Mathematical Structures in Feynman Integrals
Emmy Noether Campus, Siegen, February 15, 2023

Computer Algebra Methods for Feynman Integrals

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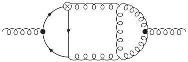
JOHANNES KEPLER UNIVERSITY LINZ

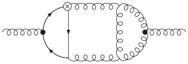
Der Wissenschaftsfonds.

Evaluation of Feynman Integrals



behavior of particles

Evaluation of Feynman Integrals



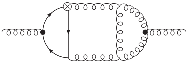
behavior of particles

$$\longrightarrow$$

$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

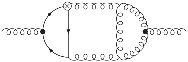
Feynman integrals

DESY

$$\sum f(N,\epsilon,k)$$

complicated
multi-sums

Evaluation of Feynman Integrals



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$$\int \Phi(N, \epsilon, x) dx$$

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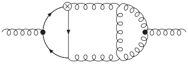
complicated multi-sums

RISC

(Sigma-package)

expression in special functions

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

DESY

LHC at CERN

$$\sum f(N,\epsilon,k)$$

complicated multi-sums

RISC

(Sigma-package)

expression in special functions

applicable

$$F(\varepsilon, N) = \iiint \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||?$$

$$F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + F_0(N)\varepsilon^0 + \dots$$

$$F(\varepsilon, N) = \iiint \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\Big(-1-\frac{3\varepsilon}{2}\Big) \times$$
$$\times B\Big(2+k, \frac{\varepsilon}{2}\Big) B(-\varepsilon+k, -\varepsilon) B\Big(1-\frac{\varepsilon}{2}+k, 1+\frac{\varepsilon}{2}\Big) \binom{N}{k}$$

where

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

$$F(\varepsilon, N) = \iiint \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{= f_{-3}(N,k)\varepsilon^{-3} + f_{-2}(N,k)\varepsilon^{-2} + f_{-1}(N,k)\varepsilon^{-1} + \ldots}$$

for general expansion methods see
J. Blümlein, CS, M. Saragnese, 2021. arXiv:2111.15501 [math-ph]

$$F(\varepsilon, N) = \iiint \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{= f_{-3}(N,k)\varepsilon^{-3}+f_{-2}(N,k)\varepsilon^{-2}+f_{-1}(N,k)\varepsilon^{-1}+\ldots}$$

$$||$$

$$\Big(\sum_{k=1}^{N} f_{-3}(N,k)\Big)\varepsilon^{-3} + \Big(\sum_{k=1}^{N} f_{-2}(N,k)\Big)\varepsilon^{-2} + \Big(\sum_{k=1}^{N} f_{-1}(N,k)\Big)\varepsilon^{-1} + \ldots$$

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$$\underbrace{\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{=f_{-3}(N,k)\varepsilon^{-3}+f_{-2}(N,k)\varepsilon^{-2}+f_{-1}(N,k)\varepsilon^{-1}+\dots}$$

$$||$$

$$\Big(\sum_{k=1}^{N} f_{-3}(N,k)\Big)\varepsilon^{-3} + \Big(\sum_{k=1}^{N} f_{-2}(N,k)\Big)\varepsilon^{-2} + \Big(\boxed{\sum_{k=1}^{N} f_{-1}(N,k)}\Big)\varepsilon^{-1} + \dots$$

Simplify

$$F_{-1}(N) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \Big)$$

where

$$S_a(N) = \sum_{i=1}^{N} \frac{\mathsf{sign}(a)^i}{i^a} \text{ and } \zeta_a = \sum_{i=1}^{\infty} \frac{1}{i^a}$$

Simplify

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$\downarrow$ (summation package `Sigma.m`)

$$\begin{aligned}&\big(16N^3+144N^2+413N+384\big)(N+1)^2F_{-1}(N)\\&\quad-(N+2)(2N+5)\big(16N^3+112N^2+221N+113\big)F_{-1}(N+1)\\&\quad+(N+3)^2\big(16N^3+96N^2+173N+99\big)F_{-1}(N+2)\\&\quad=\frac{1}{2}\big(4N^2+21N+29\big)\zeta_2+\frac{-64N^5-500N^4-1133N^3+203N^2+3516N+3090}{3(N+2)(N+3)}\end{aligned}$$

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$$\begin{aligned}&\Big\{c_1\frac{1-4N}{N+1}+c_2\frac{-14N-13}{(N+1)^2}\\&\quad+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}\\&\quad+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}\Big|c_1,c_2\in\mathbb{Q}\Big\}\end{aligned}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$$\bigcap$$

$$\Big\{c_1\frac{1-4N}{N+1}+c_2\frac{-14N-13}{(N+1)^2}$$
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$||$ (recurrence finding and solving)

$$\Big(\frac{1}{12}-\frac{1}{8}\zeta_2\Big)\frac{1-4N}{N+1}+1\,\frac{-14N-13}{(N+1)^2}$$
$$+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}$$
$$+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N) = \sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(N), \dots, a_d(N), h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N) + \cdots + a_d(N)F(N+d) = h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

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Special cases:

$$S_{2,1}(n) = \sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}$$ (harmonic sums)

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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Special cases:

$$\sum_{k=1}^{n}\frac{2^k}{k}\sum_{i=1}^{k}\frac{2^{-i}}{i}\sum_{j=1}^{i}\frac{S_1(j)}{j}$$

(generalized harmonic sums)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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Special cases:

$$\sum_{k=1}^{n}\frac{1}{(1+2k)^2}\sum_{j=1}^{k}\frac{1}{j^2}\sum_{i=1}^{j}\frac{1}{1+2i}$$

(cyclotomic harmonic sums)

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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Special cases:

$$\sum_{j=1}^{n}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2} \qquad \text{(binomial sums)}$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$\sum_{h=1}^{n} 2^{-2h}(1-\eta)^h \binom{2h}{h} \sum_{k=1}^{h} \frac{2^{2k}}{k^2\binom{2k}{k}}$$

(generalized binomial sums)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

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A more general example:

$$\sum_{k=1}^{n} \left(\prod_{i=1}^{k} \frac{1+i+i^2}{i+1} \right) \sum_{j=1}^{k} \frac{1}{j \binom{4j}{3j}^2}$$

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3. Find a "closed form"

F(N)=combined solutions in terms of indefinite nested sums.

Sigma.m is based on difference ring/field theory

1. M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
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3. M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A = B$. A. K. Peters, Wellesley, MA, 1996.
4. P. A. Hendriks and M. F. Singer. Solving difference equations in finite terms. *J. Symbolic Comput.*, 27(3):239–259, 1999.
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28. CS. Refined telescoping algorithms in R$\Pi\Sigma$-extensions to reduce the degrees of the denominators, 2023. arXiv:2302.03563 [cs.SC].

