# Amplitudes, Ansätze and Algebraic Geometry

Ben Page

CERN, Theoretical Physics Department

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In collaboration with Giuseppe De Laurentis

based on [2203.04269]

# Introduction

# Multi-scale, Multi-loop Amplitudes

- Loop amplitudes are computed via the master integral decomposition.

$$\mathcal{A}^{(l)} = \sum_k \underbrace{\mathcal{C}_k}_{\text{rational functions}} \times \underbrace{\mathcal{I}_k}_{\text{master integrals}} .$$

- Let's focus on difficulties when computing the rational functions.

(Talks on integrals by Anatonela, Claude, Christoph, David, Stefan)

- Chosen framework: use finite-field evaluations to determine $\mathcal{C}_k$.

Numerical Data: $\left\{\mathcal{C}_k(p_1^{(1)}, \dots, p_n^{(1)}), \dots, \mathcal{C}_k(p_1^{(N)}, \dots, p_n^{(N)})\right\}$

$+$

Ansatz: $\mathcal{C}_k(p_1, \dots, p_n) = \sum_{i=1}^{N} c_{ik} \mathfrak{a}_{ik}(p_1, \dots, p_n)$

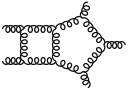
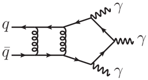
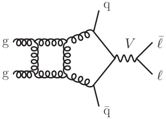
$\xrightarrow{\text{reconstruct}} \mathcal{C}_k$

[von Manteuffel, Schabinger '14; Peraro '16],

FiniteFlow [Peraro '19], Firefly [Klappert, Lange, Klein '19, '20]

- Reconstruction time dominated by sampling: $T_{\text{sample}} \sim O(\text{minute})$.

- Evaluation count for (selected) recent two-loop five-point amplitudes:

| | Three-jet | Three-photon | W + two-jets* |
|---|---|---|---|
| Process |  |  |  |
| # Samples | $\sim 10^5$ | $\sim 10^5$ | $\sim 10^6$. |

*After simplification via [Badger et al '20]

Need to better understand rational functions, build simpler Ansätze.

# The Approach of [De Laurentis, BP '22]

- Work in spinor space* to manifest gauge-theory simplifications.

$$\mathcal{C}_k(p_1, \ldots, p_n) \quad \rightarrow \quad \mathcal{C}_k(\lambda, \tilde{\lambda}).$$

*Algorithmic toolkit provided.

- Numerically study $\mathcal{C}_k$ to understand partial-fractions structure.

$$\frac{\mathcal{N}}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{\text{rest}}} = \cancel{\frac{\Delta}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{\text{rest}}}} + \frac{\Delta_1}{\mathcal{D}_2\mathcal{D}_{\text{rest}}} + \frac{\Delta_2}{\mathcal{D}_1\mathcal{D}_{\text{rest}}} ?$$

See also [De Laurentis, Maître '19].

- Construct Ansatz $\mathfrak{a}_l$ from study. Constrain $c_{kl}$ by finite field sampling.

$$\mathcal{C}_k(\lambda, \tilde{\lambda}) = \sum_{l=1}^{N_{\text{new}}} c_{kl}\mathfrak{a}_l(\lambda, \tilde{\lambda}), \qquad c_{kl} \in \mathbb{Q}, \qquad N_{\text{new}} \ll N.$$

# The Method, By Example

# A First Attempt at Numerical Partial Fractions

- Consider tree-level six-point one-quark line amplitude $\mathcal{A}_{q^+g^+g^+\overline{q}^-g^-g^-}$

$$\mathcal{A} = \frac{\mathcal{N}^*}{\langle 12\rangle\langle 23\rangle\langle 34\rangle[45][56][61]s_{345}}.$$

$^*\mathcal{N}$ is a degree 6 polynomial in spinor brackets.

- Can we rewrite without both $\langle 12\rangle$ and $\langle 23\rangle$ poles?

$$\mathcal{A} = \frac{\Delta_{12}}{\langle 23\rangle\langle 34\rangle[45][56][61]s_{345}} + \frac{\Delta_{23}}{\langle 12\rangle\langle 34\rangle[45][56][61]s_{345}}?$$

- [De Laurentis, Maître '19]: Probe $\mathcal{A}$ on points where $\langle 12\rangle$, $\langle 23\rangle$ are small.

$$\begin{aligned} \lambda_2^\alpha \sim \epsilon \quad &\Rightarrow \quad \mathcal{A} \sim \epsilon^{-2} \\ \langle 12\rangle \sim \langle 23\rangle \sim \langle 13\rangle \sim \epsilon \quad &\Rightarrow \quad \mathcal{A} \sim \epsilon^{-1}. \end{aligned}$$

