Elliptic substructure of 2-loop kites and 3-loop tadpoles

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Abstract: The generic **2-loop kite** integral has **5** internal masses. Its completion by a **sixth** propagator gives a **3-loop tadpole** whose substructure involves **12 elliptic curves**. I shall show how to compute all such kites and their tadpoles, with 200 digit precision achieved in seconds, thanks to the procedure of the **arithmetic geometric mean** for **complete elliptic integrals of the third kind**. The number theory of 3-loop tadpoles poses challenges for packages such as `HyperInt`.

Extending old-fashioned sage advice, from Gabriel Barton, 54 years ago:

GB1: If you know the **discontinuity** σ of f, use a **dispersion relation** to get f.

GB2: If you know only its **derivative** σ', integrate that against a **log**.

GB3: For the 2-loop **photon** propagator, σ' has logs, so f has **tri-logs**.

GB4: For the 2-loop **electron** propagator, σ' is **elliptic**, so f is harder to compute.

DB5: To determine a 3-loop **tadpole**, integrate an **elliptic** σ' against a **dilog**.

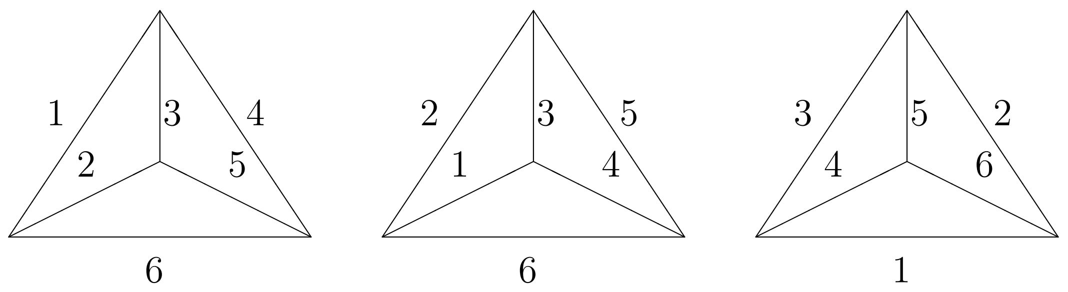
I define the 2-loop **scalar kite** integral in 4-dimensional Minkowski space as

$$I(q^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) = -\frac{q^2}{\pi^4} \int \mathrm{d}^4 l \int \mathrm{d}^4 k \prod_{j=1}^{5} \frac{1}{p_j^2 - m_j^2 - \mathrm{i}\epsilon}, \tag{1}$$

$$(p_1,\, p_2,\, p_3,\, p_4,\, p_5) = (l,\, l-q,\, l-k,\, k,\, k-q). \tag{2}$$

Suppressing masses, $I(s)$ has a **cut** $s \in [s_\mathrm{L}, \infty]$ and a leftmost **branch point** s_L that is the lowest of the **thresholds** $\{s_{1,2}, s_{4,5}, s_{2,3,4}, s_{1,3,5}\}$, where $s_{j,k} = (m_j + m_k)^2$ and $s_{i,j,k} = (m_i + m_j + m_k)^2$. On the top lip of the cut, let $\Im I(s + \mathrm{i}\epsilon) = \pi\sigma(s)$. It suffices to know the **derivative** $\sigma'(s)$, which gives the **dispersion relation**

$$I(q^2) = -\int_{s_\mathrm{L}}^{\infty} \mathrm{d}s\, \sigma'(s) \log\left(1 - \frac{q^2}{s}\right). \tag{3}$$



Consider the logarithmically divergent tetrahedral **tadpole** formed by joining the external vertices of the kite with an inverse propagator $q^2 - m_6^2 - \mathrm{i}\epsilon$. Regularization in $4 - 2\epsilon$ dimensions gives a tadpole

$$T^{5,4,6}_{1,2,3} = \left(\frac{1}{3\epsilon} + 1\right) 6\zeta_3 + 3\zeta_4 - F^{5,4,6}_{1,2,3} + O(\epsilon), \tag{4}$$