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

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In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left[2+k,\frac{\varepsilon}{2}\right]B[-\varepsilon+k,-\varepsilon]B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{N}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{N\}, \{1\}, ExpandIn \to \{\varepsilon, -3, -3\}]}$

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Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Big(-2-\frac{3\varepsilon}{2}\Big)!\,B\Big[2+k,\frac{\varepsilon}{2}\Big]B[-\varepsilon+k,-\varepsilon]B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{N\}, \{1\}, ExpandIn \to \{\varepsilon, -3, -3\}]}$

Out[5]= $\Big\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)}\Big\}$

In[1]:= $<<$ **Sigma.m**

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In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{N}(-1)^{k}e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left[2+k,\frac{\varepsilon}{2}\right]B[-\varepsilon+k,-\varepsilon]B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{N}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{N\}, \{1\}, ExpandIn \to \{\varepsilon, -3, -2\}]}$

Out[5]=
$$\Big\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)},$$
$$-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1}\Big\}$$

In[1]:= $<<$ **Sigma.m**

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In[2]:= $<<$ **HarmonicSums.m**

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In[4]:= **mySum =**

$$\sum_{k=1}^{N}(-1)^{k}e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left[2+k,\frac{\varepsilon}{2}\right]B[-\varepsilon+k,-\varepsilon]B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{N}{k};$$

In[5]:= **EvaluateMultiSum[mySum, {}, {N}, {1}, ExpandIn → {ε, −3, −1}]**

Out[5]=
$$\Big\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)},$$
$$-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1},$$
$$\left(\frac{1}{12}-\frac{1}{8}\zeta(2)\right)\frac{1-4N}{N+1}+\frac{-14N-13}{(N+1)^2}+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+$$
$$\frac{(14N+13)S_1(N)}{3(N+1)^2}+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta(2)}{8(N+1)}\Big\}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)$$

$$||$$

$$\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

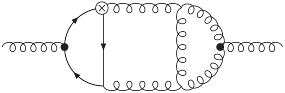
$$||$$

$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

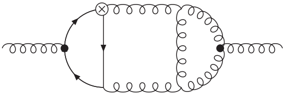
$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\boxed{\begin{aligned}&\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times\\ &\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}\\ &\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)\\ &-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))\\ &+2S_1(s-1)-2S_1(r+s)\Big]+\textbf{3 further 6–fold sums}\end{aligned}}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+\Big(-\frac{4(13N+5)}{N^2(N+1)^2} + \Big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$
$$+\big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$
$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$
$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$
$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(\cdots \frac{(17N+5)S_1(N)^3}{\cdots)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$+\Big(-\cdots \frac{\cdots 1)^N(2N+1)}{\cdots N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+\big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 - \frac{(17N+5)S_1(N)^3}{\ldots)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \Big(-\ldots\frac{\ldots 1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{\ldots} - (-1)^N\Big)S_3(N)$$

$$+ \big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N\ldots}{N^2(N+1\ldots} \ldots {}^N\big)S_2(N)^2$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(\ldots \frac{\ldots)^N}{N+1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N}\frac{1}{i^2}$$

$F_0(N) =$

$$\frac{7}{12}S_1(N)^4 \ldots \frac{(17N+5)S_1(N)^3}{\ldots} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \Big(- \ldots \frac{\ldots 1)^N(2N+1)}{\ldots N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{\ldots} - (-1)^N\Big)S_3(N)$$

$$+ \big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N \ldots}{N^2(N+1 \ldots} \ldots {}^N\big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(\ldots \frac{\ldots)^N}{N+1}\Big)$$

$$+ \Big(\frac{(-1)^N \ldots}{2N^2 \ldots} \ldots)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N - \ldots}{N(N+ \ldots} \ldots -1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N \ldots}{N(\ldots} \ldots (N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(- \ldots}{N(N+1)} \ldots \Big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N} \frac{1}{i^2}$$

$$S_{-2,1,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i^2}$$

The general tactic

Feynman integrals

The general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

The general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

$\downarrow$ symbolic summation

compact expression in terms of special functions

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 1: Expand the summand and simplify

Ablinger, Blümlein, Klein, CS, LL2010, arXiv:1006.4797 [math-ph]
Blümlein, Hasselhuhn, CS, RADCOR'10, arXiv:1202.4303 [math-ph]
CS, ACAT 2013, arXiv:1310.0160 [cs.SC]

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 2: Expand a recurrence in ε

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

$\big\downarrow$ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

$\big\downarrow$ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\big(\frac{\zeta_2}{4}+\frac{79}{24}\big)\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\big(\frac{\zeta_2}{3}+\frac{1415}{324}\big)\varepsilon^{-1}+\dots$$

$\big\downarrow$

$$\boxed{F(N)=F_{-3}(N)\,\varepsilon^{-3}+F_{-2}(N)\,\varepsilon^{-2}+F_{-1}(N)\,\varepsilon^{-1}+\dots}$$

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

↓ (summation package Sigma.m)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\big(\frac{\zeta_2}{4}+\frac{79}{24}\big)\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\big(\frac{\zeta_2}{3}+\frac{1415}{324}\big)\varepsilon^{-1}+\dots$$

↓ (summation package Sigma.m)

$$F(N)=\tfrac{4N}{3(N+1)}\varepsilon^{-3}-\Big(\tfrac{2(2N+1)}{3(N+1)}S_1(N)+\tfrac{2N(2N+3)}{3(N+1)^2}\Big)\varepsilon^{-2}$$
$$\Big(\tfrac{(1-4N)}{6(N+1)}S_1(N)^2-\tfrac{N\left(N^2-2\right)}{3(N+1)^3}+\tfrac{(3N+2)(4N+5)}{3(N+1)^2}S_1(N)+\tfrac{(1-4N)}{6(N+1)}S_2(N)+\tfrac{N\zeta_2}{2(N+1)}\Big)\varepsilon^{-1}+\dots$$

Find a recurrence for the integral

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

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The integrand is

- hyperexponential in x,y,z:

$$\frac{D_x f(\varepsilon,n,x,y,z)}{f(\varepsilon,n,x,y,z)} \in \mathbb{Q}(\varepsilon,n,x,y,z)$$

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The integrand is

- hyperexponential in x,y,z:

$$\frac{D_y f(\varepsilon,n,x,y,z)}{f(\varepsilon,n,x,y,z)} \in \mathbb{Q}(\varepsilon,n,x,y,z)$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

The integrand is

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Example: A master integral from Ladder and V-topologies

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$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