# Thinking in Terms of Polynomials

- Let's ask an equivalent question:

$$\mathcal{N} = \Delta_{12}\langle 12\rangle + \Delta_{23}\langle 23\rangle ?$$

- Mathematically, we can ask if $\mathcal{N}$ belongs to an "ideal":

$$\mathcal{N} \in \Big\langle \langle 12\rangle, \langle 23\rangle \Big\rangle ?$$

- Ideal is infinite set of polynomial combinations of generators:

$$\Big\langle \langle 12\rangle, \langle 23\rangle \Big\rangle = \Big\{ a_1\langle 12\rangle + a_2\langle 23\rangle \quad \Big| \quad a_i \text{ are any spinor polynomials} \Big\}.$$

## Zariski Nagata Theorem

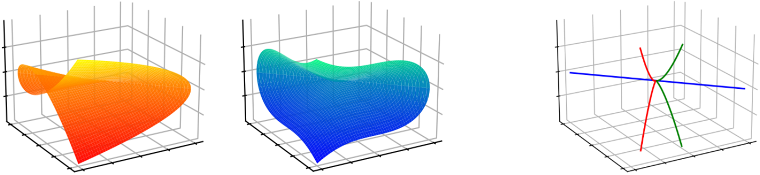
If $\mathcal{N}$ vanishes to order $k$ everywhere where $\langle 12\rangle = \langle 23\rangle = 0^{*}$ then

$$\mathcal{N} \in \Big\langle \langle 12\rangle, \langle 23\rangle \Big\rangle^{\langle k\rangle}.$$

* and $\Big\langle \langle 12\rangle, \langle 23\rangle \Big\rangle$ is radical.

# Branching of Surfaces Defined by Polynomials

- When we intersect surfaces, we may have multiple branches.



$xy^2 + y^3 - z^2 = 0$ $\quad$ $x^3 + y^3 - z^2 = 0,$ $\quad$ $xy^2 + y^3 - z^2 = x^3 + y^3 - z^2 = 0.$

- Our double denominator zero surface has two branches:

$$\langle 12 \rangle = \langle 23 \rangle = 0 \quad \Leftrightarrow \quad \langle 12 \rangle = \langle 23 \rangle = \langle 13 \rangle = 0 \quad \text{or} \quad \lambda_2^{\alpha} = 0.$$

- We compute branchings with primary decomposition techniques.

[De Laurentis, BP '22], see also [Zhang '12].

# Ansatz Construction Algorithm, Sketched

1. Construct branches of surfaces where two denominators vanish.

$$\mathcal{D}_i = \mathcal{D}_j = 0 \qquad \longrightarrow \qquad \mathcal{V} = \{U_1, U_2, \ldots\}.$$

2. Sample near surface to determine degree of numerator vanishing.

$$U: \quad \mathcal{D}_i \sim \mathcal{D}_j \sim \epsilon \qquad \Rightarrow \qquad \mathcal{N}_k \sim \epsilon^{\kappa_U}.$$

3. Ansatz is basis of intersection of associated ideals of vanishing polynomials. Ansatz constructed using Gröbner basis techniques.

$$\mathcal{N}_k \in \bigcap_{U \in \mathcal{V}} I(U)^{\langle \kappa_U \rangle}.$$

# Studying the Six-Point Tree [De Laurentis, BP; 2207.10125]

$$\mathcal{A} = \frac{\mathcal{N}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle[45][56][61]s_{345}}.$$

- Probe 108 surfaces where pairs of $\langle ij\rangle, [ij], s_{ijk}$ are small.

$$\text{e.g.} \qquad [12] \sim [13] \sim [23] \sim \mathcal{O}(\epsilon) \qquad \Rightarrow \qquad \mathcal{A} \sim \mathcal{O}(\epsilon^2).$$

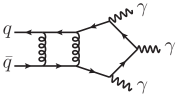
- $\mathcal{N}$ vanishes non-trivially on 28 surfaces. Many ideal memberships:

$$\mathcal{N} \quad \in \quad \Big\langle [12], [13], [23] \Big\rangle^2 \cap \Big\langle \langle 12\rangle, \langle 34\rangle \Big\rangle \cap \Big\langle \langle 12\rangle, [16] \Big\rangle \cap \text{(25 more)}.$$

- Imposing that $\mathcal{N}$ is a degree six polynomial gives one term Ansatz:

$$\mathcal{N} = c_0\Big(\langle 12\rangle[21]\langle 45\rangle[54]\langle 4|2{+}3|1] + [16]\langle 6|1{+}2|3]\langle 34\rangle s_{123}\Big), \quad c_0 \in \mathbb{Q}.$$

# Proof-of-Concept Remainders for $q\bar{q} \to \gamma\gamma\gamma$

$\mathcal{A} =$ 

$+ \cdots .$

(Simulated evaluations using analytics from [Abreu, BP, Pascual, Sotnikov '20]).