$$F^{5,4,6}_{1,2,3} = \int_{s_{\mathrm{L}}}^{\infty} \mathrm{d}s\, \sigma'(s; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) \left(\mathrm{Li}_2\left(1 - \frac{m_6^2}{s}\right) + \frac{1}{2}\log^2\left(\frac{\overline{m}^2}{s}\right)\right) \tag{5}$$

where $\overline{m}$ is the scale of **dimensional regularization** and the **dilogarithm** is the analytic continuation of the sum $\mathrm{Li}_2(z) = \sum_{n>0} z^n/n^2$, valid for $|z| < 1$. The superscripts in (5) are made **redundant** by the labelling convention: a superscript j sits above a subscript k if and only if j is congruent to k modulo 3. Subscripts identity triangles; superscripts identify vertices. Referring back to the figure, we can detect **12 elliptic curves** inherent in the tertrahedron.

Non-elliptic non-anomalous contributions: In the absence of anomalous thresholds, the non-elliptic contribution to $\sigma'(s)$ may be obtained by diligent algebra and application of Cauchy's integral theorem. Here I give a **compact** 5-parameter result.

Denote the **square root** of the symmetric **Källén function** by

$$\Delta(a, b, c) = \sqrt{a^2 + b^2 + c^2 - 2(ab + bc + ca)} \tag{6}$$

with convenient **abbreviations** $\Delta_{j,k}(s) = \Delta(s, m_j^2, m_k^2)$ and $\Delta_{i,j,k} = \Delta_{j,k}(m_i^2)$. Then

$$D_{j,k}(s) = \frac{r}{s-(m_j-m_k)^2}\log\left(\frac{1+r}{1-r}\right), \quad r = \left(\frac{s-(m_j-m_k)^2}{s-(m_j+m_k)^2}\right)^{1/2} \tag{7}$$

is analytic in the s-plane with a cut $s \in [(m_j+m_k)^2, \infty]$, where its **discontinuity** is $D_{j,k}(s+\mathrm{i}\epsilon) - D_{j,k}(s-\mathrm{i}\epsilon) = -2\pi\mathrm{i}/\Delta_{j,k}(s)$. Next, define the real constants

$$\alpha = \frac{(m_1^2-m_4^2)(m_2^2-m_5^2)}{m_3^2} - m_3^2, \quad \beta = \frac{(m_1^2m_5^2-m_2^2m_4^2)(m_1^2-m_2^2-m_4^2+m_5^2)}{m_3^2} \tag{8}$$

which help to **condense** the 5-parameter result. The (possibly complex) constants

$$s_\pm = \frac{m_1^2+m_2^2-2m_3^2+m_4^2+m_5^2-\alpha}{2} \pm \frac{\Delta_{1,3,4}\Delta_{2,3,5}}{2m_3^2} \tag{9}$$

locate **leading Landau singularities** of triangles that form the kite. With Heaviside steps denoted by Θ, the non-elliptic contribution is

$$\sigma_\mathrm{N}'(s) = \Theta(s-s_{1,2})\sigma_{1,2}'(s) + \Theta(s-s_{4,5})\sigma_{4,5}'(s). \tag{10}$$

Putting edges 1 and 2 on-shell, I obtain

$$\Delta_{1,2}(s)\sigma'_{1,2}(s) = \Re\left((s+\alpha)D_{4,5}(s) + L_{4,5} + \sum_{i=0,+,-} C_i \frac{D_{4,5}(s) - D_{4,5}(s_i)}{s - s_i}\right), \tag{11}$$

$$C_\pm = \alpha s_\pm + \beta,\ C_0 = -(m_1^2 - m_2^2)(m_4^2 - m_5^2),\ s_0 = 0,\ L_{4,5} = \log\left(\frac{m_4 m_5}{m_3^2}\right). \tag{12}$$

To obtain $\sigma'_{4,5}$, exchange (m_1, m_2) and (m_4, m_5), noting that this does not alter the coefficients C_i or the arguments s_i.

The result in (10,11) holds if both of the conditions

$$(m_1 + m_2)(m_3^2 + m_1 m_2) \ \geq\ m_1 m_5^2 + m_2 m_4^2 \tag{13}$$

$$(m_4 + m_5)(m_3^2 + m_4 m_5) \ \geq\ m_5 m_1^2 + m_4 m_2^2 \tag{14}$$

are satisfied. However, these are **not** necessary conditions. The **final** decision as to whether the non-elliptic contribution is in need of an **anomalous** correction is arbitrated by the **elliptic** contribution, which is subject to no uncertainty.

