Computing motives: an approach to irrationality proofs for periods

Daniel Juteau (LAMFA[1]: CNRS[2], UPJV[3])

Mathematical Structures in Feynman Integrals

Siegen, 15.02.2023

mathematics:	following F. Brown (Oxford), C. Dupont (Montpellier)
programming:	joint with C. Dupont (Montpellier), M. Barakat (Siegen) and kind & efficient support from the `CAP`[4] team!

[1]Laboratoire Amiénois de Mathématiques Fondamentales et Appliquées

[2]Centre National de la Recherche Scientifique

[3]Université de Picardie Jules Verne

[4]Categories, Algorithms and Programming (GAP, Julia)

Zeta values

$$\zeta(n) := \sum_{k \geq 1} \frac{1}{k^n} \qquad (n \geq 2)$$

Zeta values

$$\zeta(n) := \sum_{k\geq 1} \frac{1}{k^n} \qquad (n \geq 2)$$

Euler:

$$\sum_{n=0}^{\infty} \zeta(2n) z^{2n} = -\frac{\pi z}{2}\cot(\pi z) = -\frac{1}{2} + \frac{\pi^2}{6} z^2 + \frac{\pi^4}{90} z^4 + \frac{\pi^6}{945} z^6 + \cdots$$

Zeta values

$$\zeta(n) := \sum_{k\geq 1} \frac{1}{k^n} \qquad (n \geq 2)$$

Euler:

$$\sum_{n=0}^{\infty} \zeta(2n) z^{2n} = -\frac{\pi z}{2}\cot(\pi z) = -\frac{1}{2} + \frac{\pi^2}{6} z^2 + \frac{\pi^4}{90} z^4 + \frac{\pi^6}{945} z^6 + \cdots$$

Conjecture

$\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.

Zeta values

$$\zeta(n) := \sum_{k\geq 1} \frac{1}{k^n} \qquad (n \geq 2)$$

Euler:

$$\sum_{n=0}^{\infty} \zeta(2n) z^{2n} = -\frac{\pi z}{2}\cot(\pi z) = -\frac{1}{2} + \frac{\pi^2}{6} z^2 + \frac{\pi^4}{90} z^4 + \frac{\pi^6}{945} z^6 + \cdots$$

Conjecture

$\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.

Known facts

1. π is transcendental (Lindemann 1882)

Zeta values

$$\zeta(n) := \sum_{k \geq 1} \frac{1}{k^n} \qquad (n \geq 2)$$

Euler:

$$\sum_{n=0}^{\infty} \zeta(2n) z^{2n} = -\frac{\pi z}{2} \cot(\pi z) = -\frac{1}{2} + \frac{\pi^2}{6} z^2 + \frac{\pi^4}{90} z^4 + \frac{\pi^6}{945} z^6 + \cdots$$

Conjecture

$\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.

Known facts

1. π is transcendental (Lindemann 1882)
2. $\zeta(3)$ is irrational (Apéry 1978)

Zeta values

$$\zeta(n) := \sum_{k\geq 1} \frac{1}{k^n} \qquad (n \geq 2)$$

Euler:

$$\sum_{n=0}^{\infty} \zeta(2n) z^{2n} = -\frac{\pi z}{2}\cot(\pi z) = -\frac{1}{2} + \frac{\pi^2}{6} z^2 + \frac{\pi^4}{90} z^4 + \frac{\pi^6}{945} z^6 + \cdots$$

Conjecture

$\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.

Known facts

1. π is transcendental (Lindemann 1882)
2. $\zeta(3)$ is irrational (Apéry 1978)
3. $\dim_{\mathbb{Q}} \langle \zeta(3), \zeta(5), \zeta(7), \ldots \rangle_{\mathbb{Q}} = \infty$ (Ball-Rivoal 2000)

Zeta values

$$\zeta(n) := \sum_{k\geq 1} \frac{1}{k^n} \qquad (n \geq 2)$$

Euler:

$$\sum_{n=0}^{\infty} \zeta(2n) z^{2n} = -\frac{\pi z}{2}\cot(\pi z) = -\frac{1}{2} + \frac{\pi^2}{6} z^2 + \frac{\pi^4}{90} z^4 + \frac{\pi^6}{945} z^6 + \cdots$$

Conjecture

$\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.

Known facts

1. π is transcendental (Lindemann 1882)
2. $\zeta(3)$ is irrational (Apéry 1978)
3. $\dim_{\mathbb{Q}} \langle \zeta(3), \zeta(5), \zeta(7), \ldots \rangle_{\mathbb{Q}} = \infty$ (Ball-Rivoal 2000)
4. at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational (Zudilin 2004)

Multiple zeta values

For $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ with all $n_i \geq 1$ and $n_r \geq 2$,

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}}.$$

Its *weight* is $n = n_1 + \cdots + n_r$. MZV's span a $\mathbb{Q}$-algebra $\mathcal{Z}$.

Multiple zeta values

For $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ with all $n_i \geq 1$ and $n_r \geq 2$,

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}}.$$

Its *weight* is $n = n_1 + \cdots + n_r$. MZV's span a $\mathbb{Q}$-algebra $\mathcal{Z}$. Experiments:

n	2	3	4	5	6	7	8	9	10	11	12	13
2^{n-2}	1	2	4	8	16	32	64	128	256	512	1024	2048
d_n^{exp}	1	1	1	2	2	3	4	5	7	9	12	16

Multiple zeta values

For $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ with all $n_i \geq 1$ and $n_r \geq 2$,

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}}.$$

Its *weight* is $n = n_1 + \cdots + n_r$. MZV's span a $\mathbb{Q}$-algebra $\mathcal{Z}$. Experiments:

n	2	3	4	5	6	7	8	9	10	11	12	13
2^{n-2}	1	2	4	8	16	32	64	128	256	512	1024	2048
$d_n^{\exp}$	1	1	1	2	2	3	4	5	7	9	12	16

Conjecture (Zagier)

- If $\mathcal{Z}_n$ is the span of MZV's of weight n, then $\mathcal{Z} = \bigoplus_{n \geq 0} \mathcal{Z}_n$

Multiple zeta values

For $(n_1, \dots, n_r) \in \mathbb{Z}^r$ with all $n_i \geq 1$ and $n_r \geq 2$,

$$\zeta(n_1, \dots, n_r) = \sum_{1 \leq k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}}.$$

Its *weight* is $n = n_1 + \cdots + n_r$. MZV's span a $\mathbb{Q}$-algebra $\mathcal{Z}$. Experiments:

n	2	3	4	5	6	7	8	9	10	11	12	13
2^{n-2}	1	2	4	8	16	32	64	128	256	512	1024	2048
d_n^{exp}	1	1	1	2	2	3	4	5	7	9	12	16

Conjecture (Zagier)

- If $\mathcal{Z}_n$ is the span of MZV's of weight n, then $\mathcal{Z} = \bigoplus_{n \geq 0} \mathcal{Z}_n$
- $\dim_{\mathbb{Q}} \mathcal{Z}_n = d_n$, where $\sum_{n \geq 0} d_n t^n = \frac{1}{1 - t^2 - t^3}$,

  i.e. $d_0 = 1$, $d_1 = 0$, $d_2 = 1$, and $d_n = d_{n-2} + d_{n-3}$ for $n \geq 3$.

Multiple zeta values

For $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ with all $n_i \geq 1$ and $n_r \geq 2$,

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}}.$$

Its *weight* is $n = n_1 + \cdots + n_r$. MZV's span a $\mathbb{Q}$-algebra $\mathcal{Z}$. Experiments:

n	2	3	4	5	6	7	8	9	10	11	12	13
2^{n-2}	1	2	4	8	16	32	64	128	256	512	1024	2048
d_n^{exp}	1	1	1	2	2	3	4	5	7	9	12	16

Conjecture (Zagier)

- If $\mathcal{Z}_n$ is the span of MZV's of weight n, then $\mathcal{Z} = \bigoplus_{n \geq 0} \mathcal{Z}_n$
- $\dim_{\mathbb{Q}} \mathcal{Z}_n = d_n$, where $\sum_{n \geq 0} d_n t^n = \frac{1}{1-t^2-t^3}$,

  i.e. $d_0 = 1$, $d_1 = 0$, $d_2 = 1$, and $d_n = d_{n-2} + d_{n-3}$ for $n \geq 3$.

- Many relations, but graded dimension is predictable.
- Linear (in)dependence is easier than algebraic independence.

Structure of irrationality proofs

Strategy

- Fix MZV's $\zeta_1, \ldots, \zeta_k$. Suppose we have:

Structure of irrationality proofs

Strategy

- Fix MZV's $\zeta_1, \ldots, \zeta_k$. Suppose we have:
  1. for all $n \geq 0$, a non-zero $\mathbb{Q}$-linear combination

$$I_n = a_n^{(1)}\zeta_1 + \cdots + a_n^{(k)}\zeta_k,$$

     where $a_n^{(i)} \in \mathbb{Q}$;

Structure of irrationality proofs

Strategy

- Fix MZV's $\zeta_1, \ldots, \zeta_k$. Suppose we have:
  1. for all $n \geq 0$, a non-zero $\mathbb{Q}$-linear combination

     $$I_n = a_n^{(1)}\zeta_1 + \cdots + a_n^{(k)}\zeta_k,$$

     where $a_n^{(i)} \in \mathbb{Q}$;
  2. a bound on the linear forms I_n, e.g. find a small $\varepsilon > 0$ such that

     $$0 < I_n < \varepsilon^n;$$

Structure of irrationality proofs

Strategy

- Fix MZV's $\zeta_1, \ldots, \zeta_k$. Suppose we have:
  1. for all $n \geq 0$, a non-zero $\mathbb{Q}$-linear combination
  $$I_n = a_n^{(1)}\zeta_1 + \cdots + a_n^{(k)}\zeta_k,$$
  where $a_n^{(i)} \in \mathbb{Q}$;
  2. a bound on the linear forms I_n, e.g. find a small $\varepsilon > 0$ such that
  $$0 < I_n < \varepsilon^n;$$
  3. some control on the coefficients $a_n^{(k)}$, e.g. find r such that
  $$D_n^r a_n^{(k)} \in \mathbb{Z}, \text{ where } D_n = \mathrm{lcm}(1, 2, \ldots, n).$$

Structure of irrationality proofs

Strategy

- Fix MZV's $\zeta_1, \dots, \zeta_k$. Suppose we have:
  1. for all $n \geq 0$, a non-zero $\mathbb{Q}$-linear combination
  $$I_n = a_n^{(1)}\zeta_1 + \cdots + a_n^{(k)}\zeta_k,$$
  where $a_n^{(i)} \in \mathbb{Q}$;
  2. a bound on the linear forms I_n, e.g. find a small $\varepsilon > 0$ such that
  $$0 < I_n < \varepsilon^n;$$
  3. some control on the coefficients $a_n^{(k)}$, e.g. find r such that
  $$D_n^r a_n^{(k)} \in \mathbb{Z}, \text{ where } D_n = \mathrm{lcm}(1, 2, \dots, n).$$

  Note: prime number theorem $\Longrightarrow \lim_{n\to\infty} D_n^{1/n} = e$.