The integrand is

- hyperexponential in x,y,z:
- hypergeometric in n:

$$\frac{f(\varepsilon,n+1,x,y,z)}{f(\varepsilon,n,x,y,z)}\in\mathbb{Q}(\varepsilon,n,x,y,z)$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

$$\Big\downarrow \text{ Ablinger's MultiIntegrate.m (9 hours)}$$

$$a_0(\varepsilon, n) F(\varepsilon, n) + a_1(\varepsilon, n) F(\varepsilon, n+1) + \cdots + a_5(\varepsilon, n) F(\varepsilon, n+5) = 0$$

$$
\begin{aligned}
a_0(N,\varepsilon) = {}& (N+1)(N+2)\big(8\varepsilon^{10} + 104\varepsilon^9(N+3) + 4\varepsilon^8\big(96N^2 + 601N + 887\big) \\
&+ 4\varepsilon^7\big(12N^3 + 414N^2 + 1583N + 1393\big) \\
&- 8\varepsilon^6\big(264N^4 + 2436N^3 + 8643N^2 + 14518N + 9947\big) \\
&- 16\varepsilon^5\big(156N^5 + 1690N^4 + 6847N^3 + 12661N^2 + 9537N + 717\big) \\
&+ 32\varepsilon^4\big(68N^6 + 1158N^5 + 8155N^4 + 30114N^3 + 61712N^2 + 67616N + 31693\big) \\
&+ 64\varepsilon^3\big(40N^7 + 560N^6 + 2755N^5 + 3729N^4 - 14194N^3 - 61920N^2 - 89140N - 46600\big) \\
&- 128\varepsilon^2(N+2)\big(12N^7 + 254N^6 + 2249N^5 + 10758N^4 + 30173N^3 + 50610N^2 \\
&+ 49122N + 22706\big) \\
&+ 256\varepsilon(N+2)^2(N+3)(N+4)\big(44N^4 + 501N^3 + 2044N^2 + 3455N + 1976\big) \\
&- 512(N+1)(N+2)^3(N+3)^2(N+4)\big(6N^2 + 47N + 95\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_1(N,\varepsilon) = (N+2)\big(&-22\varepsilon^{11} - 2\varepsilon^{10}(157N+435) - \varepsilon^9\big(1500N^2+8611N+11745\big)\\
&- \varepsilon^8\big(2548N^3+22936N^2+63597N+54229\big)\\
&+ 4\varepsilon^7\big(266N^4+1857N^3+6065N^2+14351N+15987\big)\\
&+ 8\varepsilon^6\big(994N^5+12961N^4+67246N^3+174692N^2+226821N+116092\big)\\
&+ 16\varepsilon^5\big(336N^6+5348N^5+33569N^4+104918N^3+165290N^2+108259N+6100\big)\\
&- 16\varepsilon^4\big(404N^7+7578N^6+61778N^5+284762N^4+802660N^3+1382074N^2\\
&+ 1340455N+560287\big)\\
&- 64\varepsilon^3\big(94N^8+1823N^7+14305N^6+55870N^5+96299N^4-37256N^3\\
&- 447044N^2-704959N-379338\big)\\
&+ 128\varepsilon^2(N+3)\big(30N^8+715N^7+7667N^6+48253N^5+194086N^4+507439N^3\\
&+ 835393N^2+785327N+320382\big)\\
&- 256\varepsilon(N+2)(N+3)^2\big(107N^6+2070N^5+16342N^4+67226N^3+151557N^2\\
&+ 176932N+83196\big)\\
&+ 256(N+2)^3(N+3)^3(N+4)\big(30N^3+331N^2+1193N+1386\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_2(N,\varepsilon) = \big(&12\varepsilon^{12} + 12\varepsilon^{11}(17N+45) + 2\varepsilon^{10}\big(620N^2+3553N+4795\big) \\
&+ 2\varepsilon^9\big(1504N^3+14190N^2+41901N+38907\big) \\
&+ 4\varepsilon^8\big(172N^4+4983N^3+30942N^2+69119N+50850\big) \\
&- 4\varepsilon^7\big(1996N^5+24056N^4+113313N^3+269119N^2+337198N+185290\big) \\
&- 16\varepsilon^6\big(450N^6+8210N^5+59749N^4+227386N^3+486841N^2+563176N+275664\big) \\
&+ 16\varepsilon^5\big(340N^7+4314N^6+19137N^5+25532N^4-55105N^3-206516N^2-191528N \\
&- 23458\big) \\
&+ 32\varepsilon^4\big(140N^8+2940N^7+26550N^6+139926N^5+493839N^4+1240186N^3 \\
&+ 2161699N^2+2304248N+1100084\big) \\
&+ 64\varepsilon^3\big(4N^9+506N^8+8651N^7+63510N^6+236215N^5+395334N^4-105413N^3 \\
&- 1551017N^2-2362944N-1217770\big) \\
&- 128\varepsilon^2(N+3)\big(12N^9+314N^8+3782N^7+29105N^6+160727N^5+640273N^4 \\
&+ 1750874N^3+3052505N^2+3017094N+1276604\big) \\
&+ 256\varepsilon(N+2)(N+3)^2(N+4)\big(26N^6+825N^5+8967N^4+46529N^3+125411N^2 \\
&+ 168628N+88652\big) \\
&- 512(N+1)(N+2)^2(N+3)^3(N+4)^2\big(6N^3+98N^2+459N+655\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_3(N,\varepsilon) = \big(&- 64\varepsilon^{12} - 8\varepsilon^{11}(113N + 298) - 8\varepsilon^{10}\big(519N^2 + 2948N + 3896\big) \\
&- 4\varepsilon^9\big(1444N^3 + 13839N^2 + 39746N + 34305\big) \\
&+ 4\varepsilon^8\big(1948N^4 + 17868N^3 + 63837N^2 + 112966N + 84655\big) \\
&+ 16\varepsilon^7\big(1456N^5 + 20460N^4 + 112365N^3 + 304963N^2 + 412258N + 221769\big) \\
&- 8\varepsilon^6\big(320N^6 + 2050N^5 + 4192N^4 + 27408N^3 + 174901N^2 + 411759N + 324872\big) \\
&- 16\varepsilon^5\big(1756N^7 + 33154N^6 + 265889N^5 + 1186719N^4 + 3218059N^3 + 5349388N^2 \\
&+ 5071913N + 2113696\big) \\
&+ 32\varepsilon^4\big(188N^8 + 4802N^7 + 59527N^6 + 439922N^5 + 2025336N^4 + 5813984N^3 \\
&+ 10076450N^2 + 9621283N + 3878602\big) \\
&+ 64\varepsilon^3\big(140N^9 + 2768N^8 + 22500N^7 + 99545N^6 + 287700N^5 + 723136N^4 \\
&+ 1854572N^3 + 3714620N^2 + 4272517N + 2031600\big) \\
&- 128\varepsilon^2\big(24N^{10} + 830N^9 + 14362N^8 + 152630N^7 + 1053620N^6 + 4834279N^5 \\
&+ 14824351N^4 + 29964399N^3 + 38244797N^2 + 27875896N + 8824032\big) \\
&+ 256\varepsilon(N+2)(N+3)(N+4)\big(118N^7 + 2639N^6 + 24247N^5 + 118311N^4 + 329565N^3 \\
&+ 520306N^2 + 426076N + 136854\big) \\
&- 512(N+1)(N+2)^2(N+3)^2(N+4)^2(N+5)\big(12N^3 + 97N^2 + 230N + 144\big)\big),
\end{aligned}
$$