Elliptic contribution: This comes from **3-particle** intermediate states, giving

$$\sigma'_{\rm E}(s) = \Theta(s - s_{2,3,4})\sigma'_{2,3,4}(s) + \Theta(s - s_{1,3,5})\sigma'_{1,3,5}(s). \tag{15}$$

It contains **complete** elliptic integrals of the **third kind**, which I shall divide by complete integrals of the first kind. For real $k^2 < 1$, let

$$P(n,k) = \frac{\Pi(n,k)}{\Pi(0,k)}, \quad \Pi(n,k) = \int_0^{\pi/2} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \tag{16}$$

where $\Pi(0,k) = (\pi/2)/\mathrm{AGM}\,(1,\sqrt{1-k^2})$ is determined by the arithmetic-geometric mean of Gauss. Then $P(n,k)$ is analytic in the n-plane with a cut $n \in [1,\infty]$ on which its principal value is $1 - P(k^2/n,k)$.

With $s = w^2$, an integration over the phase space of particles 2, 3 and 4 determines

$$k^2 = 1 - \frac{16 m_2 m_3 m_4 w}{W}, \quad W = (w_+^2 - m_+^2)(w_-^2 - m_-^2) \tag{17}$$

with $w_\pm = w \pm m_2$ and $m_\pm = m_3 \pm m_4$. Then I obtain

$$\sigma'_{2,3,4}(w^2) = \frac{4\pi m_3 m_4}{\mathrm{AGM}\,(\sqrt{16 m_2 m_3 m_4 w},\sqrt{W})} \Re\left(\sum_{i=+,-} E_i \frac{P(n_i,k) - P(n_1,k)}{t_i - t_1}\right) \tag{18}$$

with coefficients and arguments given, as **compactly** as possible, by

$$E_\pm = \frac{m_2^2 - m_3^2 + m_5^2}{2m_5^2} \pm \left(\frac{m_4^2 - m_5^2 - w^2}{2m_5^2}\right) \frac{\Delta_{2,3,5}}{\Delta_{4,5}(w^2)}, \tag{19}$$

$$t_\pm = \frac{\gamma \pm \Delta_{2,3,5}\Delta_{4,5}(w^2)}{2m_5^2}, \quad t_1 = m_1^2, \quad n_i = \frac{(w_-^2 - m_+^2)(t_i - m_-^2)}{(w_-^2 - m_-^2)(t_i - m_+^2)}, \tag{20}$$

$$\gamma = (m_2^2 + m_3^2 + m_4^2 - m_5^2 + w^2)m_5^2 + (m_2^2 - m_3^2)(m_4^2 - w^2). \tag{21}$$

An **AGM procedure** speedily evaluates $P(n,k) = \Pi(n,k)/\Pi(0,k)$ to high precision:

1. **Initialize** $[a,\, b,\, p,\, q] = [1,\, \sqrt{1-k^2},\, \sqrt{1-n},\, n/(2-2n)]$. Then set $f = 1 + q$.
2. Set $m = ab$ and then $r = p^2 + m$. Compute a vector of **new values** as follows: $[(a+b)/2,\, \sqrt{m},\, r/(2p),\, (r-2m)q/(2r)]$. Then replace $[a,\, b,\, p,\, q]$ by those new values. Then add q to f.
3. If $|q/f|$ is sufficiently **small**, then return $P(n,k) = f$, else go to step 2.

This converges **very** quickly, for $n \notin [1,\infty]$. On the cut with $n \geq 1$, replace n by $n' = k^2/n < 1$, to obtain the **principal value** $\Re P(n,k) = 1 - P(n',k)$.