Structure of irrationality proofs

Strategy

- Fix MZV's $\zeta_1, \ldots, \zeta_k$. Suppose we have:
  1. for all $n \geq 0$, a non-zero $\mathbb{Q}$-linear combination
  $$I_n = a_n^{(1)}\zeta_1 + \cdots + a_n^{(k)}\zeta_k,$$
  where $a_n^{(i)} \in \mathbb{Q}$;
  2. a bound on the linear forms I_n, e.g. find a small $\varepsilon > 0$ such that
  $$0 < I_n < \varepsilon^n;$$
  3. some control on the coefficients $a_n^{(k)}$, e.g. find r such that
  $$D_n^r a_n^{(k)} \in \mathbb{Z}, \text{ where } D_n = \operatorname{lcm}(1, 2, \ldots, n).$$
  Note: prime number theorem $\Longrightarrow \lim_{n\to\infty} D_n^{1/n} = e$.
- Now assume $\zeta_1, \ldots, \zeta_k \in \mathbb{Q}$, say in $\frac{1}{q}\mathbb{Z}$.

Structure of irrationality proofs

Strategy

- Fix MZV's $\zeta_1, \ldots, \zeta_k$. Suppose we have:
  1. for all $n \geq 0$, a non-zero $\mathbb{Q}$-linear combination
     $$I_n = a_n^{(1)}\zeta_1 + \cdots + a_n^{(k)}\zeta_k,$$
     where $a_n^{(i)} \in \mathbb{Q}$;
  2. a bound on the linear forms I_n, e.g. find a small $\varepsilon > 0$ such that
     $$0 < I_n < \varepsilon^n;$$
  3. some control on the coefficients $a_n^{(k)}$, e.g. find r such that
     $$D_n^r a_n^{(k)} \in \mathbb{Z}, \text{ where } D_n = \mathrm{lcm}(1, 2, \ldots, n).$$

     Note: prime number theorem $\Longrightarrow \lim_{n\to\infty} D_n^{1/n} = e$.
- Now assume $\zeta_1, \ldots, \zeta_k \in \mathbb{Q}$, say in $\frac{1}{q}\mathbb{Z}$. Then
  $$qD_n^r I_n \geq 1, \text{ hence } e^r \varepsilon \geq 1.$$

Structure of irrationality proofs

Strategy

- Fix MZV's $\zeta_1, \dots, \zeta_k$. Suppose we have:
  1. for all $n \geq 0$, a non-zero $\mathbb{Q}$-linear combination
  $$I_n = a_n^{(1)} \zeta_1 + \cdots + a_n^{(k)} \zeta_k,$$
  where $a_n^{(i)} \in \mathbb{Q}$;
  2. a bound on the linear forms I_n, e.g. find a small $\varepsilon > 0$ such that
  $$0 < I_n < \varepsilon^n;$$
  3. some control on the coefficients $a_n^{(k)}$, e.g. find r such that
  $$D_n^r a_n^{(k)} \in \mathbb{Z}, \text{ where } D_n = \operatorname{lcm}(1, 2, \dots, n).$$
  Note: prime number theorem $\Longrightarrow \lim_{n \to \infty} D_n^{1/n} = e$.
- Now assume $\zeta_1, \dots, \zeta_k \in \mathbb{Q}$, say in $\frac{1}{q}\mathbb{Z}$. Then
$$q D_n^r I_n \geq 1, \text{ hence } e^r \varepsilon \geq 1.$$

Contradiction if r and ε are sufficiently small, so that $e^r \varepsilon < 1$.

Irrationality of $\zeta(3)$

Beuker's integral:

$$\begin{aligned} I_n &= \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-xy)z)^{n+1}} \, dx \, dy \, dz \\ &= a_n \zeta(3) + b_n \end{aligned}$$

with $a_n \in \mathbb{Z}$ and $D_n^3 b_n \in \mathbb{Z}$, bounded by

$$0 < I_n < \varepsilon^n, \qquad \varepsilon = (\sqrt{2}-1)^4.$$

Irrationality of $\zeta(3)$

Beuker's integral:

$$I_n = \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-xy)z)^{n+1}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

$$= a_n \zeta(3) + b_n$$

with $a_n \in \mathbb{Z}$ and $D_n^3 b_n \in \mathbb{Z}$, bounded by

$$0 < I_n < \varepsilon^n, \qquad \varepsilon = (\sqrt{2}-1)^4.$$

Numerical application:

$$e^3 \varepsilon = 0.591 \cdots < 1,$$

Irrationality of $\zeta(3)$

Beuker's integral:

$$I_n = \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-xy)z)^{n+1}} \, dx \, dy \, dz$$

$$= a_n \zeta(3) + b_n$$

with $a_n \in \mathbb{Z}$ and $D_n^3 b_n \in \mathbb{Z}$, bounded by

$$0 < I_n < \varepsilon^n, \qquad \varepsilon = (\sqrt{2} - 1)^4.$$

Numerical application:

$$e^3 \varepsilon = 0.591 \cdots < 1,$$

hence $\zeta(3)$ is irrational!

The moduli space $\mathcal{M}_{0,N}$

$$
\begin{aligned}
\mathcal{M}_{0,N} &= \{\text{curves of genus } 0 \text{ with } N \text{ ordered marked points}\} \\
&= \{N \text{ ordered marked points on } \mathbb{P}^1\}/\operatorname{PGL}_2 \\
&= \{(t_1, \ldots, t_{N-3}) \in \mathbb{A}^{N-3} \mid t_i \neq t_j, t_i \neq 0, 1\}
\end{aligned}
$$

The moduli space $\mathcal{M}_{0,N}$

$$\begin{aligned}\mathcal{M}_{0,N} &= \{\text{curves of genus 0 with } N \text{ ordered marked points}\}\\ &= \{N \text{ ordered marked points on } \mathbb{P}^1\}/\operatorname{PGL}_2\\ &= \{(t_1,\ldots,t_{N-3}) \in \mathbb{A}^{N-3} \mid t_i \neq t_j, t_i \neq 0,1\}\end{aligned}$$

Let $n := N-3$. A connected component of $\mathcal{M}_{0,N}(\mathbb{R})$ is the simplex

$$\delta_n = \{0 < t_1 < \cdots < t_n < 1\} \subset \mathbb{R}^n.$$

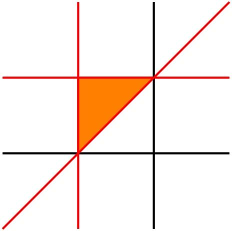
The moduli space $\mathcal{M}_{0,N}$

$$\begin{aligned}
\mathcal{M}_{0,N} &= \{\text{curves of genus 0 with } N \text{ ordered marked points}\}\\
&= \{N \text{ ordered marked points on } \mathbb{P}^1\}/\operatorname{PGL}_2\\
&= \{(t_1,\ldots,t_{N-3}) \in \mathbb{A}^{N-3} \mid t_i \neq t_j, t_i \neq 0,1\}
\end{aligned}$$

Let $n := N-3$. A connected component of $\mathcal{M}_{0,N}(\mathbb{R})$ is the simplex

$$\delta_n = \{0 < t_1 < \cdots < t_n < 1\} \subset \mathbb{R}^n.$$

Example: $N = 5$, $n = 2$



A recipe: periods of moduli spaces $\mathcal{M}_{0,N}$

Examples of period integrals on $\mathcal{M}_{0,N}$:

$$\int_{\delta_n} \prod_i t_i^{a_i} \prod_j (1-t_j)^{b_j} \prod_{i<j} (t_i - t_j)^{c_{i,j}} \, \mathrm{d}t_1 \ldots \mathrm{d}t_n$$

for some $a_i, b_j, c_{i,j} \in \mathbb{Z}$ such that the integral converges.

A recipe: periods of moduli spaces $\mathcal{M}_{0,N}$

Examples of period integrals on $\mathcal{M}_{0,N}$:

$$\int_{\delta_n} \prod_i t_i^{a_i} \prod_j (1 - t_j)^{b_j} \prod_{i<j} (t_i - t_j)^{c_{i,j}} \, \mathrm{d}t_1 \dots \mathrm{d}t_n$$

for some $a_i, b_j, c_{i,j} \in \mathbb{Z}$ such that the integral converges.

Theorem (Brown)

The periods of moduli spaces $\mathcal{M}_{0,N}$ are $\mathbb{Q}[2\pi i]$-linear combinations of multiple zeta values of total weight $\leq n = N - 3$.

A recipe: periods of moduli spaces $\mathcal{M}_{0,N}$

Examples of period integrals on $\mathcal{M}_{0,N}$:

$$\int_{\delta_n} \prod_i t_i^{a_i} \prod_j (1-t_j)^{b_j} \prod_{i<j} (t_i - t_j)^{c_{i,j}} \, \mathrm{d}t_1 \dots \mathrm{d}t_n$$

for some $a_i, b_j, c_{i,j} \in \mathbb{Z}$ such that the integral converges.