- Analyze remainder, reconstruct pentagon function coefficients.

- Reconstruction now requires 317 $\mathbb{Q}_p$ + 566 $\mathbb{F}_p$ samples.

| Amplitude | $R^{(2,0)}_{-++}$ | $R^{(2,N_f)}_{-++}$ | $R^{(2,0)}_{+++}$ | $R^{(2,N_f)}_{+++}$ |
|---|---|---|---|---|
| Ansatz Dim [Abreu et al '20] | 41301 | 2821 | 7905 | 1045 |
| Ansatz Dim [De Laurentis, BP '22] | 566 | 20 | 18 | 6 |

# Some Thoughts on Bottlenecks

# Ansatz Basis Construction in [2203.04269]

- $\mathcal{N}$ lives in space of polynomials of fixed mass-dimension/little group:

$$\mathcal{N} \in \mathcal{M}_{d,\vec{\phi}}.$$

- Approach used in [2203.04269] is to avoid intersecting ideals:

$$\text{Ansatz} = \text{basis}\left(\bigcap_{U \in \mathcal{V}} \left[\mathcal{M}_{d,\vec{\phi}} \cap I(U)^{\langle \kappa_U \rangle}\right]\right).$$

- Bottlenecks:
  - Many, large vector space intersections.
  - Ansatz picked by RREF. $c_i$ reconstruction needs many finite fields.

$$\mathcal{N} = \sum_i c_i \mathfrak{a}_i, \qquad (c_i \mod p_1, c_i \mod p_2, \ldots) \xrightarrow{\text{CRT}} c_i.$$

Improvement path: Can we analytically intersect ideals in controlled way?

# Regular Sequences

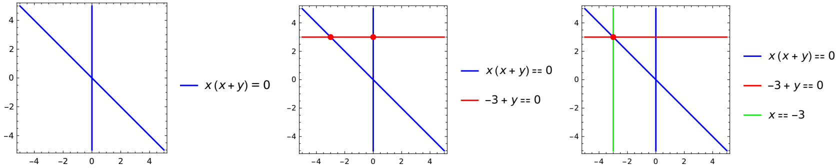
- Ideal intersection is combinatoric when using a regular sequence:

$\{q_1, \ldots, q_n\}$ where $q_{i+1}$ is **non-zero divisor** mod $\{q_1, \ldots, q_i\}$, $i \in [0, n-1]$.

- $\{\langle 12\rangle, \langle 13\rangle, \langle 14\rangle\}$ is not a regular sequence because of Schouten.

$$\langle 14\rangle\langle 23\rangle = \langle 12\rangle\langle 34\rangle - \langle 13\rangle\langle 24\rangle.$$

- Geometrically, sequential intersections decrease dimension.
  E.g. $\{x(x+y), y-3\}$ is regular sequence, $\{x(x+y), y-3, x+3\}$ is not:



# Improving Basis Construction

- Regular sequence $\{q_1, \ldots, q_n\}$. Monomial ideals intersect as

$$\left\langle \prod_i q_i^{\alpha_i} : \vec{\alpha} \in A \right\rangle \cap \left\langle \prod_i q_i^{\beta_i} : \vec{\beta} \in B \right\rangle = \left\langle \mathrm{lcm}\left( \prod_i q_i^{\alpha_i}, \prod_i q_i^{\beta_i} \right) : \vec{\alpha} \in A, \vec{\beta} \in B \right\rangle . \quad (*)$$

- E.g., $\{\langle 12\rangle, [12], s_{123}\}$ is regular sequence:

$$\langle \langle 12\rangle, s_{123}^2 \rangle \cap \langle [12], s_{123} \rangle = \langle \langle 12\rangle[12], s_{123}^2, \langle 12\rangle s_{123}, [12]s_{123}^2 \rangle.$$

- **Strategy**:
  1. Organize intersection into denominator ideals.

$$\bigcap_{U \in \mathcal{V}} I(U)^{\langle \kappa_U \rangle} = \bigcap_{(a,b)} \langle \mathcal{D}_a, \mathcal{D}_b \rangle^{k_{ab}}.$$

  2. Apply (*) to subsets of intersectands built from regular sequences.

# Closing Remarks

**Summary:**

- Rational functions in amplitudes have poorly understood structure.
- We study them in singular limits to characterize that structure. We interpret this behavior in terms of ideals from which we build Ansätze.