$$\begin{aligned}
a_4(N,\varepsilon) = \Big(&64\varepsilon^{12} + 192\varepsilon^{11}(5N+14) + 16\varepsilon^{10}\big(297N^2 + 1769N + 2451\big) \\
&+ 16\varepsilon^9\big(453N^3 + 4462N^2 + 13094N + 11244\big) \\
&- 8\varepsilon^8\big(1084N^4 + 11117N^3 + 47258N^2 + 103981N + 94650\big) \\
&- 8\varepsilon^7\big(3304N^5 + 51138N^4 + 311957N^3 + 948722N^2 + 1440105N + 858544\big) \\
&+ 16\varepsilon^6\big(420N^6 + 5507N^5 + 36275N^4 + 169650N^3 + 536911N^2 + 952507N + 694370\big) \\
&+ 16\varepsilon^5\big(1828N^7 + 38868N^6 + 353301N^5 + 1801014N^4 + 5604391N^3 + 10664390N^2 \\
&+ 11433064N + 5260048\big) \\
&- 32\varepsilon^4\big(316N^8 + 8356N^7 + 105800N^6 + 802421N^5 + 3836854N^4 + 11588223N^3 \\
&+ 21401558N^2 + 22066744N + 9745752\big) \\
&- 64\varepsilon^3\big(116N^9 + 2424N^8 + 19923N^7 + 82966N^6 + 208191N^5 + 530980N^4 + 1847484N^3 \\
&+ 4687014N^2 + 6120858N + 3111104\big) \\
&+ 128\varepsilon^2\big(24N^{10} + 826N^9 + 14897N^8 + 172000N^7 + 1314686N^6 + 6710299N^5 \\
&+ 22873183N^4 + 51298261N^3 + 72551278N^2 + 58573022N + 20544948\big) \\
&- 256\varepsilon(N+2)(N+3)\big(106N^8 + 3278N^7 + 42903N^6 + 310942N^5 + 1366350N^4 \\
&+ 3729418N^3 + 6173159N^2 + 5657732N + 2191212\big) \\
&+ 512(N+1)(N+2)^2(N+3)^2(N+4)(N+5)(N+6)\big(12N^3 + 121N^2 + 396N + 431\big)\Big),
\end{aligned}$$

$$
\begin{aligned}
a_5(N,\varepsilon) = (N+5)\big(&- 128\varepsilon^{11} - 128\varepsilon^{10}(11N+26) - 32\varepsilon^{9}\big(115N^2+592N+647\big) \\
&+ 32\varepsilon^{8}\big(63N^3+430N^2+1665N+2384\big) \\
&+ 16\varepsilon^{7}\big(714N^4+7881N^3+33802N^2+66225N+47654\big) \\
&- 16\varepsilon^{6}\big(234N^5+2444N^4+13989N^3+50862N^2+104083N+87848\big) \\
&- 16\varepsilon^{5}\big(580N^6+10181N^5+76586N^4+319207N^3+772120N^2+1012046N+547832\big) \\
&+ 16\varepsilon^{4}\big(244N^7+5456N^6+61605N^5+401216N^4+1536277N^3+3408574N^2 \\
&+ 4066436N+2026928\big) \\
&+ 64\varepsilon^{3}\big(26N^8+357N^7+583N^6-11139N^5-65193N^4-120264N^3+11864N^2 \\
&+ 272830N+222624\big) \\
&- 64\varepsilon^{2}(N+3)\big(12N^8+298N^7+4684N^6+49024N^5+306907N^4+1122441N^3 \\
&+ 2350650N^2+2607576N+1185072\big) \\
&+ 256\varepsilon(N+2)(N+3)\big(25N^7+743N^6+8856N^5+55358N^4+197497N^3+404131N^2 \\
&+ 439902N+196128\big) \\
&- 256(N+1)(N+2)^2(N+3)^2(N+4)(N+6)(N+7)\big(6N^2+35N+54\big)\big).
\end{aligned}
$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon, n) F(\varepsilon, n) + a_1(\varepsilon, n) F(\varepsilon, n+1) + \cdots + a_5(\varepsilon, n) F(\varepsilon, n+5) = 0$$

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Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

`Sigma.m` $\downarrow$ (2 hours)

$$F(\varepsilon,n)=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\cdots+F_4(n)\varepsilon^4+O(\varepsilon^5)$$

We get
$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$

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$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

$$F_{-2}(n) = -\frac{4(-1)^n\left(3n^3+18n^2+31n+18\right)}{3(n+1)^3(n+2)^2} - \frac{4\left(6n^3+32n^2+51n+26\right)}{3(n+1)^3(n+2)^2}$$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

$$F_{-2}(n) = -\frac{4(-1)^n\left(3n^3+18n^2+31n+18\right)}{3(n+1)^3(n+2)^2} - \frac{4\left(6n^3+32n^2+51n+26\right)}{3(n+1)^3(n+2)^2}$$

$$\begin{aligned}F_{-1}(n) = (-1)^n &\left(\frac{2\left(9n^5+81n^4+295n^3+533n^2+500n+204\right)}{3(n+1)^4(n+2)^3} + \frac{\zeta_2}{(n+1)(n+2)}\right)\\ &+ \frac{2\left(18n^5+150n^4+490n^3+755n^2+536n+132\right)}{3(n+1)^4(n+2)^3} + \frac{(2n+3)\zeta_2}{(n+1)^2(n+2)}\\ &+ \left(-\frac{4}{(n+1)^2(n+2)} + \frac{4(-1)^n}{(n+1)(n+2)}\right) S_2(n)\\ &+ \left(\frac{4(-1)^n}{3(n+1)(n+2)} - \frac{4(n+9)}{3(n+1)^2(n+2)}\right) S_{-2}(n)\end{aligned}$$

Find a recurrence for the integral

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

Find a recurrence for the integral

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
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ε-recurrence solver

multivariate Almquist/Zeilberger (J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \ldots, i_7)$$

Wegschaider's MultiSum Package (F. Stan)

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

Find a recurrence for the integral

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots$$

ε-recurrence solver

multivariate Almquist/Zeilberger (J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \ldots, i_7)$$

Wegschaider's MultiSum Package (F. Stan)

Holonomic/difference field approach (M. Round)

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 2: Expand a recurrence in ε

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

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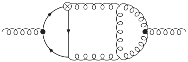
Evaluation of Feynman Integrals

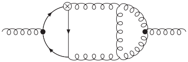


Behavior of particles

$$\longrightarrow \int \Phi(N, \epsilon, x) dx$$

Feynman integrals

Evaluation of Feynman Integrals



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DESY

$$Dy = A\, y$$

coupled systems of linear DEs

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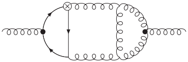