Theorem (Brown)

The periods of moduli spaces $\mathcal{M}_{0,N}$ are $\mathbb{Q}[2\pi i]$-linear combinations of multiple zeta values of total weight $\leq n = N-3$.

General recipe for linear forms in MZV's

Consider family of convergent integrals

$$I_{f,\omega}(k) = \int_{\delta_n} f^k \omega$$

where $\omega \in \Omega^n(\mathcal{M}_{0,N}, \mathbb{Q})$ is a regular n-form and $f \in \Omega^0(\mathcal{M}_{0,N}, \mathbb{Q})$.

Vanishing of coefficients

In view of irrationality proofs, we want the additional requirement:

4. Vanishing theorems for some of the coefficients $a_j^{(i)}$.

Vanishing of coefficients

In view of irrationality proofs, we want the additional requirement:

4. Vanishing theorems for some of the coefficients $a_j^{(i)}$.

- Generic period integral on $\mathcal{M}_{0,6}$ gives 1, $\zeta(2)$ and $\zeta(3)$...
  Get rid of $\zeta(2)$! Then we obtain exactly the Apéry sequence.

Vanishing of coefficients

In view of irrationality proofs, we want the additional requirement:

4. Vanishing theorems for some of the coefficients $a_j^{(i)}$.

- Generic period integral on $\mathcal{M}_{0,6}$ gives 1, $\zeta(2)$ and $\zeta(3)$...
  Get rid of $\zeta(2)$! Then we obtain exactly the Apéry sequence.
- Ball-Rivoal: "very well-poised hypergeometric series"
  $\Longrightarrow$ odd zeta values only.

Vanishing of coefficients

In view of irrationality proofs, we want the additional requirement:

4. Vanishing theorems for some of the coefficients $a_j^{(i)}$.

- Generic period integral on $\mathcal{M}_{0,6}$ gives 1, $\zeta(2)$ and $\zeta(3)$...
  Get rid of $\zeta(2)$! Then we obtain exactly the Apéry sequence.
- Ball-Rivoal: "very well-poised hypergeometric series"
  $\Longrightarrow$ odd zeta values only.

In terms of algebraic geometry: consider the (mixed Tate) motive

$$\mathrm{H}_{A,B} := \mathrm{H}^n(\overline{\mathcal{M}}_{0,N} \setminus A, B \setminus A), \text{ where}$$

- $\overline{\mathcal{M}}_{0,N}$ is the Deligne-Mumford compactification;
- A is a divisor where differential forms are allowed to have poles;
- B is a divisor containing the boundary of the domain of integration.

Vanishing of coefficients

In view of irrationality proofs, we want the additional requirement:

4. Vanishing theorems for some of the coefficients $a_j^{(i)}$.

- Generic period integral on $\mathcal{M}_{0,6}$ gives 1, $\zeta(2)$ and $\zeta(3)$...
  Get rid of $\zeta(2)$! Then we obtain exactly the Apéry sequence.
- Ball-Rivoal: "very well-poised hypergeometric series"
  $\Longrightarrow$ odd zeta values only.

In terms of algebraic geometry: consider the (mixed Tate) motive

$$\mathrm{H}_{A,B} := \mathrm{H}^n(\overline{\mathcal{M}}_{0,N} \setminus A, B \setminus A), \text{ where}$$

- $\overline{\mathcal{M}}_{0,N}$ is the Deligne-Mumford compactification;
- A is a divisor where differential forms are allowed to have poles;
- B is a divisor containing the boundary of the domain of integration.

Then $\mathrm{gr}_{2k}^W \mathrm{H}_{A,B} = 0 \Longrightarrow$ vanishing of coefficients $a_j^{(i)}$ in weight k.

Periods and cohomology

For a smooth algebraic variety defined over $\mathbb{Q}$, we have:

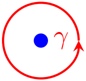
- the Betti cohomology groups (singular cohomology) $\mathsf{H}^k_{\mathrm{B}}(X)$;
- the algebraic de Rham cohomology groups $\mathsf{H}^k_{\mathrm{dR}}(X)$;
- the comparison isomorphism $\mathsf{H}^k_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathsf{H}^k_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$, whose coefficients are *periods*. Equivalently: Betti / de Rham *pairing*.

Periods and cohomology

For a smooth algebraic variety defined over $\mathbb{Q}$, we have:

- the Betti cohomology groups (singular cohomology) $\mathsf{H}^k_{\mathrm{B}}(X)$;
- the algebraic de Rham cohomology groups $\mathsf{H}^k_{\mathrm{dR}}(X)$;
- the comparison isomorphism $\mathsf{H}^k_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathsf{H}^k_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$, whose coefficients are *periods*. Equivalently: Betti / de Rham *pairing*.

$\mathbb{Q}(-1) = \mathsf{H}^1(\mathbb{C}^*)$



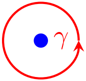
	$\mathrm{d}z/z$
γ	$(2\pi i)$

Periods and cohomology

For a smooth algebraic variety defined over $\mathbb{Q}$, we have:

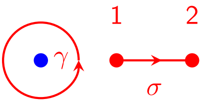
- the Betti cohomology groups (singular cohomology) $\mathrm{H}^k_{\mathrm{B}}(X)$;
- the algebraic de Rham cohomology groups $\mathrm{H}^k_{\mathrm{dR}}(X)$;
- the comparison isomorphism $\mathrm{H}^k_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathrm{H}^k_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$, whose coefficients are *periods*. Equivalently: Betti / de Rham *pairing*.

$\mathbb{Q}(-1) = \mathrm{H}^1(\mathbb{C}^*)$



$$\begin{array}{cc} & dz/z \\ \gamma & (2\pi i) \end{array}$$

$K_2 = \mathrm{H}^1(\mathbb{C}^*, \{1,2\})$



$$\begin{array}{cc} & \begin{array}{cc} dz & dz/z \end{array} \\ \begin{array}{c} \sigma \\ \gamma \end{array} & \begin{pmatrix} 1 & \log 2 \\ 0 & 2\pi i \end{pmatrix} \end{array}$$

$0 \to \mathbb{Q}(0) \to K_2 \to \mathbb{Q}(-1) \to 0,$ “ramified at 2”

Mixed Tate motives over $\mathbb{Q}$, over $\mathbb{Z}$ ("non-ramified")

Category $MTM(\mathbb{Q})$: abelian, rigid tensor category (symmetric, duals), exact faithful tensor functors $\omega_{\mathrm{dR}}, \omega_{\mathrm{B}} : MTM(\mathbb{Q}) \to \mathrm{Vect}$ (Tannakian)

Mixed Tate motives over $\mathbb{Q}$, over $\mathbb{Z}$ ("non-ramified")

Category $MTM(\mathbb{Q})$: abelian, rigid tensor category (symmetric, duals), exact faithful tensor functors $\omega_{\mathrm{dR}}, \omega_{\mathrm{B}} : MTM(\mathbb{Q}) \to \mathrm{Vect}$ (Tannakian)

- For each $n \in \mathbb{Z}$, simple object $\mathbb{Q}(-n) = \mathbb{Q}(-1)^{\otimes n}$.

Mixed Tate motives over $\mathbb{Q}$, over $\mathbb{Z}$ ("non-ramified")

Category $MTM(\mathbb{Q})$: abelian, rigid tensor category (symmetric, duals), exact faithful tensor functors $\omega_{\mathrm{dR}}, \omega_{\mathrm{B}} : MTM(\mathbb{Q}) \to \mathrm{Vect}$ (Tannakian)

- For each $n \in \mathbb{Z}$, simple object $\mathbb{Q}(-n) = \mathbb{Q}(-1)^{\otimes n}$.
  Period matrix: $\left((2\pi i)^n \right)$.

Mixed Tate motives over $\mathbb{Q}$, over $\mathbb{Z}$ ("non-ramified")

Category $MTM(\mathbb{Q})$: abelian, rigid tensor category (symmetric, duals), exact faithful tensor functors $\omega_{\mathrm{dR}}, \omega_{\mathrm{B}} : MTM(\mathbb{Q}) \to \mathrm{Vect}$ (Tannakian)

- For each $n \in \mathbb{Z}$, simple object $\mathbb{Q}(-n) = \mathbb{Q}(-1)^{\otimes n}$.
  Period matrix: $\left((2\pi i)^n \right)$. Effective if $n \geq 0$: $\mathbb{Q}(-n) = \mathrm{H}^n((\mathbb{C}^*)^n)$.

Mixed Tate motives over $\mathbb{Q}$, over $\mathbb{Z}$ ("non-ramified")

Category $MTM(\mathbb{Q})$: abelian, rigid tensor category (symmetric, duals), exact faithful tensor functors $\omega_{\mathrm{dR}}, \omega_{\mathrm{B}} : MTM(\mathbb{Q}) \to \mathrm{Vect}$ (Tannakian)

- For each $n \in \mathbb{Z}$, simple object $\mathbb{Q}(-n) = \mathbb{Q}(-1)^{\otimes n}$.
  Period matrix: $\begin{pmatrix} (2\pi i)^n \end{pmatrix}$. Effective if $n \geq 0$: $\mathbb{Q}(-n) = \mathrm{H}^n((\mathbb{C}^*)^n)$.
- for each $a \in \mathbb{Q}_{>0}$, we have an extension in $MTM(\mathbb{Q})$

$$0 \to \mathbb{Q}(0) \to K_a \to \mathbb{Q}(-1) \to 0$$

$$\text{Period matrix: } \begin{pmatrix} 1 & \log a \\ 0 & 2\pi i \end{pmatrix}$$

(Kummer motive, trivial extension iff $a = 1$)

Mixed Tate motives over $\mathbb{Q}$, over $\mathbb{Z}$ ("non-ramified")

Category $MTM(\mathbb{Q})$: abelian, rigid tensor category (symmetric, duals), exact faithful tensor functors $\omega_{\mathrm{dR}}, \omega_{\mathrm{B}} : MTM(\mathbb{Q}) \to \mathrm{Vect}$ (Tannakian)

