$

$$0 = \int d^D k_1 \ldots d^D k_L \, \frac{\partial}{\partial k_\mu} k_\mu \left(\frac{1}{\left(k^2 - m_1^2 + i\epsilon\right) \, \left((k+p)^2 - m_2^2 + i\epsilon\right)} \right)$$

$$= (D-3)\, I(1,1) \; - \; 2m_1^2 \, I(2,1) \; - \; (2m_2^2 - p^2)\, I(1,2) \; - \; I(0,2)$$

$$0 = \int d^D k_1 \ldots d^D k_L \, \frac{\partial}{\partial k_\mu} p_\mu \left(\frac{1}{\left(k^2 - m_1^2 + i\epsilon\right) \, \left((k+p)^2 - m_2^2 + i\epsilon\right)} \right)$$

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Terms with higher powers of propagators

$$0 = \int d^D k_1 \ldots d^D k_L \, \frac{\partial}{\partial k_\mu} p_\mu \left(\frac{1}{\left(k^2 - m_1^2 + i\epsilon\right) \, \left((k+p)^2 - m_2^2 + i\epsilon\right)} \right)$$

$$= p^2 \boxed{I(2,1)} - \; I(2,0) \; + \; I(0,2) \; - \; p^2 \boxed{I(1,2)}$$

Integration-By-Parts Reduction

It is desirable to avoid such doubled propagators i.e. propagators with exponents higher than 1:

- Such integrals do not frequently appear in scattering amplitudes, hence reductions are not required
- IBP systems become much larger in size e.g. $\sim 10^8$ equations for $gg \to ZZ$

How to avoid these doubled propagators? Multiple approaches:

- Suitable choices for generating vectors v_μ using Gröbner bases [Gluza, Kajda, Kosower (2010)]
- Linear algebra based approach [Schabinger (2011)]
- Differential geometry [Zhang (2014)]

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Spoiler: This reduces to a much smaller number

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Integration-By-Parts Reduction

General scalar Feynman integral in Baikov representation [Baikov (1996)] :

$$I(\nu_1, \ldots, \nu_N) = \mathcal{N}_0 \int dz_1 \ldots dz_N \left(\prod_{i=1}^{N} \frac{1}{z_i^{\nu_i}} \right) P^{\frac{D-L-E-1}{2}}$$

z_i : Baikov variable for edge i

P : Baikov polynomial

E : Independent external momenta

IBPs in Baikov representation:

$$0 = \int \left(\prod_i^L dz_i \right) \sum_i^N \left(\frac{\partial f}{\partial z_i} + \frac{d-L-E-1}{2P} f_i \frac{\partial P}{\partial z_i} - \frac{\nu_i f_i}{z_i} \right) P^{(d-L-E-1)/2}$$

f_i are arbitrary polynomials in z_i and kinematics (similar to v_μ)

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Doubled propagators

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Dimension shifting term (NEW) Doubled propagators

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Syzygies

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Syzygy constraint

Polynomials f_i and g solving the above constraint form a "Syzygy" module.

Note that P and $\partial_{z_i} P$ are themselves polynomials in z_i and kinematics.

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Syzygies

A syzygy is simply a relation or constraint between certain polynomials.

Some properties of syzygies:

1. If $s = \{f_i, g\}$ is a solution to the above constraint, then $p \cdot s = \{p\, f_i, p\, g\}$ for any polynomial p in z_i and kinematics is also a solution.
2. If s_1 and s_2 are two solutions, any combination $p_1 \cdot s_1 + p_2 \cdot s_2$ with p_1, p_2 being polynomials in z_i and kinematics is also a solution.

E.g. $\mathrm{syz}(x + y, x^2 + y^2, x^3 + y^3) = \{\{y^2, -x, 1\}, \{x^2 + y^2, -x + y, 0\}\}$

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Syzygy constraint

- Explicit solutions known and straightforward to write [Boehm, Georgoudis, Larsen, Schulze, Zhang (2017)] [Abreu, Cordero, Ita, Page, Zeng (2017)]
- Polynomials of degree 1 in Baikov parameters z_i and kinematics
- Generate exactly the IBPs obtained in the momentum space representation

Syzygies

E.g. no dimension-shift syzygies for a $gg \to ZZ$ integral family:

M_{no-dim}=

$\{\{-mt2 - z_2 + z_3 - z_5 + z_8, - z_2 + z_3, - z_2 + z_3, - s - z_2 + z_3, 0,0,0,0, - mz2 + t - z_2 + z_3, 0\},$

$\{mt2 + z_2 - z_4 + z_6 - z_8, z_2 - z_4, s + z_2 - z_4, z_2 - z_4, 0,0,0,0, mz2 - t + z_2 - z_4, 0\},$

$\{mt2 + z_3 - z_5 + z_7 - z_9, mt2 + t + z_3 - z_9, mt2 + mz2 + z_3 - z_9, mt2 + mz2 - s + z_3 - z_9, 0,0,0,0, mt2 - mz2 + z_3 - z_9, 0\}$

$,\{mt2 + z_1 + z_2 - z_8, 2(mt2 + z_2), 2mt2 + z_2 + z_3, 2mt2 + z_2 + z_4, 0,0,0,0, mt2 - t + z_2 + z_9, - 2\},$

….