Behavior of particles → $\int \Phi(N, \epsilon, x) dx$

Feynman integrals

↓ **DESY**

$$Dy = A\,y$$

coupled systems of linear DEs

← **RISC** (new coupled system solver)

expression in special functions

Tactic 3: Solve coupled systems of differential equations

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

given

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$\big\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \cdots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

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2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \text{LinComb}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \text{LinComb}(..., D^i\hat{R}_i(x), \dots)$$

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$\swarrow$

DE-solver

A differential equation solver (`HarmonicSums.m`)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \dots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

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DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

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Special cases of iterated integrals over hyperexponential functions:

$$H_{1,-1}(x) = \int_0^x \frac{1}{1-\tau_1} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1$$ (harmonic polylogarithms)

E. Remiddi, E. and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) [arXiv:hep-ph/9905237]

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Special cases of iterated integrals over hyperexponential functions:

$$H_{2,-2}(x) = \int_0^x \frac{1}{2-\tau_1} \int_0^{\tau_1} \frac{1}{2+\tau_2} d\tau_2 d\tau_1$$ (generalized polylogarithms)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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$$\int_0^x \frac{1}{1+\tau_1+\tau_1^2}\int_0^{\tau_1}\frac{1}{1+\tau_2^2}d\tau_2 d\tau_1$$ (cyclotomic polylogarithms)

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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$$\int_0^x \frac{1}{\sqrt{1+\tau_1}}\int_0^{\tau_1}\frac{1}{1+\tau_2}d\tau_2 d\tau_1 \qquad \text{(radical integrals)}$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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(generalized radical integrals)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

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A more general example:

$$\int_0^x e^{\int_1^{\tau_1} \frac{1}{1+y+y^2} dy} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1$$

HarmonicSums can also deal with Liouvillian solutions (i.e., it contains Kovacic's algorithm):

$$(11+20x)f'(x)+(1+x)(35+134x)f''(x) + 3(1+x)^2(4+37x)f^{(3)}(x)+18x(1+x)^3f^{(4)}(x)=0$$

$$\downarrow$$

$$\Big\{c_1+c_2\int_0^x\frac{1}{1+\tau_1}\,d\tau_1+c_3\int_0^x\frac{1}{1+\tau_1}\int_0^{\tau_1}\frac{\sqrt[3]{1+\sqrt{1+\tau_2}}}{1+\tau_2}d\tau_2 d\tau_1 + c_4\int_0^x\frac{1}{1+\tau_1}\int_0^{\tau_1}\frac{\sqrt[3]{1-\sqrt{1+\tau_2}}}{1+\tau_2}d\tau_2 d\tau_1 \mid c_1,c_2,c_3,c_4\in\mathbb{K}\Big\}$$

Connection: DE $\longleftrightarrow$ REC

Let

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

be a (formal) power series. Then:

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$$\Updownarrow$$

There exist $a_0(x), \ldots, a_\delta(x) \in \mathbb{K}[x]$ with

$$a_0(n) F(n) + \cdots + a_\delta(n) F(n+\delta) = 0.$$

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$\Updownarrow$ algorithmic

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Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

for

$$-\left(x^4 - 64x^3\right) f^{(4)}(x) - 2\left(5x^3 - 144x^2\right) f^{(3)}(x) \\ - \left(25x^2 - 208x\right) f''(x) - (15x - 8) f'(x) - f(x) = 0$$

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for

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Example 2: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

for

$$\begin{aligned}&\left(x^6 - 32x^5 + 256x^4\right) f^{(6)}(x) + \left(23x^5 - 528x^4 + 2560x^3\right) f^{(5)}(x)\\&+ \left(171x^4 - 2552x^3 + 6272x^2\right) f^{(4)}(x) + 2\left(245x^3 - 2002x^2 + 1728x\right) f^{(3)}(x)\\&\qquad + 2\left(253x^2 - 786x + 72\right) f''(x) + 4(35x - 12) f'(x) + 4f(x) = 0\end{aligned}$$

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Example 2: Find a power series solution

$$f(x) = c_1 \cdot {}_3F_2\left[\begin{matrix}1,1,1\\ \frac{1}{2},\frac{1}{2}\end{matrix};\frac{x}{16}\right] + c_2 \sum_{n=0}^{\infty} \frac{S_1(n)}{\binom{2n}{n}^2} x^n$$

for

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DE-solver

REC-solver

Tactic 3: the DE-REC approach

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

Tactic 3: the DE-REC approach

DE system	OreSys package (S. Gerhold) → uncoupling algorithm	uncoupled DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$ $\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$

Tactic 3: the DE-REC approach

DE system	OreSys package (S. Gerhold) → uncoupling algorithm	uncoupled DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$ $\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

Tactic 3: the DE-REC approach

DE system

$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$

OreSys package (S. Gerhold)

uncoupling algorithm

uncoupled DE system

$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$

$\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

holonomic closure prop.

linear recurrence

$\sum_i a'_i(N) I_1(N+i) = r'(N)$

Tactic 3: the DE-REC approach

DE system
$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

→ OreSys package (S. Gerhold) — uncoupling algorithm →

uncoupled DE system
$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$
$$\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)),\, k > 1$$

↓ holonomic closure prop. ↓

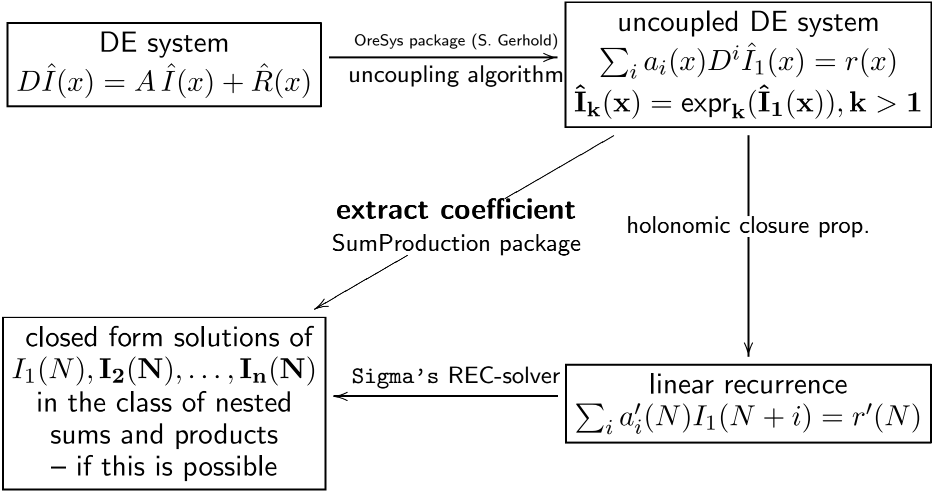
$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

linear recurrence
$$\sum_i a'_i(N) I_1(N+i) = r'(N)$$

← `Sigma`'s REC-solver ←

closed form solutions of $I_1(N)$ in the class of nested sums and products – if this is possible