- For each $n \in \mathbb{Z}$, simple object $\mathbb{Q}(-n) = \mathbb{Q}(-1)^{\otimes n}$.
  Period matrix: $\left((2\pi i)^n \right)$. Effective if $n \geq 0$: $\mathbb{Q}(-n) = \mathrm{H}^n((\mathbb{C}^*)^n)$.
- for each $a \in \mathbb{Q}_{>0}$, we have an extension in $MTM(\mathbb{Q})$

$$0 \to \mathbb{Q}(0) \to K_a \to \mathbb{Q}(-1) \to 0$$

$$\text{Period matrix: } \begin{pmatrix} 1 & \log a \\ 0 & 2\pi i \end{pmatrix}$$

  (Kummer motive, trivial extension iff $a = 1$)
- For $n = 3, 5, 7, \ldots$, we have a non-trivial extension

$$0 \to \mathbb{Q}(0) \to Z_n \to \mathbb{Q}(-n) \to 0$$

$$\text{Period matrix: } \begin{pmatrix} 1 & \zeta(n) \\ 0 & (2\pi i)^n \end{pmatrix}$$

Mixed Tate motives over $\mathbb{Q}$, over $\mathbb{Z}$ ("non-ramified")

Category $MTM(\mathbb{Q})$: abelian, rigid tensor category (symmetric, duals), exact faithful tensor functors $\omega_{\mathrm{dR}}, \omega_{\mathrm{B}} : MTM(\mathbb{Q}) \to \mathrm{Vect}$ (Tannakian)

- For each $n \in \mathbb{Z}$, simple object $\mathbb{Q}(-n) = \mathbb{Q}(-1)^{\otimes n}$. Period matrix: $\begin{pmatrix} (2\pi i)^n \end{pmatrix}$. Effective if $n \geq 0$: $\mathbb{Q}(-n) = \mathrm{H}^n((\mathbb{C}^*)^n)$.
- for each $a \in \mathbb{Q}_{>0}$, we have an extension in $MTM(\mathbb{Q})$

$$0 \to \mathbb{Q}(0) \to K_a \to \mathbb{Q}(-1) \to 0$$

$$\text{Period matrix: } \begin{pmatrix} 1 & \log a \\ 0 & 2\pi i \end{pmatrix}$$

  (Kummer motive, trivial extension iff $a = 1$)
- For $n = 3, 5, 7, \ldots$, we have a non-trivial extension

$$0 \to \mathbb{Q}(0) \to Z_n \to \mathbb{Q}(-n) \to 0$$

$$\text{Period matrix: } \begin{pmatrix} 1 & \zeta(n) \\ 0 & (2\pi i)^n \end{pmatrix}$$

- All Ext^i with $i \geq 2$ vanish.

Mixed Tate motives over $\mathbb{Q}$, over $\mathbb{Z}$ ("non-ramified")

Category $MTM(\mathbb{Q})$: abelian, rigid tensor category (symmetric, duals), exact faithful tensor functors $\omega_{\mathrm{dR}}, \omega_{\mathrm{B}} : MTM(\mathbb{Q}) \to \mathrm{Vect}$ (Tannakian)

- For each $n \in \mathbb{Z}$, simple object $\mathbb{Q}(-n) = \mathbb{Q}(-1)^{\otimes n}$.
  Period matrix: $\left((2\pi i)^n \right)$. Effective if $n \geq 0$: $\mathbb{Q}(-n) = \mathrm{H}^n((\mathbb{C}^*)^n)$.
- for each $a \in \mathbb{Q}_{>0}$, we have an extension in $MTM(\mathbb{Q})$

$$0 \to \mathbb{Q}(0) \to K_a \to \mathbb{Q}(-1) \to 0$$

$$\text{Period matrix: } \begin{pmatrix} 1 & \log a \\ 0 & 2\pi i \end{pmatrix}$$

  (Kummer motive, trivial extension iff $a = 1$)
- For $n = 3, 5, 7, \ldots$, we have a non-trivial extension

$$0 \to \mathbb{Q}(0) \to Z_n \to \mathbb{Q}(-n) \to 0$$

$$\text{Period matrix: } \begin{pmatrix} 1 & \zeta(n) \\ 0 & (2\pi i)^n \end{pmatrix}$$

- All Ext^i with $i \geq 2$ vanish.
- In $MTM(\mathbb{Z})$, the extensions K_a are not allowed.

Mixed Tate motives over $\mathbb{Q}$, over $\mathbb{Z}$ ("non-ramified")

Category $MTM(\mathbb{Q})$: abelian, rigid tensor category (symmetric, duals), exact faithful tensor functors $\omega_{\mathrm{dR}}, \omega_{\mathrm{B}} : MTM(\mathbb{Q}) \to \mathrm{Vect}$ (Tannakian)

- For each $n \in \mathbb{Z}$, simple object $\mathbb{Q}(-n) = \mathbb{Q}(-1)^{\otimes n}$.
  Period matrix: $\left({\scriptstyle (2\pi i)^n} \right)$. Effective if $n \geq 0$: $\mathbb{Q}(-n) = \mathrm{H}^n((\mathbb{C}^*)^n)$.
- for each $a \in \mathbb{Q}_{>0}$, we have an extension in $MTM(\mathbb{Q})$

$$0 \to \mathbb{Q}(0) \to K_a \to \mathbb{Q}(-1) \to 0$$

$$\text{Period matrix: } \begin{pmatrix} 1 & \log a \\ 0 & 2\pi i \end{pmatrix}$$

  (Kummer motive, trivial extension iff $a = 1$)
- For $n = 3, 5, 7, \ldots$, we have a non-trivial extension

$$0 \to \mathbb{Q}(0) \to Z_n \to \mathbb{Q}(-n) \to 0$$

$$\text{Period matrix: } \begin{pmatrix} 1 & \zeta(n) \\ 0 & (2\pi i)^n \end{pmatrix}$$

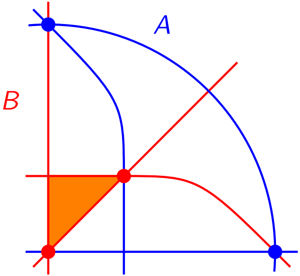
- All Ext^i with $i \geq 2$ vanish.
- In $MTM(\mathbb{Z})$, the extensions K_a are not allowed.
- $\mathrm{Per}(MTM(\mathbb{Z})) = \bigcup_N \mathrm{Per}(\mathcal{M}_{0,N}) = \mathbb{Q}[2\pi i][\mathrm{MZV}] = \mathbb{Q}[2\pi i][\mathrm{MZV}_{2,3}]$

Example: $\zeta(2)$

$$\zeta(2) = \sum_{k \geq 1} \frac{1}{k^2} = \iint_{0<x<y<1} \frac{dx\, dy}{(1-x)y}.$$

Example: $\zeta(2)$

$$\zeta(2) = \sum_{k \geq 1} \frac{1}{k^2} = \iint_{0<x<y<1} \frac{dx\, dy}{(1-x)y}.$$

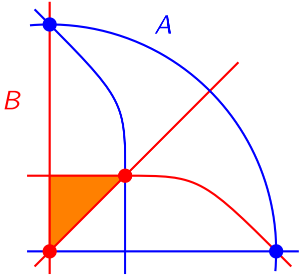


$\mathbb{P}^2$

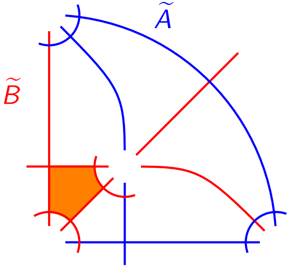
6 lines, 7 points

Example: $\zeta(2)$

$$\zeta(2) = \sum_{k \geq 1} \frac{1}{k^2} = \iint_{0<x<y<1} \frac{dx\,dy}{(1-x)y}.$$



π ← blow-up

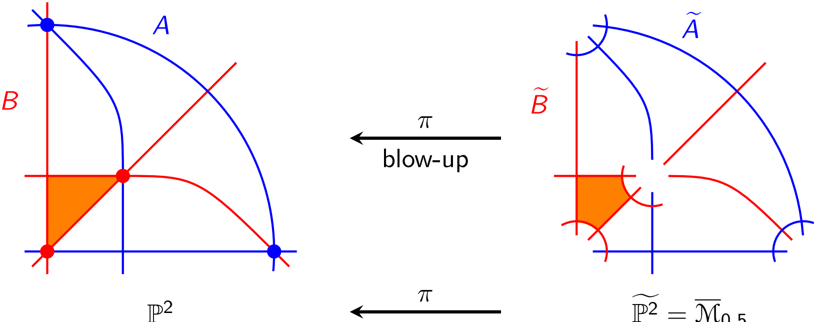


$\mathbb{P}^2$ $\xleftarrow{\pi}$ $\widetilde{\mathbb{P}^2} = \overline{\mathcal{M}}_{0,5}$

6 lines, 7 points

10 lines, 15 points

$$\zeta(2) = \sum_{k\geq 1} \frac{1}{k^2} = \iint_{0<x<y<1} \frac{dx\,dy}{(1-x)y}.$$



$\mathbb{P}^2$ $\qquad$ $\widetilde{\mathbb{P}^2} = \overline{\mathcal{M}}_{0,5}$

6 lines, 7 points $\qquad$ 10 lines, 15 points

$\mathrm{H} := \mathrm{H}^2(\widetilde{\mathbb{P}^2} \setminus \widetilde{A}, \widetilde{B} \setminus \widetilde{A})$. Period matrix: $\begin{pmatrix} 1 & \zeta(2) \\ 0 & (2\pi i)^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & (2\pi i)^2 \end{pmatrix}$.