Syzygies

Require no dimension-shifting terms and no terms introducing doubled propagators → simultaneous solution to both constraints.

- No double propagator term:

$$\nu_i \frac{f_i}{z_i} \quad \Rightarrow \quad f_i \sim z_i$$

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Trivial to write down solutions to the above constraint. Simply $f_i = z_i$.

Note that this is required only for the propagators where we wish to avoid higher exponents. For the rest, $f_i = 1$ suffices.

Still highly non-trivial to solve the constraints simultaneously.

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Constructing Syzygies

Solving both constraints simultaneously; approach one:

- Compute module intersection of the two syzygy modules [Gluza, Kajda, Kosower (2010)] [Zhang (2014)] [Larsen, Zhang (2015)] [Boehm, Georgoudis, Larsen, Schoenemann, Zhang (2018)]
- Use of computer algebra packages such as **Singular** for this task.
- Computation of complete syzygy module often not feasible; impose a degree bound on the polynomials ⇒ not necessarily a problem.

See talks by Janko Böhm and Yang Zhang for more details.

Constructing Syzygies

Solving both constraints simultaneously; approach two:

- Linear algebra based approach [Schabinger (2011)] [Agarwal, Jones, von Manteuffel (2020)]
- Map the problem of module intersection to row reduction of a matrix; construct linear combinations of no dimension-shift syzygies that also satisfy the no doubled propagator constraint
- Use finite field methods [von Manteuffel, Schabinger (2014)] [Peraro (2016)] to speed up the linear algebra
- Solutions produced up to a requested degree in z_i
- Can run in a highly distributed manner

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Brief description of the algorithm:

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   Try to find linear combinations satisfying no doubled propagator constraint.

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And repeat to required degree

Note that there is a “trivial” solution: $(z_2 z_3) \cdot M_{no-dim}$

Constructing Syzygies

Implementation using Linear algebra:

- Convert the system to a matrix
  - Each polynomial $z_i z_j M_{no-dim,k}$ corresponding to a row of the matrix
  - Each monomial $e_i \sum z_j^{\nu_j}$ corresponding to a column; e_i is the ith basis vector of the module e.g.

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  - Order the columns such that "inadmissible" terms such as $e_3 z_1 z_5$ are to the left of "admissible" terms