Tactic 3: the DE-REC approach (SolveCoupledSystem package)



J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

General strategy: physical problem $\hat{P}(x)$

$\downarrow$ IBP methods

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$

General strategy: physical problem $\hat{P}(x)$

$\downarrow$ IBP methods

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$

$\downarrow$ solver for $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N) x^N$

$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) + \varepsilon^{0} F_0(N) + \ldots$$

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

$$\Big\downarrow \text{solver for } \hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N) x^N$$

$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) + \varepsilon^{0} F_0(N) + \dots$$

$$\Big\downarrow \text{plug into } \hat{P}(x) = \sum_{N=0}^{\infty} P(N) x^N$$

$$P(N) = \varepsilon^{-3} P_{-3}(N) + \varepsilon^{-2} P_{-2}(N) + \varepsilon^{-1} P_{-1}(N) + \varepsilon^{0} P_0(N) + \dots$$

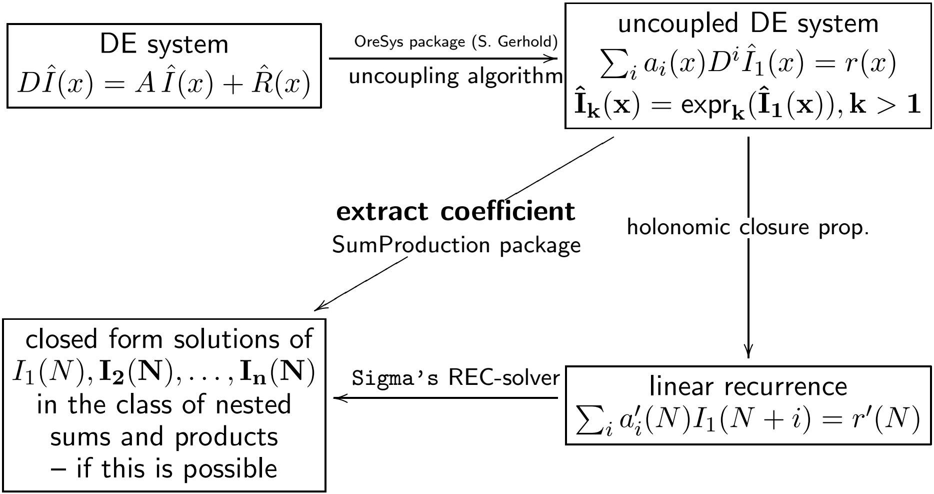
Calculations based on Tactic 3:

- J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, pp. 263-288. 2014.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, pp. 280-317. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. Nuclear Physics B 886, pp. 733-823. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, pp. 48-151. 2015.
- A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, pp. 612-644. 2015.
- A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x, Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), pp. 1-19. 2015.
- A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel, CS. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+-W^-}(x, Q^2)$ and $F_2^{W^+-W^-}(x, Q^2)$. Physical Review D 94(11), pp. 1-19. 2016.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Manteuffel, CS. Calculating Three Loop Ladder and V-Topologies for Massive Operator Matrix Elements by Computer Algebra. Comput. Phys. Comm. 202, pp. 33-112. 2016.
- J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, N. Rana, CS. The Heavy Quark Form Factors at Two Loops. Physical Review D 97(094022), pp. 1-44. 2018.
- J. Ablinger, J. Blümlein, A. De Freitas, CS, K. Schönwald. The two-mass contribution to the three-loop pure singlet operator matrix element. Nucl. Phys. B(927), pp. 339-367. 2018. ISSN 0550-3213.
- J. Blümlein, A. De Freitas, CS, K. Schönwald. The Variable Flavor Number Scheme at Next-to-Leading Order. Physics Letters B 782, pp. 362-366. 2018.
- J. Ablinger, J. Blümlein, P. Marquard, N. Rana, CS. Heavy Quark Form Factors at Three Loops in the Planar Limit. Physics Letters B 782, pp. 528-532. 2018.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Goedicke, A. von Manteuffel, CS, K. Schönwald. The Unpolarized and Polarized Single-Mass Three-Loop Heavy Flavor Operator Matrix Elements $A_{gg,Q}$ and $\Delta A_{gg,Q}$. Journal of High Energy Physics 2022(12), pp. 1-55. 2022.

Tactic 4: Compute large moments and guessing recurrences

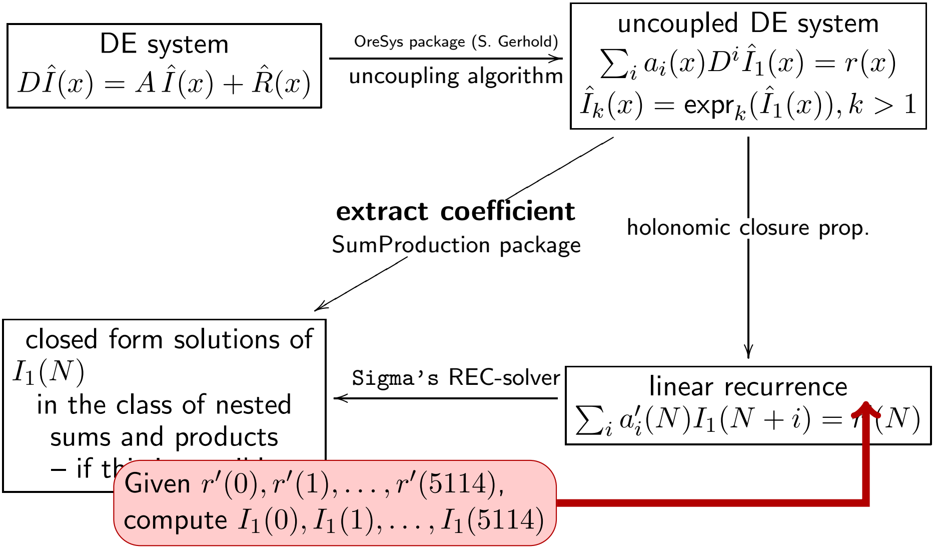
[coming, e.g., from IBP methods]

Tactic 3: the DE-REC approach (SolveCoupledSystem package)



J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

Tactic 4: compute large moments (SolveCoupledSystem package)



J. Blümlein, CS, 2017, arXiv:1701.04614 [hep-ph].

General strategy: physical problem $\hat{P}(x)$

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$$I_i(N) = \underbrace{\varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) + \varepsilon^{0} F_{0}(N) + \dots}_{\text{only numbers}}$$

$N = 0, 1, \dots, 8000$

$\big\downarrow$ plug into $\hat{P}(x) = \sum_{N=0}^{\infty} P(N) x^N$

$$P(N) = \underbrace{\varepsilon^{-3} P_{-3}(N) + \varepsilon^{-2} P_{-2}(N) + \varepsilon^{-1} P_{-1}(N)}_{\text{numbers}} + \underbrace{\varepsilon^{0} P_{0}(N)}_{\text{numbers}} + \dots$$