Bi-arrangements of hyperplanes

Definition (Dupont 2014)

A *projective bi-arrangement of hyperplanes* is a triple $(\mathcal{L}, \mathcal{M}, \chi)$ where

- $\mathcal{L} = \{L_1, \ldots, L_l\}$ is a set of hyperplanes in $\mathbb{P}^n$;
- $\mathcal{M} = \{M_1, \ldots, M_m\}$ is a set of hyperplanes in $\mathbb{P}^n$;
- $\chi : \mathcal{S} = \mathrm{Flats}(\mathcal{L} \cup \mathcal{M}) \to \{\lambda, \mu\}$ is a coloring function, satisfying
$$\chi(L_i) = \lambda \text{ and } \chi(M_j) = \mu \text{ for all } i, j$$

Bi-arrangements of hyperplanes

Definition (Dupont 2014)

A *projective bi-arrangement of hyperplanes* is a triple $(\mathcal{L}, \mathcal{M}, \chi)$ where

- $\mathcal{L} = \{L_1, \ldots, L_l\}$ is a set of hyperplanes in $\mathbb{P}^n$;
- $\mathcal{M} = \{M_1, \ldots, M_m\}$ is a set of hyperplanes in $\mathbb{P}^n$;
- $\chi : \mathcal{S} = \mathrm{Flats}(\mathcal{L} \cup \mathcal{M}) \to \{\lambda, \mu\}$ is a coloring function, satisfying

$$\chi(L_i) = \lambda \text{ and } \chi(M_j) = \mu \text{ for all } i, j$$

Definition (Dupont 2014)

The *motive* of the bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$ is the collection of relative cohomology groups (mixed Hodge structures)

$$H^\bullet(\widetilde{\mathbb{P}^n} \setminus \widetilde{\mathcal{L}}, \widetilde{\mathcal{M}} \setminus \widetilde{\mathcal{L}}) .$$

Bi-arrangements of hyperplanes

Definition (Dupont 2014)

A *projective bi-arrangement of hyperplanes* is a triple $(\mathcal{L}, \mathcal{M}, \chi)$ where

- $\mathcal{L} = \{L_1, \dots, L_l\}$ is a set of hyperplanes in $\mathbb{P}^n$;
- $\mathcal{M} = \{M_1, \dots, M_m\}$ is a set of hyperplanes in $\mathbb{P}^n$;
- $\chi : \mathcal{S} = \mathrm{Flats}(\mathcal{L} \cup \mathcal{M}) \to \{\lambda, \mu\}$ is a coloring function, satisfying

$$\chi(L_i) = \lambda \text{ and } \chi(M_j) = \mu \text{ for all } i, j$$

Definition (Dupont 2014)

The *motive* of the bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$ is the collection of relative cohomology groups (mixed Hodge structures)

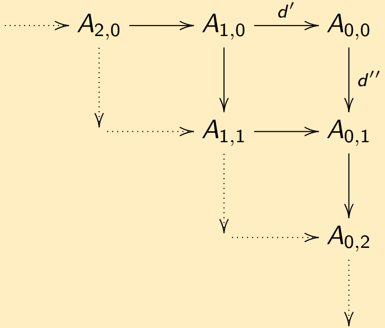
$$H^\bullet(\widetilde{\mathbb{P}^n} \setminus \widetilde{\mathcal{L}}, \widetilde{\mathcal{M}} \setminus \widetilde{\mathcal{L}}) \ .$$

Inspired by (Aomoto 1977, 1982) and (Beilinson-Goncharov-Schechtman-Varchenko, 1989).

The Orlik-Solomon bicomplex

Definition

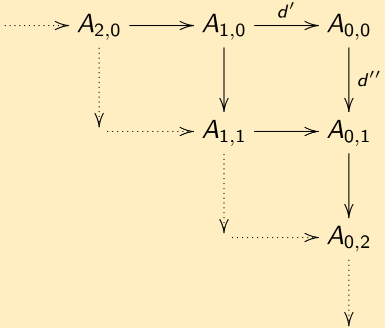
We define the *Orlik-Solomon bicomplex* $A_{\bullet,\bullet} = A_{\bullet,\bullet}(\mathcal{L}, \mathcal{M}, \chi)$:



The Orlik-Solomon bicomplex

Definition

We define the *Orlik-Solomon bicomplex* $A_{\bullet,\bullet} = A_{\bullet,\bullet}(\color{blue}{\mathcal{L}}, \color{red}{\mathcal{M}}, \chi)$:



We define $A_{i,j} = \bigoplus_{S \in \mathcal{S}_{i+j}} A_{i,j}^S$ and the differentials d' and d'' by induction on the codimension $i + j$. Here $\mathcal{S}_k =$ flats of codimension k.

The Orlik-Solomon bicomplex

Base step of the induction : $A_{0,0} = \mathbb{Q}$.

The Orlik-Solomon bicomplex

Base step of the induction : $A_{0,0} = \mathbb{Q}$.

Inductive step :

- For a flat Σ such that $\chi(\Sigma) = \lambda$, we define $A^{\Sigma}_{i,j}$ as a *kernel*:

$$0 \to A^{\Sigma}_{i,j} \xrightarrow{d'} \bigoplus_{S \supset \Sigma} A^{S}_{i-1,j} \xrightarrow{d'} \bigoplus_{T \supset \Sigma} A^{T}_{i-2,j} \ .$$

The Orlik-Solomon bicomplex

Base step of the induction : $A_{0,0} = \mathbb{Q}$.

Inductive step :

- For a flat Σ such that $\chi(\Sigma) = \lambda$, we define $A^{\Sigma}_{i,j}$ as a *kernel*:

$$0 \to A^{\Sigma}_{i,j} \xrightarrow{d'} \bigoplus_{S \supset \Sigma} A^{S}_{i-1,j} \xrightarrow{d'} \bigoplus_{T \supset \Sigma} A^{T}_{i-2,j} \ .$$

- For a flat Σ such that $\chi(\Sigma) = \mu$, we define $A^{\Sigma}_{i,j}$ as a *cokernel*:

$$0 \leftarrow A^{\Sigma}_{i,j} \xleftarrow{d''} \bigoplus_{S \supset \Sigma} A^{S}_{i,j-1} \xleftarrow{d''} \bigoplus_{T \supset \Sigma} A^{T}_{i,j-2} \ .$$

The Orlik-Solomon bicomplex

Base step of the induction : $A_{0,0} = \mathbb{Q}$.

Inductive step :

- For a flat Σ such that $\chi(\Sigma) = \lambda$, we define $A^{\Sigma}_{i,j}$ as a *kernel*:

$$0 \to A^{\Sigma}_{i,j} \xrightarrow{d'} \bigoplus_{S \supset \Sigma} A^{S}_{i-1,j} \xrightarrow{d'} \bigoplus_{T \supset \Sigma} A^{T}_{i-2,j} \;.$$

- For a flat Σ such that $\chi(\Sigma) = \mu$, we define $A^{\Sigma}_{i,j}$ as a *cokernel*:

$$0 \leftarrow A^{\Sigma}_{i,j} \xleftarrow{d''} \bigoplus_{S \supset \Sigma} A^{S}_{i,j-1} \xleftarrow{d''} \bigoplus_{T \supset \Sigma} A^{T}_{i,j-2} \;.$$

- In any case, we complete the squares by the universal property.

The Orlik-Solomon bicomplex

Base step of the induction : $A_{0,0} = \mathbb{Q}$.

Inductive step :

- For a flat Σ such that $\chi(\Sigma) = \lambda$, we define $A^{\Sigma}_{i,j}$ as a *kernel*:

$$0 \to A^{\Sigma}_{i,j} \xrightarrow{d'} \bigoplus_{S \supset \Sigma} A^{S}_{i-1,j} \xrightarrow{d'} \bigoplus_{T \supset \Sigma} A^{T}_{i-2,j} \, .$$

- For a flat Σ such that $\chi(\Sigma) = \mu$, we define $A^{\Sigma}_{i,j}$ as a *cokernel*:

$$0 \leftarrow A^{\Sigma}_{i,j} \xleftarrow{d''} \bigoplus_{S \supset \Sigma} A^{S}_{i,j-1} \xleftarrow{d''} \bigoplus_{T \supset \Sigma} A^{T}_{i,j-2} \, .$$

- In any case, we complete the squares by the universal property.

Hence we use:

- `KernelObject`, `KernelMorphism`, `KernelLift` and dual versions,

The Orlik-Solomon bicomplex

Base step of the induction : $A_{0,0} = \mathbb{Q}$.

Inductive step :

- For a flat Σ such that $\chi(\Sigma) = \lambda$, we define $A^{\Sigma}_{i,j}$ as a *kernel*:

$$0 \to A^{\Sigma}_{i,j} \xrightarrow{d'} \bigoplus_{S \supset \Sigma} A^{S}_{i-1,j} \xrightarrow{d'} \bigoplus_{T \supset \Sigma} A^{T}_{i-2,j} \ .$$

- For a flat Σ such that $\chi(\Sigma) = \mu$, we define $A^{\Sigma}_{i,j}$ as a *cokernel*:

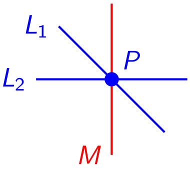
$$0 \leftarrow A^{\Sigma}_{i,j} \xleftarrow{d''} \bigoplus_{S \supset \Sigma} A^{S}_{i,j-1} \xleftarrow{d''} \bigoplus_{T \supset \Sigma} A^{T}_{i,j-2} \ .$$

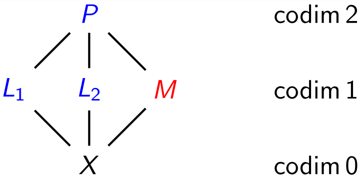
- In any case, we complete the squares by the universal property.

Hence we use:

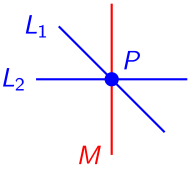
- `KernelObject`, `KernelMorphism`, `KernelLift` and dual versions,
- `MorphismBetweenDirectSums`,
  `ComponentOfMorphismIntoDirectSum`,
  `ComponentOfMorphismFromDirectSum`...