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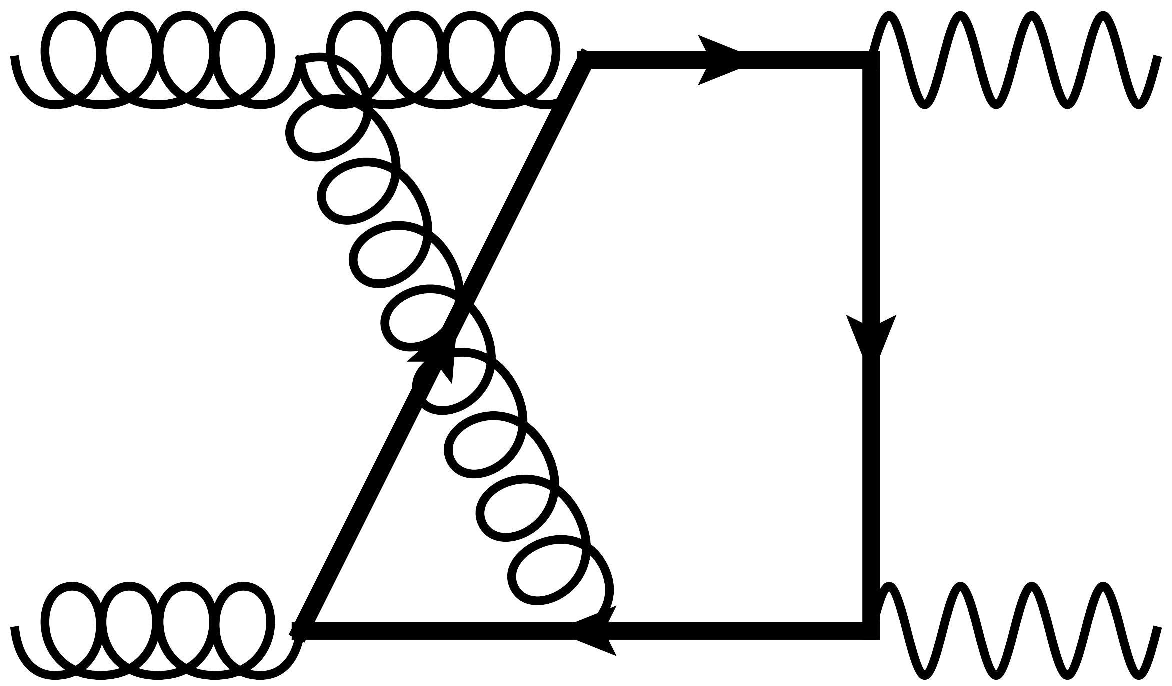
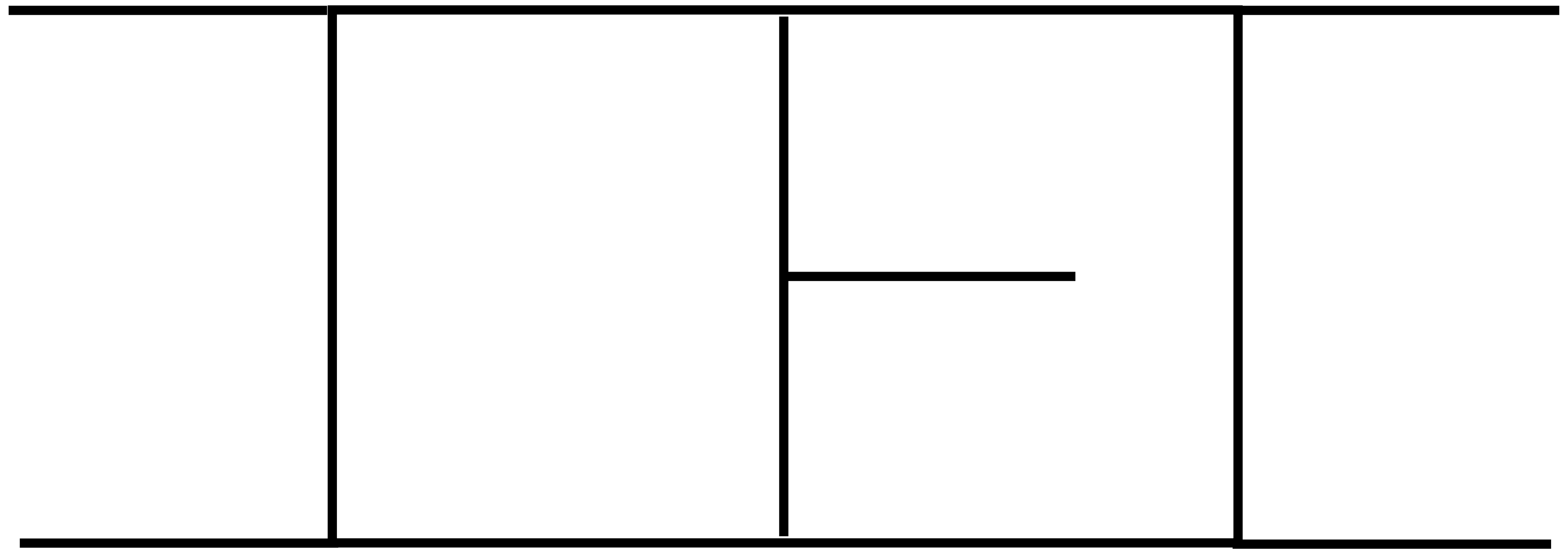
$$e_3$$

  - Order the columns such that "inadmissible" terms such as $e_3 z_1 z_5$ are to the left of "admissible" terms
  - Use Gaussian elimination to eliminate these inadmissible terms

Constructing Syzygies

Examples of successful use:





- Total size of syzygies $\sim 2GB$
- Largest syzygy $\sim 230MB$ (max degree 5)
- Up to $s = 4$ integrals
- 2 scales s, t $(m_t, m_Z$ set to numbers)
- Extremely complicated due to internal masses

- Total size of syzygies $\sim 1GB$
- Largest syzygy $\sim 40MB$ (max degree 4)
- Up to $s = 5$ integrals
- 4 scales $s_{23}, s_{34}, s_{45}, s_{51}$ $(s_{12} = 1)$

Advantages and limitations

Advantages:

- Linear systems generated from syzygies are much smaller than those from conventional approaches. E.g. for $gg \to ZZ$, only ~$3 \cdot 10^5$ equations compared to 10^8 before
- Made several previously inaccessible processes possible

Limitations:

- Generating syzygies a very complicated task in general. A complete calculation of the intersection between the syzygy modules would be ideal, however this has been found to be unfeasible in most cases
- The degree bound used to generate the syzygies in either approach implies incomplete solution at times; in many cases syzygies with high degree bounds are needed which are very challenging
- Even with syzygies, analytic solution of the linear system not feasible in more complicated cases

Summary

- Syzygies are a powerful tool in modern loop amplitude calculations
- Construct much smaller linear systems compared to traditional IBPs. In essence, "partially pre-solve" the system to altogether avoid the appearance of unnecessary terms/equations
- No-numerator syzygies to construct equations without any irreducible scalar products [Lee (2013)] [Bitoun, Bogner, Klausen, Panzer (2017)] [von Manteuffel, Schabinger (2019)]; useful for reduction of systems with high number of dots
- Still unanswered questions; more work needed