**Looking forward:**

- Can we get even better at intersecting these ideals?

(Does there exist a "next-to-regular" sequence?)

- What is the physical origin of the partial fractions structure?

# Backup

# Lorentz Invariance

- Coefficients are Lorentz invariant functions of spinor brackets.

$$\mathcal{C}(\lambda, \tilde{\lambda}) = \mathcal{C}(\langle\rangle, []).$$

- Relevant ring is Lorentz invariant subring of $S_n$.

$$\mathcal{S}_n = \mathbb{F}\Big[\langle 12\rangle, \ldots, \langle (n-1)n\rangle, [12], \ldots [(n-1)n]\Big].$$

- Variables are brackets, now have "Schouten identities".

$$\mathcal{J}_{\Lambda_n} = \left\langle \sum_{j=1}^{n} \langle ij\rangle[jk], \langle ij\rangle\langle kl\rangle - \langle ik\rangle\langle jl\rangle - \langle il\rangle\langle kj\rangle, \langle\rangle \leftrightarrow [] \right\rangle.$$

- Physical spinor bracket functions also form a quotient ring.

$$\mathcal{R}_n = \mathcal{S}_n / \mathcal{J}_{\Lambda_n}.$$

# Bases of Spinor Space and Polynomial Reduction

- Numerators are $\mathbb{Q}-$linear combinations of spinor monomials.

$$m_\alpha = \prod_i v_i^{\alpha_i} \qquad \text{where} \quad \vec{v} = \{\langle 12\rangle, \langle 23\rangle, \ldots [12], [23], \ldots\}.$$

- Polynomial reduction writes $p$ in terms of generators $g_i$.

$$p = \Delta_{\{g_1,\ldots,g_k\}}(p) + \sum_{i=1}^{k} c_i g_i.$$

- Polynomial in ideal if and only if Groebner remainder is 0.

$$\Delta_{\mathcal{G}(J)}(p) = 0 \qquad \Leftrightarrow \qquad p \in J.$$

- Monomials irreducible by $\mathcal{G}(\mathcal{J}_{\Lambda_n})$ form basis. Related [Zhang '12]

$$\text{basis} = \{m_\alpha \text{ such that } \Delta_{\mathcal{G}(\mathcal{J}_{\Lambda_n})}(m_\alpha) = m_\alpha\}.$$

# How To Perform Numerical Investigations?

- Need to find phase-space points $(\lambda^\epsilon, \tilde{\lambda}^\epsilon)$ where $\mathcal{D}_i$ are small.

$$\mathcal{D}_i(\lambda^\epsilon, \tilde{\lambda}^\epsilon) \quad \sim \quad \mathcal{D}_j(\lambda^\epsilon, \tilde{\lambda}^\epsilon) \quad \sim \quad \epsilon.$$

- Conflict with modern techniques: no small elements in a finite field.

$$|0|_{\mathbb{F}_p} = 0, \qquad \text{and} \qquad a \neq 0 \;\Rightarrow\; |a|_{\mathbb{F}_p} = 1.$$

- Approaching with complex numbers would be plagued by instabilities.

Enter the p-adic numbers – a middle ground between finite fields and $\mathbb{C}$.

# Introduction to the $p$-adic Numbers

- The $p$-adic numbers roughly correspond to Laurent series in $p$.

$$x = \sum_{i=\nu}^{\infty} a_i p^i = a_\nu p^\nu + a_{\nu+1} p^{\nu+1} + \cdots, \qquad \left( \begin{smallmatrix} a_i \in [0, p-1], \\ a_\nu \neq 0. \end{smallmatrix} \right).$$

- The $p$-adic numbers form a field. $x, y \in \mathbb{Q}_p \Rightarrow$

$$x + y \in \mathbb{Q}_p, \quad -x \in \mathbb{Q}_p, \quad x \times y \in \mathbb{Q}_p, \quad \frac{1}{x} \in \mathbb{Q}_p \text{ (if } x \neq 0).$$

- The $p$-adic absolute value allows for small numbers ($p \sim \epsilon$).

$$|x|_p = p^{-\nu}, \quad \Rightarrow \quad |p|_p < |1|_p.$$

# Computing with $p$-adic Numbers

- For computing purposes* we truncate to finite order.

$$x = p^{\nu(x)}\Big( \underbrace{\tilde{x}}_{\text{mantissa}} + \mathcal{O}(p^k)\Big).$$

*Try [https://github.com/GDeLaurentis/pyadic] to investigate yourselves.