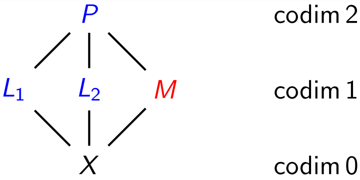
Example





Example



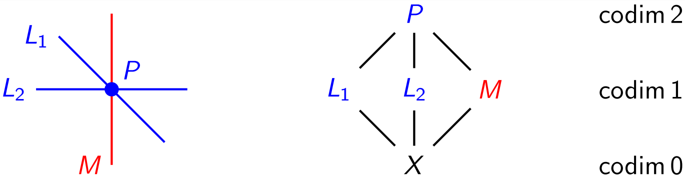


$A^{\leq X}_{\bullet,\bullet}$

$\mathbb{Q}$

$$A^X_{0,0} = \mathbb{Q}$$

Example

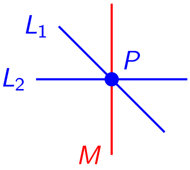


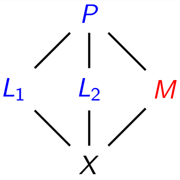
$A^{<L_i}_{\bullet,\bullet}$

$\mathbb{Q}$

$$A^X_{0,0} = \mathbb{Q}$$

Example





codim 2

codim 1

codim 0

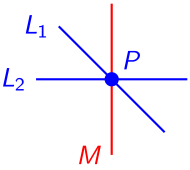
$$A^{\leq L_i}_{\bullet,\bullet}$$

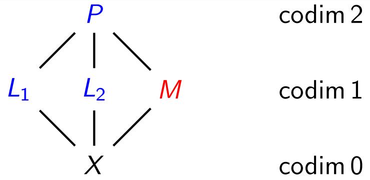
$$\mathbb{Q} \xrightarrow{(\,1\,)} \mathbb{Q}$$

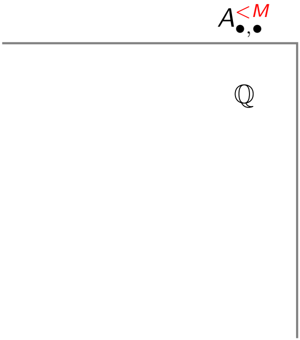
$$A^X_{0,0} = \mathbb{Q}$$

$$A^{L_i}_{1,0} = \mathbb{Q} \qquad d'^{X,L_i}_{1,0} = (1)$$

Example



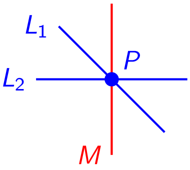


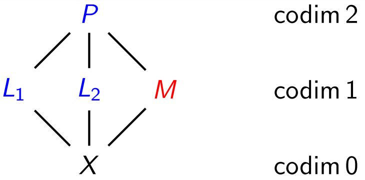


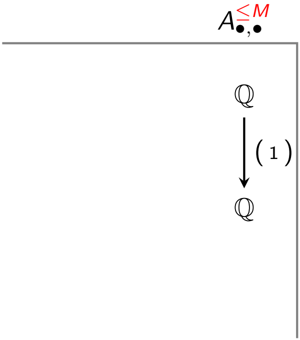
$$A^X_{0,0} = \mathbb{Q}$$

$$A^{L_i}_{1,0} = \mathbb{Q} \qquad d'^{X,L_i}_{1,0} = (1)$$

Example



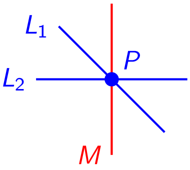


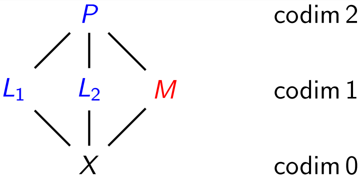


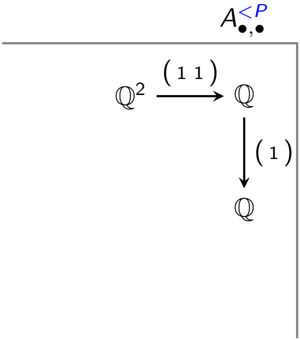
$$A^{X}_{0,0} = \mathbb{Q}$$

$$A^{L_i}_{1,0} = \mathbb{Q} \qquad d'^{X,L_i}_{1,0} = (1)$$

$$A^{M}_{0,1} = \mathbb{Q} \qquad d''^{M,X}_{0,0} = (1)$$





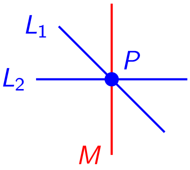


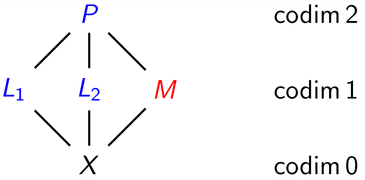
$$A^X_{0,0} = \mathbb{Q}$$

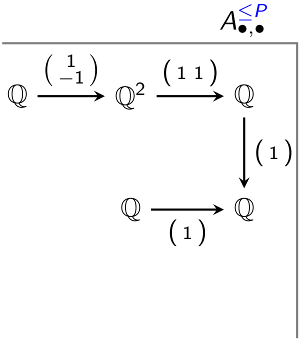
$$A^{L_i}_{1,0} = \mathbb{Q} \qquad d'^{X,L_i}_{1,0} = (1)$$

$$A^{M}_{0,1} = \mathbb{Q} \qquad d''^{M,X}_{0,0} = (1)$$

Example







$A^X_{0,0} = \mathbb{Q}$

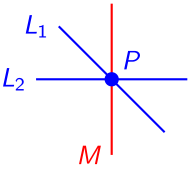
$A^{L_i}_{1,0} = \mathbb{Q}$ $\quad d'^{X,L_i}_{1,0} = (1)$

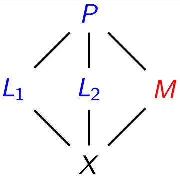
$A^M_{0,1} = \mathbb{Q}$ $\quad d''^{M,X}_{0,0} = (1)$

$A^P_{2,0} = \mathbb{Q}^2$ $\quad d'^{L_1,P}_{2,0} = (1)$ $\quad d'^{L_2,P}_{2,0} = (-1)$

$A^P_{1,1} = \mathbb{Q}$ $\quad d'^{M,P}_{1,1} = (1)$

Example

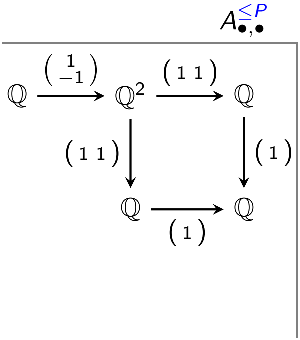




codim 2

codim 1

codim 0



$A^{X}_{0,0} = \mathbb{Q}$

$A^{L_i}_{1,0} = \mathbb{Q}$ $\quad d'^{X,L_i}_{1,0} = (1)$

$A^{M}_{0,1} = \mathbb{Q}$ $\quad d''^{M,X}_{0,0} = (1)$

$A^{P}_{2,0} = \mathbb{Q}^2$ $\quad d'^{L_1,P}_{2,0} = (1)$ $\quad d'^{L_2,P}_{2,0} = (-1)$

$A^{P}_{1,1} = \mathbb{Q}$ $\quad d'^{M,P}_{1,1} = (1)$

$d''^{P,L_1}_{1,0} = (1)$ $\quad d''^{P,L_2}_{1,0} = (1)$

Exactness

Definition

A bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$ is *exact* if the above exact sequences can be continued to long exact sequences

$$0 \to A^{\Sigma}_{i,j} \xrightarrow{d'} \bigoplus_{S \supset \Sigma} A^{S}_{i-1,j} \xrightarrow{d'} \bigoplus_{T \supset \Sigma} A^{T}_{i-2,j} \xrightarrow{d'} \cdots$$

or

$$0 \leftarrow A^{\Sigma}_{i,j} \xleftarrow{d''} \bigoplus_{S \supset \Sigma} A^{S}_{i,j-1} \xleftarrow{d''} \bigoplus_{T \supset \Sigma} A^{T}_{i,j-2} \xleftarrow{d''} \cdots$$

Exactness

Definition

A bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$ is *exact* if the above exact sequences can be continued to long exact sequences

$$0 \to A^{\Sigma}_{i,j} \xrightarrow{d'} \bigoplus_{S \supset \Sigma} A^{S}_{i-1,j} \xrightarrow{d'} \bigoplus_{T \supset \Sigma} A^{T}_{i-2,j} \xrightarrow{d'} \cdots$$

or

$$0 \leftarrow A^{\Sigma}_{i,j} \xleftarrow{d''} \bigoplus_{S \supset \Sigma} A^{S}_{i,j-1} \xleftarrow{d''} \bigoplus_{T \supset \Sigma} A^{T}_{i,j-2} \xleftarrow{d''} \cdots$$

Remark

- All arrangements of hyperplanes $(\mathcal{A}, \varnothing, \lambda)$ are exact, $A_{\bullet,0}(\mathcal{A}, \varnothing, \lambda) = A_{\bullet}(\mathcal{A})$.

Exactness

Definition

A bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$ is *exact* if the above exact sequences can be continued to long exact sequences

$$0 \to A^{\Sigma}_{i,j} \xrightarrow{d'} \bigoplus_{S \supset \Sigma} A^{S}_{i-1,j} \xrightarrow{d'} \bigoplus_{T \supset \Sigma} A^{T}_{i-2,j} \xrightarrow{d'} \cdots$$

or

$$0 \leftarrow A^{\Sigma}_{i,j} \xleftarrow{d''} \bigoplus_{S \supset \Sigma} A^{S}_{i,j-1} \xleftarrow{d''} \bigoplus_{T \supset \Sigma} A^{T}_{i,j-2} \xleftarrow{d''} \cdots$$

Remark

- All arrangements of hyperplanes $(\mathcal{A}, \varnothing, \lambda)$ are exact, $A_{\bullet,0}(\mathcal{A}, \varnothing, \lambda) = A_{\bullet}(\mathcal{A})$.
- Deletion and restriction formalism for exact bi-arrangements of hyperplanes.

The main theorem

Theorem (Dupont 2014)

For an *exact* bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$ in $\mathbb{P}^n$, "the Orlik-Solomon bicomplex $A_{\bullet,\bullet}(\mathcal{L}, \mathcal{M}, \chi)$ computes the motive $H^\bullet(\mathcal{L}, \mathcal{M}, \chi)$".