Criterion for an anomalous contribution: If there is an anomalous contribution, it occurs above the higher of the two-particle thresholds. Without loss of generality, suppose that $s_{4,5} \geq s_{1,2}$. Then

$$\sigma'(s) = \sigma'_{\rm N}(s) + \sigma'_{\rm E}(s) + C_{\rm A}\frac{\Theta(s - s_{4,5})}{\Delta_{4,5}(s)}\,\Re\left(\frac{2\pi{\rm i}\Delta_{4,5}(s_-)}{s - s_-}\right) \tag{22}$$

with $C_{\rm A} \neq 0$ **if and only if** $(m_1{+}m_2)(m_3^2{+}m_1m_2) < m_1m_5^2{+}m_2m_4^2$ **and** at least one of $\Delta_{1,3,4}$ and $\Delta_{2,3,5}$ is imaginary, in which case $C_{\rm A} = \pm 1$ is the sign of the imaginary part of $\Delta_{4,5}(s_-)$.

This value of $C_{\rm A}$ is required by the **elliptic** contribution at high energy. With $L_k = m_k^2 \log(s/m_k^2)$, the large-$s$ behaviour

$$s^2\sigma'(s) = 2L_3 + \sum_{k=1,2,4,5}(L_k + m_k^2) \,+\, O\left(\frac{\log(s)}{s}\right) \tag{23}$$

invariably holds. The elliptic contribution $\sigma'_{\rm E}$ in (22) is oblivious to the anomalous threshold problem. Its high-energy behaviour determines $C_{\rm A}$, ensuring (23).

Zero-mass limits: As $m_3 \to 0$ with $m_1 \neq m_4$ and $m_2 \neq m_5$,

$$\Delta_{1,2}(s)\sigma'_{1,2}(s) \to \Re\left((2s - s_3)D_{4,5}(s) + \widehat{L}_{4,5} + \sum_{i=0,3} C_i \, \frac{D_{4,5}(s) - D_{4,5}(s_i)}{s - s_i}\right), \tag{24}$$

$$s_3 = -\frac{(m_1^2 m_5^2 - m_2^2 m_4^2)(m_1^2 - m_2^2 - m_4^2 + m_5^2)}{M}, \quad M = (m_1^2 - m_4^2)(m_2^2 - m_5^2), \tag{25}$$

$$\widehat{L}_{4,5} = \log\left(\frac{m_4^2 m_5^2}{M}\right), \quad C_3 = -\left(\frac{m_1^2}{u} - m_2^2 u\right)\left(\frac{m_4^2}{u} - m_5^2 u\right), \quad u = \frac{m_1^2 - m_4^2}{m_2^2 - m_5^2}. \tag{26}$$

As $m_3 \to 0$ with $m_1 = m_4$ and $m_2 \neq m_5$

$$\Delta_{2,4}(s)\sigma'_{1,2}(s) \to \Re\Bigg((3s - m_2^2 - 2m_4^2 - m_5^2)D_{4,5}(s) + \log\left(\frac{m_4 m_5^3}{(m_2^2 - m_5^2)^2}\right) + (m_2^2 - m_4^2)(m_4^2 - m_5^2)\,\frac{D_{4,5}(s) - D_{4,5}(0)}{s}\Bigg). \tag{27}$$

The degenerate case with $m_1 = m_4$ and $m_2 = m_5$ will be considered after adding contributions from three-particle cuts.

As $m_3 \to 0$, the three-particle cuts yield logarithms:

$$\sigma'_{2,3,4}(w^2) \to \Re\left(\sum_{i=\pm} E_i \, \frac{\widehat{P}_{2,4}(t_i, w) - \widehat{P}_{2,4}(m_1^2, w)}{t_i - m_1^2}\right), \tag{28}$$

$$\widehat{P}_{j,k}(t, w) = \frac{(m_k^2 - t)v(t)}{(w - m_j)^2 - t} \log\left(\frac{v(t) + v(m_k^2)}{v(t) - v(m_k^2)}\right), \; v(t) = \left(\frac{(w - m_j)^2 - t}{(w + m_j)^2 - t}\right)^{1/2}. \tag{29}$$

With $m_1 = m_4$ and $m_2 = m_5$ all four thresholds collide as $m_3 \to 0$, giving [DB1990]

$$\sigma'(s) \to \Theta(s - s_{4,5}) \, \frac{2\mu(y_4) + 2\mu(y_5) - 8\mu(y_4 y_5)}{\Delta_{4,5}(s)}, \tag{30}$$

$$\mu(y) = \log|1 - y| + \frac{y \log|y|}{1 - y}, \quad y_k = \frac{-2m_k^2}{s - m_4^2 - m_5^2 + \Delta_{4,5}(s)}. \tag{31}$$