$N = 0, 1, \dots, 8000$

coupled systems for $P(N)$

`SolveCoupledSystem.m` ↓

large no. of moments

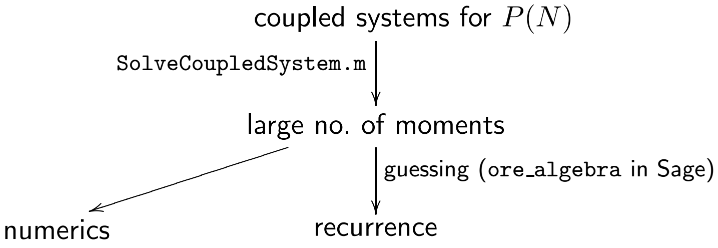
coupled systems for $P(N)$

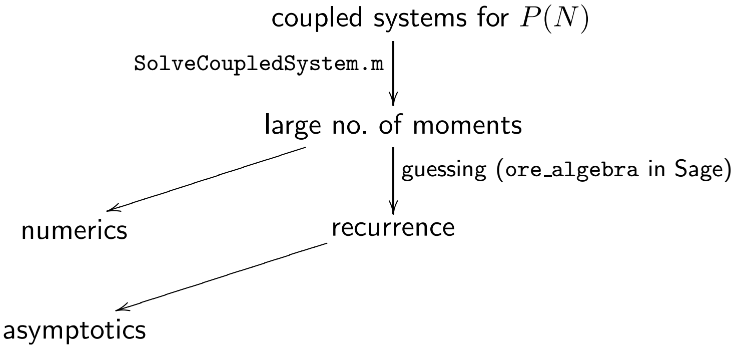
`SolveCoupledSystem.m` ↓

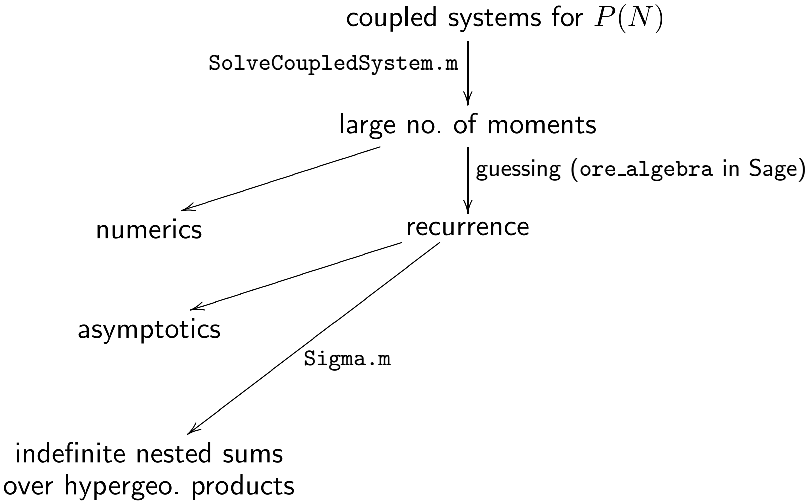
large no. of moments

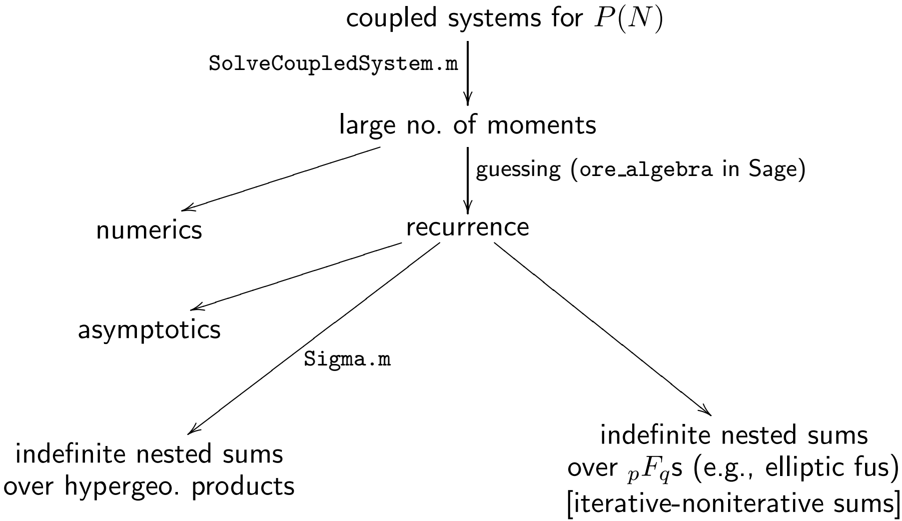
↙

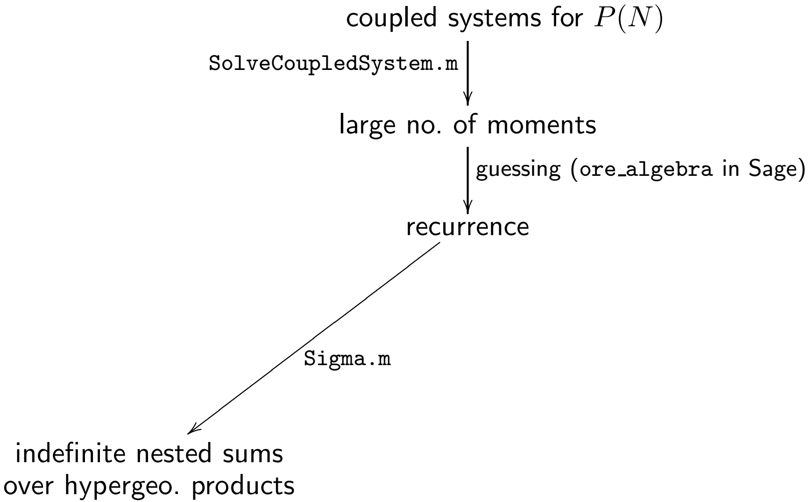
numerics


coupled systems for $P(N)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence


coupled systems for $P(N)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics


coupled systems for $P(N)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
Sigma.m
indefinite nested sums
over hypergeo. products


coupled systems for $P(N)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
Sigma.m
indefinite nested sums over hypergeo. products
indefinite nested sums over ${}_pF_q$s (e.g., elliptic fus) [iterative-noniterative sums]


coupled systems for $P(N)$
SolveCoupledSystem.m
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$N = 0, 1, \dots, 8000$

$\downarrow$ plug into $\hat{P}(x) = \sum_{N=0}^{\infty} P(N) x^N$

$$P(N) = \underbrace{\varepsilon^{-3} P_{-3}(N) + \varepsilon^{-2} P_{-2}(N) + \varepsilon^{-1} P_{-1}(N)}_{\text{numbers}} + \underbrace{\varepsilon^{0} P_{0}(N)}_{\text{numbers}} + \dots$$