The main theorem

Theorem (Dupont 2014)

For an *exact* bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$ in $\mathbb{P}^n$, "the Orlik-Solomon bicomplex $A_{\bullet,\bullet}(\mathcal{L}, \mathcal{M}, \chi)$ computes the motive $H^\bullet(\mathcal{L}, \mathcal{M}, \chi)$". More precisely, for each $k = 0, \ldots, n$:

- we consider the double complex $A_{0 \leq \bullet \leq k, 0 \leq \bullet \leq n-k}$;
- we let ${}^{(k)}A_\bullet$ be its total complex ;
- then $\mathrm{gr}^W_{2k} H^r(\mathcal{L}, \mathcal{M}, \chi) \cong H_{2k-r}({}^{(k)}A_\bullet)$ ($W =$ the weight filtration coming from mixed Hodge theory).

The main theorem

Theorem (Dupont 2014)

For an *exact* bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$ in $\mathbb{P}^n$, "the Orlik-Solomon bicomplex $A_{\bullet,\bullet}(\mathcal{L}, \mathcal{M}, \chi)$ computes the motive $H^\bullet(\mathcal{L}, \mathcal{M}, \chi)$". More precisely, for each $k = 0, \ldots, n$:

- we consider the double complex $A_{0 \le \bullet \le k, 0 \le \bullet \le n-k}$;
- we let ${}^{(k)}A_\bullet$ be its total complex ;
- then $\mathrm{gr}^W_{2k} H^r(\mathcal{L}, \mathcal{M}, \chi) \cong H_{2k-r}({}^{(k)}A_\bullet)$ ($W =$ the weight filtration coming from mixed Hodge theory).

Remark

- For arrangements of hyperplanes, we recover the (projective) Brieskorn-Orlik-Solomon theorem, with only weight $\mathrm{gr}_{2k} H^k$.

The main theorem

Theorem (Dupont 2014)

For an *exact* bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$ in $\mathbb{P}^n$, "the Orlik-Solomon bicomplex $A_{\bullet,\bullet}(\mathcal{L}, \mathcal{M}, \chi)$ computes the motive $H^\bullet(\mathcal{L}, \mathcal{M}, \chi)$". More precisely, for each $k = 0, \ldots, n$:

- we consider the double complex $A_{0 \leq \bullet \leq k, 0 \leq \bullet \leq n-k}$;
- we let ${}^{(k)}A_\bullet$ be its total complex ;
- then $\mathrm{gr}^W_{2k} H^r(\mathcal{L}, \mathcal{M}, \chi) \cong H_{2k-r}({}^{(k)}A_\bullet)$ ($W =$ the weight filtration coming from mixed Hodge theory).

Remark

- For arrangements of hyperplanes, we recover the (projective) Brieskorn-Orlik-Solomon theorem, with only weight $\mathrm{gr}_{2k} H^k$.
- The weight-graded quotients $\mathrm{gr}^W_{2k} H^\bullet(\mathcal{L}, \mathcal{M}, \chi)$ are combinatorial invariants, but not the whole motive $H^\bullet(\mathcal{L}, \mathcal{M}, \chi)$.

Explicit computations: the tame case

Combinatorial notion of *tame* bi-arrangements of hyperplanes.

- Generic bi-arrangements are tame
- tame $\Longrightarrow$ exact.

Explicit computations: the tame case

Combinatorial notion of *tame* bi-arrangements of hyperplanes.

- Generic bi-arrangements are tame
- tame $\Longrightarrow$ exact.

Proposition

For a *tame* bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$, the Orlik-Solomon bicomplex $A_{\bullet,\bullet}(\mathcal{L}, \mathcal{M}, \chi)$ is an explicit sub-quotient of $A_\bullet(\mathcal{L}) \otimes A_\bullet(\mathcal{M})^\vee$.

Explicit computations: the tame case

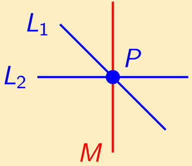
Combinatorial notion of *tame* bi-arrangements of hyperplanes.

- Generic bi-arrangements are tame
- tame $\Longrightarrow$ exact.

Proposition

For a *tame* bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$, the Orlik-Solomon bicomplex $A_{\bullet,\bullet}(\mathcal{L}, \mathcal{M}, \chi)$ is an explicit sub-quotient of $A_\bullet(\mathcal{L}) \otimes A_\bullet(\mathcal{M})^\vee$.

Example



$$A_{\bullet,\bullet} = \Lambda^\bullet(e_1, e_2) \otimes \Lambda^\bullet(f_1^\vee)/(d(e_1 \wedge e_2) \otimes f_1^\vee)$$

Explicit computations: the tame case

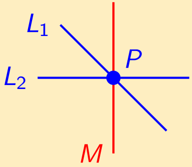
Combinatorial notion of *tame* bi-arrangements of hyperplanes.

- Generic bi-arrangements are tame
- tame $\Longrightarrow$ exact.

Proposition

For a *tame* bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$, the Orlik-Solomon bicomplex $A_{\bullet,\bullet}(\mathcal{L}, \mathcal{M}, \chi)$ is an explicit sub-quotient of $A_\bullet(\mathcal{L}) \otimes A_\bullet(\mathcal{M})^\vee$.

Example



$$A_{\bullet,\bullet} = \Lambda^\bullet(e_1, e_2) \otimes \Lambda^\bullet(f_1^\vee)/(d(e_1 \wedge e_2) \otimes f_1^\vee)$$

Example

One can define multiple zeta bi-arrangements $\mathcal{Z}(n_1, \ldots, n_r)$ that are tame.

Basic cellular integrals

Given a permutation $\sigma \in \mathfrak{S}_N$, define on $\mathbb{P}^N \setminus \bigcup\{z_i = z_j\}$:

$$\tilde{f}_\sigma = \prod_{i \in \mathbb{Z}/N\mathbb{Z}} \frac{z_i - z_{i+1}}{z_{\sigma(i)} - z_{\sigma(i+1)}} \quad \text{and} \quad \tilde{\omega}_\sigma = \frac{\mathrm{d}z_1 \ldots \mathrm{d}z_N}{\prod_{i \in \mathbb{Z}/N\mathbb{Z}} \left(z_{\sigma(i)} - z_{\sigma(i+1)}\right)},$$

both PGL_2-invariant, hence we get $f_\sigma \in \mathcal{O}(\mathcal{M}_{0,N})$, and $\omega_\sigma \in \Omega^n(\mathcal{M}_{0,N})$ after dividing by an invariant volume form on PGL_2.

Basic cellular integrals

Given a permutation $\sigma \in \mathfrak{S}_N$, define on $\mathbb{P}^N \setminus \bigcup\{z_i = z_j\}$:

$$\tilde{f}_\sigma = \prod_{i \in \mathbb{Z}/N\mathbb{Z}} \frac{z_i - z_{i+1}}{z_{\sigma(i)} - z_{\sigma(i+1)}} \quad \text{and} \quad \tilde{\omega}_\sigma = \frac{dz_1 \ldots dz_N}{\prod_{i \in \mathbb{Z}/N\mathbb{Z}} \left(z_{\sigma(i)} - z_{\sigma(i+1)}\right)},$$

both PGL_2-invariant, hence we get $f_\sigma \in \mathcal{O}(\mathcal{M}_{0,N})$, and $\omega_\sigma \in \Omega^n(\mathcal{M}_{0,N})$ after dividing by an invariant volume form on PGL_2.

Basic cellular integral:

$$I_\sigma(k) = \int_{\delta_n} f_\sigma^k \omega_\sigma$$

It converges iff σ is a *convergent permutation* ("dinner party problem").

Basic cellular integrals

Given a permutation $\sigma \in \mathfrak{S}_N$, define on $\mathbb{P}^N \setminus \bigcup\{z_i = z_j\}$:

$$\tilde{f}_\sigma = \prod_{i \in \mathbb{Z}/N\mathbb{Z}} \frac{z_i - z_{i+1}}{z_{\sigma(i)} - z_{\sigma(i+1)}} \quad \text{and} \quad \tilde{\omega}_\sigma = \frac{dz_1 \ldots dz_N}{\prod_{i \in \mathbb{Z}/N\mathbb{Z}} (z_{\sigma(i)} - z_{\sigma(i+1)})},$$

both PGL_2-invariant, hence we get $f_\sigma \in \mathcal{O}(\mathcal{M}_{0,N})$, and $\omega_\sigma \in \Omega^n(\mathcal{M}_{0,N})$ after dividing by an invariant volume form on PGL_2.

Basic cellular integral:

$$I_\sigma(k) = \int_{\delta_n} f_\sigma^k \omega_\sigma$$

It converges iff σ is a *convergent permutation* ("dinner party problem").

Number of convergent configurations, up to dihedral symmetries:

N	4	5	6	7	8	9	10	11
C_N	0	1	1	5	17	105	771	7028

Basic cellular integrals

Given a permutation $\sigma \in \mathfrak{S}_N$, define on $\mathbb{P}^N \setminus \bigcup\{z_i = z_j\}$:

$$\tilde{f}_\sigma = \prod_{i \in \mathbb{Z}/N\mathbb{Z}} \frac{z_i - z_{i+1}}{z_{\sigma(i)} - z_{\sigma(i+1)}} \quad \text{and} \quad \tilde{\omega}_\sigma = \frac{\mathrm{d}z_1 \ldots \mathrm{d}z_N}{\prod_{i \in \mathbb{Z}/N\mathbb{Z}} \left(z_{\sigma(i)} - z_{\sigma(i+1)}\right)},$$

both PGL_2-invariant, hence we get $f_\sigma \in \mathcal{O}(\mathcal{M}_{0,N})$, and $\omega_\sigma \in \Omega^n(\mathcal{M}_{0,N})$ after dividing by an invariant volume form on PGL_2.

Basic cellular integral:

$$I_\sigma(k) = \int_{\delta_n} f_\sigma^k \omega_\sigma$$

It converges iff σ is a *convergent permutation* ("dinner party problem").