Next, consider cases with $m_3 > 0$ and one of the other masses vanishing. Without loss of generality, take it to be m_4. As $m_4 \to 0$, logarithms from (29) appear in

$$\sigma'_{2,3,4}(w^2) \to \Re\left(\sum_{i=\pm} E_i \, \frac{\widehat{P}_{2,3}(t_i, w) - \widehat{P}_{2,3}(m_1^2, w)}{t_i - m_1^2}\right). \tag{32}$$

The logarithms for two-particle cuts are modified, as $m_4 \to 0$, to give

$$\Delta_{1,2}(s)\sigma'_{1,2}(s) \to \Re\left((s+\alpha)\widehat{D}_5(s) + \widehat{L}_5 + \sum_{i=0,+,-} C_i\,\frac{\widehat{D}_5(s)-\widehat{D}_5(s_i)}{s-s_i}\right), \tag{33}$$

$$\widehat{D}_5(s) = \frac{1}{s-m_5^2}\log\left(1-\frac{s}{m_5^2}\right), \quad \widehat{L}_5 = \log\left(\frac{m_5^2}{m_3^2}\right). \tag{34}$$

An elliptic contribution persists if two **non-adjacent** edges have vanishing mass. As $m_1 \to 0$ and $m_5 \to 0$,

$$(w^2-m_4^2)\sigma'_{2,3,4}(w^2) \;\to\; -\frac{4\pi m_3 m_4 \Re R(w^2,m_2^2,m_3^2,m_4^2)}{\mathrm{AGM}\left(\sqrt{16 m_2 m_3 m_4 w},\sqrt{W}\right)}, \tag{35}$$

$$R(s,b,c,d) = P(\widehat{n},k) - \rho P(n_0,k) + (\rho-1)P(n_3,k), \tag{36}$$

$$\widehat{n} = \frac{w_-^2-m_+^2}{w_-^2-m_-^2}, \quad \frac{n_0}{\widehat{n}} = \frac{m_-^2}{m_+^2}, \quad \frac{n_3}{\widehat{n}} = \frac{t_3-m_-^2}{t_3-m_+^2}, \quad t_3 = \frac{(bd-cs)(b-c+d-s)}{(b-c)(d-s)}, \tag{37}$$

$$\rho = \left(\frac{d-s}{b-c+d-s}\right)\left(\frac{(b+c)(d-s)+(b-c)(b+d)}{bd-cs}\right), \tag{38}$$

$$R(s,c,c,d) = 2P(\widehat{n},k) - 2P(n_0,k), \quad R(s,d,d,d) = \frac{s-9d}{6d}, \tag{39}$$

with a **rational** result for R in the QED case $m_2 = m_3 = m_4$ [DB1990].

Tadpoles and number theory

The **rescaling** $m_k \to \kappa\, m_k$ gives $F \to F + 12\zeta_3 \log(\kappa)$ for the **finite** part F.
To standardize, I set $\overline{m} = \max(m_k) = 1$.

I define a tetrahedral tadpole to be **perfect** if and only if the Källén function **vanishes** at each of its 4 vertices, thereby avoiding all resolutions of square roots. Promoting the subscripts and superscripts of F to arguments that denote the 6 masses, I define the two-parameter **family of perfect tadpoles**:

$$\widehat{F}(x,y) = F^{(1-y,1-x,|x-y|)}_{(x,y,1)} = \widehat{F}(y,x) = \widehat{F}(1-x,1-y) \tag{40}$$

with symmetries restricting distinct cases to $x \ge y \ge 1-x \ge 0$ and hence $x \in [\frac{1}{2}, 1]$.
In [DB1999] I identified **tetralogarithms** in two perfect **binary** tadpoles, obtaining

$$\widehat{F}(1,0) = F^{(1,1,0)}_{(1,1,0)} = 17\zeta_4 + 16U_{3,1}, \quad \widehat{F}(1,1) = F^{(0,0,0)}_{(1,1,1)} = 12\zeta_4, \tag{41}$$

$$U_{3,1} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \tfrac{1}{2}\zeta_4 + \tfrac{1}{2}\zeta_2 \log^2(2) - \tfrac{1}{12}\log^4(2) - 2\,\mathrm{Li}_4(\tfrac{1}{2}). \tag{42}$$