- Truncation reduces to finite field case for $\nu = 0, k = 1$.

- Arithmetic $(+ - / *)$ is essentially performed modulo $p^k$, e.g.

$$x \times y = p^{\nu(x)+\nu(y)}\left(\tilde{x}\tilde{y} + \mathcal{O}(p^k)\right).$$

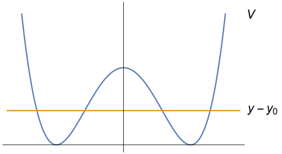
- Mantissa inverse computed with extended euclidean algorithm.

# P-adic (Integer) Points Near an Irreducible Variety

- Want to find $(\lambda^{(\epsilon)}, \tilde{\lambda}^{(\epsilon)})$ "close" to $U = V(\langle q_1, \ldots, q_m \rangle_{R_n})$:

$$q_i\left(\lambda^{(\epsilon)}, \tilde{\lambda}^{(\epsilon)}\right) = pc_i + \mathcal{O}(p^k), \qquad \sum_{i=1}^{n} \lambda^{(\epsilon)}_{i\alpha} \tilde{\lambda}^{(\epsilon)}_{i\dot{\alpha}} = 0 + \mathcal{O}(p^k).$$

- First, find finite field $x \in U$ by intersecting with random plane.




- Arbitrarily extend $\mathbb{F}_p$ point $(\lambda, \tilde{\lambda})$ to $k$ digits. Trivially near $U$.

- To satisfy momentum conservation, perturb by $(p\delta, p\tilde{\delta})$.

$$(\lambda^{(\epsilon)}, \tilde{\lambda}^{(\epsilon)}) = (\lambda + p\delta, \tilde{\lambda} + p\tilde{\delta}).$$

# Polynomials that Vanish on a Variety

- Polynomials that vanish on all points of $U$ form an ideal

$$I(U) = \left\{ q \in S_n \quad \text{where} \quad q(x) = 0 \quad \text{for all } x \in U \right\}.$$

- Consider if $\mathcal{N}_i$ vanishes to order $k_U$ on $U$,

$$\mathcal{N}_i(x^{(\epsilon)}) = \mathcal{O}(\epsilon^{k_U}), \quad \text{where} \quad |x - x^{(\epsilon)}| \leq \epsilon \quad \text{and} \quad x \in U.$$

- It turns out that $\mathcal{N}_i$ still belongs to an ideal!

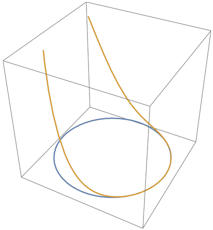
## Zariski-Nagata Theorem

Polynomials vanishing to $\mathcal{O}(k_U)$ on $U$ belong to $I(U)^{\langle k_U \rangle}$ – the $k_U$th "symbolic power" of $I(U)$.

- Computed from primary decomposition of ideal power $I(U)^{k_U}$.

# Examples of Symbolic Powers

- A function vanishing to fourth order at a point on the circle:

$$\langle x-1\rangle^{\langle 4\rangle}_{\mathbb{F}[x,y]/\langle x^2+y^2-1\rangle} \sim$$  .

- Often the symbolic power coincides with standard power, e.g.

$$\langle\langle 12\rangle,[12]\rangle^{\langle 2\rangle}_{R_5} = \langle\langle 12\rangle,[12]\rangle^{2}_{R_5} = \langle\langle 12\rangle^2,\langle 12\rangle[12],[12]^2\rangle_{R_5}.$$

- Symbolic/standard power may not coincide. E.g. in $\mathbb{F}[x,y,z]$

$$\langle xy,xz,yz\rangle^{\langle 2\rangle} = \langle x^2y^2,x^2z^2,y^2z^2,xyz\rangle \neq \langle xy,xz,yz\rangle^2$$

# The $p$-adic Logarithm

- Over the $p-adic$ numbers, one can define converging power series.
- The power series for a logarithm converges for $|x|_p < 1$.

$$\log_p(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$$

- To map to radius of convergence, use Fermat's little theorem.

$$w^{p-1} = 1 \mod p \qquad \Rightarrow \qquad |w^{p-1} - 1|_p < 1$$

- Logarithm relations then $p$-adically analytically continue $\log_p$.

$$\log_p(w) = \frac{1}{p-1} \log(w^{p-1})$$