Number of convergent configurations, up to dihedral symmetries:

N	4	5	6	7	8	9	10	11
C_N	0	1	1	5	17	105	771	7028

$N = 5$: only ${}_5\pi = [5, 2, 4, 1, 3]$, $\qquad N = 6$: only ${}_6\pi = [6, 2, 4, 1, 5, 3]$

Vanishing for basic cellular integrals

Theorem (Brown 2016)

Suppose that $A, B \subset \mathcal{M}_{0,N}$ are cellular boundary divisors with no common irreducible components. Let $n = N - 3$. Then

$$\mathrm{gr}_2^W \mathrm{H}_{A,B} = \mathrm{gr}_{2n-2}^W \mathrm{H}_{A,B} = 0$$

and $\mathrm{gr}_0^W \mathrm{H}_{A,B}$ and $\mathrm{gr}_{2n}^W \mathrm{H}_{A,B}$ are both 1-dimensional.

Vanishing for basic cellular integrals

Theorem (Brown 2016)

Suppose that $A, B \subset \mathcal{M}_{0,N}$ are cellular boundary divisors with no common irreducible components. Let $n = N - 3$. Then

$$\mathrm{gr}_2^W \mathrm{H}_{A,B} = \mathrm{gr}_{2n-2}^W \mathrm{H}_{A,B} = 0$$

and $\mathrm{gr}_0^W \mathrm{H}_{A,B}$ and $\mathrm{gr}_{2n}^W \mathrm{H}_{A,B}$ are both 1-dimensional.

Hence for the unique convergent configurations for $N = 5, 6$, we must have

$$\mathrm{gr}_\bullet^W \mathrm{H}_{A,B} = \begin{cases} \mathbb{Q}(0) \oplus \mathbb{Q}(-2) & \text{for N} = 5, \\ \mathbb{Q}(0) \oplus \mathbb{Q}(-3) & \text{for N} = 6. \end{cases}$$

Vanishing for basic cellular integrals

Theorem (Brown 2016)

Suppose that $A, B \subset \mathcal{M}_{0,N}$ are cellular boundary divisors with no common irreducible components. Let $n = N - 3$. Then

$$\operatorname{gr}_2^W \mathrm{H}_{A,B} = \operatorname{gr}_{2n-2}^W \mathrm{H}_{A,B} = 0$$

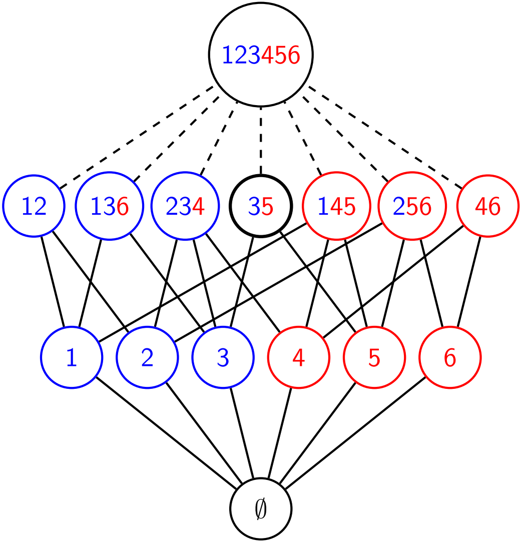
and $\operatorname{gr}_0^W \mathrm{H}_{A,B}$ and $\operatorname{gr}_{2n}^W \mathrm{H}_{A,B}$ are both 1-dimensional.

Hence for the unique convergent configurations for $N = 5, 6$, we must have

$$\operatorname{gr}_\bullet^W \mathrm{H}_{A,B} = \begin{cases} \mathbb{Q}(0) \oplus \mathbb{Q}(-2) & \text{for N} = 5, \\ \mathbb{Q}(0) \oplus \mathbb{Q}(-3) & \text{for N} = 6. \end{cases}$$

Those are the Apéry motives! They give the linear combinations of 1 and $\zeta(2)$ for $N = 5$, resp. 1 and $\zeta(3)$ for $N = 6$, used in the irrationality proofs.

Flat poset for $\zeta(2)$



35 may be set red or blue

morphism red $\to$ blue

`KernelObjectFunctorial`
`TotalComplexFunctorial`

Take the image!

Irrelevant for $\zeta(2)$:
$101 \twoheadrightarrow 101 \hookrightarrow 101$

Relevant for $\zeta(3)$:

$1011 \twoheadrightarrow 1001 \hookrightarrow 1101$

More basic cellular integrals

$N = 7$

Two dual pairs and one self-dual configuration:

$$
\begin{array}{ccc}
{}_7\pi_1 = [7,2,4,1,6,3,5] & \longleftrightarrow & {}_7\pi_1^\vee = [7,2,5,1,4,6,3] \\
{}_7\pi_2 = [7,2,4,6,1,3,5] & \longleftrightarrow & {}_7\pi_1^\vee = [7,3,6,2,5,1,4]
\end{array}
$$
$$
{}_7\pi_3 = [7,2,5,1,3,6,4] = {}_7\pi_3^\vee
$$

More basic cellular integrals

$N = 7$

Two dual pairs and one self-dual configuration:

$$
\begin{aligned}
{}_7\pi_1 = [7,2,4,1,6,3,5] \quad &\longleftrightarrow \quad {}_7\pi_1^\vee = [7,2,5,1,4,6,3] \\
{}_7\pi_2 = [7,2,4,6,1,3,5] \quad &\longleftrightarrow \quad {}_7\pi_1^\vee = [7,3,6,2,5,1,4] \\
{}_7\pi_3 = [7,2,5,1,3,6,4] &= {}_7\pi_3^\vee
\end{aligned}
$$

Experimentally, all give linear combinations of 1, $\zeta(2)$ and $\zeta(4)$.

More basic cellular integrals

$N = 7$

Two dual pairs and one self-dual configuration:

$$\begin{aligned} {}_7\pi_1 = [7,2,4,1,6,3,5] \quad &\longleftrightarrow \quad {}_7\pi_1^\vee = [7,2,5,1,4,6,3] \\ {}_7\pi_2 = [7,2,4,6,1,3,5] \quad &\longleftrightarrow \quad {}_7\pi_1^\vee = [7,3,6,2,5,1,4] \end{aligned}$$
$${}_7\pi_3 = [7,2,5,1,3,6,4] = {}_7\pi_3^\vee$$

Experimentally, all give linear combinations of 1, $\zeta(2)$ and $\zeta(4)$.
With `MotivesForBiarrangements`, based on `CAP`, we can confirm:

$$\mathrm{gr}^W_\bullet \mathrm{H}_{A,B} = \mathbb{Q}(0) \oplus \mathbb{Q}(-2) \oplus \mathbb{Q}(-4)$$

More basic cellular integrals

$N = 7$

Two dual pairs and one self-dual configuration:

$$
\begin{aligned}
{}_7\pi_1 = [7,2,4,1,6,3,5] \quad &\longleftrightarrow \quad {}_7\pi_1^\vee = [7,2,5,1,4,6,3] \\
{}_7\pi_2 = [7,2,4,6,1,3,5] \quad &\longleftrightarrow \quad {}_7\pi_1^\vee = [7,3,6,2,5,1,4] \\
{}_7\pi_3 = [7,2,5,1,3,6,4] &= {}_7\pi_3^\vee
\end{aligned}
$$

Experimentally, all give linear combinations of 1, $\zeta(2)$ and $\zeta(4)$.
With `MotivesForBiarrangements`, based on `CAP`, we can confirm:

$$\mathrm{gr}_\bullet^W \mathrm{H}_{A,B} = \mathbb{Q}(0) \oplus \mathbb{Q}(-2) \oplus \mathbb{Q}(-4)$$

$N = 8$

Among the 17 convergent configurations, let us note

$${}_8\pi_8 = [8,2,5,1,6,4,7,3] \longleftrightarrow {}_8\pi_8^\vee = [8,2,4,1,7,5,3,6]$$

More basic cellular integrals

$N = 7$

Two dual pairs and one self-dual configuration:

$$
\begin{gathered}
{}_7\pi_1 = [7,2,4,1,6,3,5] \quad \longleftrightarrow \quad {}_7\pi_1^\vee = [7,2,5,1,4,6,3] \\
{}_7\pi_2 = [7,2,4,6,1,3,5] \quad \longleftrightarrow \quad {}_7\pi_1^\vee = [7,3,6,2,5,1,4] \\
{}_7\pi_3 = [7,2,5,1,3,6,4] = {}_7\pi_3^\vee
\end{gathered}
$$

Experimentally, all give linear combinations of 1, $\zeta(2)$ and $\zeta(4)$.
With `MotivesForBiarrangements`, based on `CAP`, we can confirm:

$$\mathrm{gr}_\bullet^W \mathrm{H}_{A,B} = \mathbb{Q}(0) \oplus \mathbb{Q}(-2) \oplus \mathbb{Q}(-4)$$

$N = 8$

Among the 17 convergent configurations, let us note

$${}_8\pi_8 = [8,2,5,1,6,4,7,3] \longleftrightarrow {}_8\pi_8^\vee = [8,2,4,1,7,5,3,6]$$

With `MotivesForBiarrangements`, based on `CAP`, we can confirm:

$$\mathrm{gr}_\bullet^W \mathrm{H}_{A,B} = \begin{cases} \mathbb{Q}(0) \oplus \mathbb{Q}(-3) \oplus \mathbb{Q}(-5) & \text{for } {}_8\pi_8, \\ \mathbb{Q}(0) \oplus \mathbb{Q}(-2) \oplus \mathbb{Q}(-5) & \text{for } {}_8\pi_8^\vee. \end{cases}$$

Bibliography

Francis Brown

- *Multiple zeta values and periods of moduli spaces $\mathcal{M}_{0,n}$*, Ann. Scient. Éc. Norm. Sup. (2009)
- *Mixed Tate motives over $\mathbb{Z}$*, Annals of Mathematics (2012)
- *Irrationality proofs for zeta values, moduli spaces and dinner parties*, Mosc. J. Comb. Number Theory (2016)

Clément Dupont

- *Relative cohomology of bi-arrangements*, Trans. Amer. Math. Soc. (2017)
- *Odd zeta motive and linear forms in odd zeta values*, Compos. Math. (2018), with appendix by Don Zagier

Javier Fresán & José Burgos Gil

Multiple zeta values: from numbers to motives, Clay Mathematics Proceedings, to appear.