Fast elliptic determination of a perfect tadpole

Now consider the elliptic route to evaluating $\widehat{F}(\frac{1}{2},\frac{1}{2})$. With $(m_3,m_6)=(1,0)$ and $m_1=m_2=m_4=m_5=\frac{1}{2}$, I obtained

$$\widehat{F}(\tfrac{1}{2},\tfrac{1}{2}) = \tfrac{1}{2}\int_1^\infty \mathrm{d}s\,(\widehat{\sigma}'_{\mathrm{N}}(s)+\widehat{\sigma}'_{\mathrm{E}}(s))\log^2(s), \tag{43}$$

$$w^2\widehat{\sigma}'_{\mathrm{N}}(w^2) = \Theta(w-1)\left(2\log\left(\frac{r+1}{r-1}\right)-4r\log(2)\right),\ r=\frac{w}{\sqrt{w^2-1}}, \tag{44}$$

$$w^2\widehat{\sigma}'_{\mathrm{E}}(w^2) = \frac{4\pi(1-P(n,k))\Theta(w-2)}{\mathrm{AGM}(2\sqrt{w},(w-1)\sqrt{w^2+2w})},\ n=\frac{w^2-2w}{(w-1)^2},\ \frac{k^2}{n}=\frac{(w+1)^2}{w^2+2w} \tag{45}$$

and readily discovered a **new** reduction of a perfect tadpole to tetralogarithms

$$\widehat{F}(\tfrac{1}{2},\tfrac{1}{2}) = 30\zeta_3\log(2)-16\zeta_4-32U_{3,1}. \tag{46}$$

Relations between tadpoles

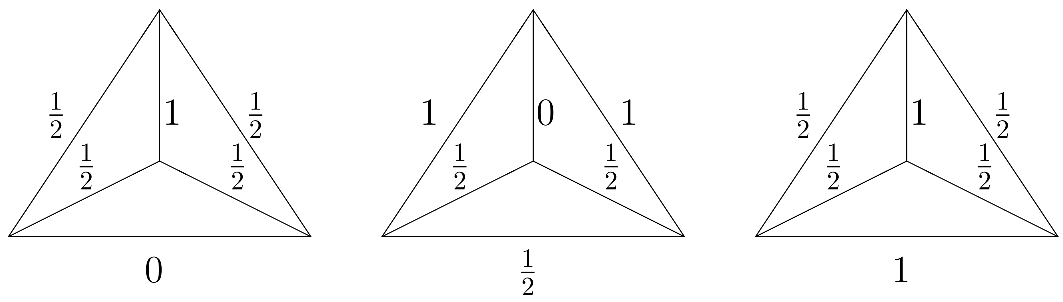


Figure 2: The perfect tadpoles $\widehat{F}(\frac{1}{2},\frac{1}{2})$, $\widehat{F}(1,\frac{1}{2})$ and $\widehat{G}(\frac{1}{2})$ in relation (48)

In addition to the two-parameter family $\widehat{F}(x,y)$ in (40) there is a one-parameter family $\widehat{G}(x) = F^{(x,1-x,1)}_{(x,1-x,1)}$ of perfect tadpoles, with $x \in [0,\frac{1}{2}]$ and $\widehat{G}(0) = 17\zeta_4 + 16U_{3,1}$. I used the efficient AGM of Gauss to obtain **200 digits** of

$$\widehat{G}(\tfrac{1}{2}) = -\int_1^\infty \mathrm{d}s\,(\widehat{\sigma}'_{\mathrm{N}}(s) + \widehat{\sigma}'_{\mathrm{E}}(s))\,\mathrm{Li}_2(1-s) \tag{47}$$

to which all routes are **elliptic**. This revealed the intriguing **empirical** relation

$$2\widehat{F}(\tfrac{1}{2},\tfrac{1}{2}) + 2\widehat{F}(1,\tfrac{1}{2}) + \widehat{G}(\tfrac{1}{2}) = 42\zeta_4 + 24\zeta_3\log(2). \tag{48}$$

A non-elliptic route to $\widehat{F}(1, \frac{1}{2})$ leads to **multiple polylogarithms** in an alphabet of forms, $\mathrm{d}x/(x-a_i)$, with $a_i \in \{0, 1, -1, -2\}$. Then the linear relation determines

$$\widehat{G}(\tfrac{1}{2}) = 47\zeta_4 + 40U_{3,1} + 4\zeta_3 \log(2) - 12\overline{G} \tag{49}$$

$$\overline{G} = 2\zeta_2 \mathrm{Li}_2(\tfrac{1}{4}) + 3\zeta_2 \log^2(2) + 4 \sum_{m>n>0} \frac{(-1)^m(-\frac{1}{2})^n}{m^2 n^2} \tag{50}$$

with 10000 digits now obtainable in 2 seconds.

Binary tadpoles, with $m_k \in \{0, 1\}$, evaluate to multiple polylogarithms in an alphabet containing **sixth roots** of unity, with $\lambda = (1 + \sqrt{-3})/2$ appearing if three massive edges meet at a vertex, where $\Delta_{i,j,k} = \sqrt{-3}$. For example, with 5 unit edges

$$F^{(1,1,0)}_{(1,1,1)} = \frac{109}{6}\left(\frac{\pi}{3}\right)^4 + 16\Re\left(\frac{\mathrm{Li}_2^2(\lambda)}{6} + \sum_{m>n>0} \frac{\lambda^{3m+2n}}{m^3 n}\right). \tag{51}$$

There are **linear relations** between binary tadpoles, as here:

$$3F^{(1,1,1)}_{(0,0,0)} = F^{(0,0,0)}_{(1,1,1)} + 2F^{(1,0,0)}_{(1,1,0)}, \tag{52}$$

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$$F^{(1,1,1)}_{(1,1,1)} + F^{(1,0,0)}_{(1,0,0)} = F^{(1,1,0)}_{(1,1,0)} + F^{(1,1,1)}_{(0,0,0)}. \tag{54}$$

Number fields of the alphabets of tadpoles

So far, one might **guess** that a tadpole with rational masses evaluates to multiple tetralogarithms in an alphabet whose number field is no larger than the **compositum** of the **quadratic** number fields associated by Gunnar Källén to the vertices of the tetrahedron, namely the field $Q(\Delta_{1,3,4}, \Delta_{2,3,5}, \Delta_{1,2,6}, \Delta_{4,5,6})$.

Yet that is **not** the case. The **imperfect** binary tadpole $F^{(1,0,0)}_{(1,1,0)}$ involves $\Re\mathrm{Li}_2^2(\lambda)$, but the Källén function vanishes at each of its 4 vertices.

Faced with this rather limited, yet potent, evidence, I arrive at three **suggestions**, each too weak to be dignified as a well-tested conjecture.

1. Every tetrahedral tadpole with rational masses reduces to multiple or single tetralogarithms whose alphabet lies in an algebraic number field.
2. If the tadpole is perfect, the alphabet is rational.
3. If the tadpole is imperfect, the alphabetic field may include the Källén field.

Experimentum crucis: I evaluated the totally massive imperfect tadpole $F^{(1,1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ with Källén field $Q(\sqrt{-3})$. Seeking a reduction to multiple tetralogarithms in an alphabet $\{0,1,-1,-2,\lambda\}$, with $\lambda=(1+\sqrt{-3})/2$, I achieved an empirical evaluation

$$F^{(1,1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} = 3\zeta_3\log(2) - 4U_{3,1} + 10\zeta_4 + 10\mathrm{Cl}_2^2(\pi/3) - \tfrac{1}{2}\widehat{G}(\tfrac{1}{2}) \tag{55}$$

with a Clausen value $\mathrm{Cl}_2(\pi/3) = \Im\mathrm{Li}_2(\lambda)$, from the Källén field, and a perfect tadpole $\widehat{G}(\frac{1}{2})$ already evaluated in the rational alphabet $\{0,1,-1,-2\}$. Then, thanks to Gauss, it took less than a minute to validate (55) at 600-digit precision.

Tests and benchmarks for kites and tadpoles

1. Elliptic terms do not depend on the order of phase-space integrations.
2. The derivative of the discontinuity of a kite satisfies the sum rule

$$\int_{s_{\rm L}}^{\infty} {\rm d}s\,\sigma'(s)\log\left(\frac{s}{s_{\rm L}}\right) = 6\zeta_3. \tag{56}$$

3. The high energy behaviour of $s^2\sigma'(s)$ holds irrespective of anomalous thresholds.
4. The same tadpole is obtained by integrating over 6 distinct kites.

These tests were invariably passed, at high precision, in a plethora of cases.

Benchmark 1: A useful benchmark was established by Stefan Bauberger and Manfred Böhm, who gave 6 decimal digits of $B_1 = I(50+\mathrm{i}\epsilon; 1, 2, 3, 4, 5)/50$, with all 4 cuts opened. For B_1, I obtain the value

```
+0.173901219069555460362391997806756419040779085211744093645075
-0.118080028202009293890731446888246675922194086181504660940640*I
```

Benchmark 2: Stephen Martin computed 8 digits of $B_2 = -I(10+\mathrm{i}\epsilon; 1, 3, 5, 2, 4)/10$, in a non-anomalous case with only one open cut. For B_2, I obtain the value

```
+0.718335353533534129653528554796276560425262176802655670356407
+0.390162199972762321424365961074218884677858368327292408622989*I
```

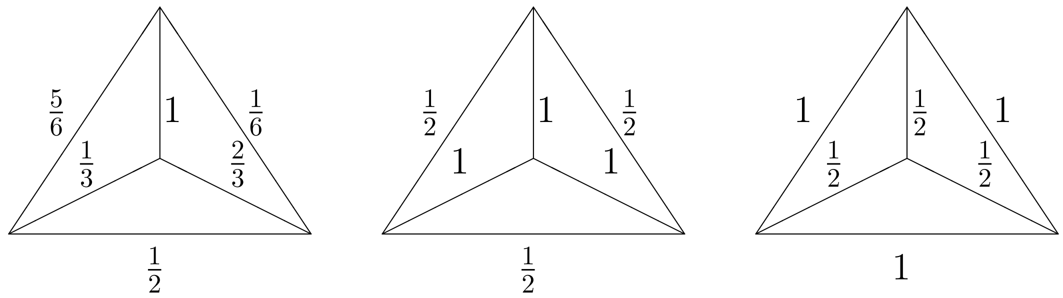

Figure 3: Tadpoles for benchmarks B_3, B_4 and B_5

The benchmarks of Figure 3 are ambitious targets for adept users of `HyperInt`.

Benchmark 3: The first example in Figure 3 is the simplest perfect tadpole with 6 distinct non-zero rational masses. I suggest that its alphabet may be rational. For its finite part $B_3 = \widehat{F}(\frac{5}{6}, \frac{1}{3})$, I obtain

`13.3861455348739022697615450327228552185248654855497464708212`

Benchmark 4: The second example was conjecturally evaluated in the alphabet $\{0, 1, -1, -2, \lambda\}$. The benchmark for $B_4 = F^{(1,1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ is

`16.6059542811980228081648880073141697347243824321176643541089`

Benchmark 5: The third example has two imperfections. I suggest that its alphabetic field may include $Q(\sqrt{-3}, \sqrt{5})$. The benchmark for $B_5 = F^{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}_{(1,1,1)}$ is

`16.5896999071871022548891317280131669711968061643643361121466`

Comments and summary

1. Elliptic substructure of 2-loop kites and 3-loop tadpoles is not a problem. The time taken to evaluate a complete elliptic integral, of whatever kind, is commensurate with the time for a logarithm and less than the time for a dilogarithm. Thanks to Gauss, elliptic integrals should be embraced, not feared.
2. Anomalous terms are not problematic. They submit to Gauss, at high energy.
3. The number theory of tadpoles is subtle. They may be polylogarithmic, even in totally massive cases to which every route is elliptic.
4. I have given far-reaching suggestions on the number theory of tadpoles and benchmarks for users of `HyperInt` to investigate those suggestions analytically.