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General strategy: physical problem $\hat{P}(x)$

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$$I_i(N) = \underbrace{\varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) + \varepsilon^{0} F_{0}(N) + \dots}_{\text{only numbers}}$$

$\big\downarrow$ plug into $\hat{P}(x) = \sum_{N=0}^{\infty} P(N) x^N$

$$P(N) = \underbrace{\varepsilon^{-3} P_{-3}(N) + \varepsilon^{-2} P_{-2}(N) + \varepsilon^{-1} P_{-1}(N)}_{\text{nice}} + \underbrace{\varepsilon^{0} P_{0}(N)}_{\text{partially nice}} + \dots$$

all N solution

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[8]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[9]:= **initial =<< iFile16**

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[8]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[9]:= **initial =<< iFile16**

Out[9]= {37, 34577/1296, 7598833/151875, 13675395569/230496000,

475840076183/7501410000, 1432950323678333/21965628762000,

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52869784323778576751/802218994536960000,

49422862094045523994231/753773992230616156800,

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498938690219595294505102809199154550783080767/8468883667852979813171262304054002720000000, In[10]:= **rec =<< rFile16**

Out[10]=
$$(n+1)^4(n+2)^2(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)\Big(309237645312n^{32} + 38256884318208n^{31} + 2282100271087616n^{30} + 87428170197762048n^{29} + 2417273990256001024n^{28} + 51388547929265405952n^{27} + 873862324676687036416n^{26} + 12209268055143308328960n^{25} + 142860861222820240162816n^{24} + 1419883954103469621510144n^{23} + 12115561235109256405319680n^{22} + 89479384946084038000803840n^{21} + 575561340618928527623274496n^{20} + 3239547818363227419971647488n^{19} + 16009805333085271423330779136n^{18} + 69631814641718655426881659392n^{17} + 266892117418348771052573667328n^{16} + 901901113782416884441719270144n^{15} + 2685821385767154471801366647296n^{14} + 7038702625583766161604414471744n^{13} + 16195069575749412648646633248128n^{12} + 32602540883321212533013752639288n^{11} + 57154680141624618025310553466704n^{10} + 86710462147941775492301231896818n^{9} + 112917328975807075881545543668548n^{8} + 124873767581470867343743078943772n^{7} + 115624836314544572769501784072647n^{6} + 87938536330971046886456627610048n^{5} + 53481897815980319933589323279298n^{4} + 25000430622737750756669804052204n^{3} + 8430930497463933665464836129855n^{2} + 1825177817831282261293155379650n + 190428196025667395685609855000\Big)(2n+1)^4 P[n]$$

$$-(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11)\Big(12369505812480n^{38} + 1613151061671936n^{37} +$$

$$101748284195864576n^{36} + 4135139115563745280n^{35} + 121713599527855849472n^{34} +$$

$$2765050919624810430464n^{33} + 50453046277771391664128n^{32} + 759760507477065230974976n^{31} +$$

$$9628262076527899425374208n^{30} + 104191253579306374131613696n^{29} + 973595596739520084325171200n^{28} +$$

$$7924537790312611436520013824n^{27} + 56571687381518195331462463488n^{26} +$$

$$356133102136059681954436399104n^{25} + 1985507231916669869451824553984n^{24} +$$

$$9836060321685410187563260035072n^{23} + 43406506634905372676489415905280n^{22} +$$

$$170945808151999530921656848106496n^{21} + 601507760131008511164113355409920n^{20} +$$

$$1892149418896523531194676203153920n^{19} + 5321173806292333448534132495165440n^{18} +$$

$$13370912745727662541153592039812160n^{17} + 29987002021632029091547005084057760n^{16} +$$

$$59921270253255984811455083696758912n^{15} + 106434458966741189159011567116493072n^{14} +$$

$$167533688453539238956436945725341004n^{13} + 232781742346547554435545097479210510n^{12} +$$

$$284125621128876904663642986868770746n^{11} + 302806836393712159148051277734975424n^{10} +$$

$$279679164311116651162116055961513301n^{9} + 221781415386984655607595031093415136n^{8} +$$

$$149214365004640710156345950062395186n^{7} + 83882523964213110328265187672574356n^{6} +$$

$$38609679702395410742361774562392789n^{5} + 14149471988638475521561721269939086n^{4} +$$

$$3963748138857399502678254252169734n^{3} + 795659668131014454843348852372480n^{2} +$$

$$101701393436276172443717692853400n + 6204709909986751913151675960000\Big)\,\mathsf{P}[n+1]$$

$$
\begin{aligned}
&+2(n+3)^2(2n+5)^3(2n+9)(2n+11)\big(24739011624960n^{40}+3317836466356224n^{39}+215508170284466176n^{38}+9032884062187945984n^{37}+\\
&274636134389959884800n^{36}+6455501959255126179840n^{35}+122094572934385260036096n^{34}+1909387225793663151898624n^{33}+\\
&25180108291969215434326016n^{32}+284171960705270647479074816n^{31}+2775794400720227034854326272n^{30}+\\
&23677622163992853854566219776n^{29}+177624312783583749157935120384n^{28}+1178515602115604757944201871360n^{27}+\\
&6947091965313419323781358354432n^{26}+36515023100308314818702129258496n^{25}+171621148571344894953594594017280n^{24}+\\
&722837793013976317556258102507520n^{23}+2732534027077907914497042720534528n^{22}+9281028665970648470895368668485120n^{21}+\\
&28337819215557708948254385336117248n^{20}+77786125749274632150536464583130752n^{19}+191877161455672780973502244537632256n^{18}+\\
&424953221702140663089937921965135648n^{17}+843818276409975584824720931649555264n^{16}+\\
&1499359936674956711935311062995422344n^{15}+2378007025570977662661938772843220240n^{14}+\\
&3355671771434535852147325502571953770n^{13}+4196375762867184563407432891655585484n^{12}+\\
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&3761696365025837909581516781307249585n^{9}+2726553473467254373993685951699145492n^{8}+\\
&1683383212304999468664293798012773485n^{7}+871926653651504419744271839781064837n^{6}+\\
&371307437598003570058538796122994147n^{5}+126427972742886389602285855482966072n^{4}+33048762330145623969058704448697313n^{3}+\\
&6217924746857741077419160100404560n^{2}+748298077423337427195946099994100n+43181089548034246077698611794000\big)\mathsf{P}[n+2]
\end{aligned}
$$

$$
\begin{aligned}
&-2(n+4)^2(2n+5)(2n+7)^3(2n+11)\Big(24739011624960n^{40}+3322784268681216n^{39}+216160919414112256n^{38}+9074528155284275200n^{37}+\\
&276348048819456311296n^{36}+6506479077331107315712n^{35}+123266585640616142569472n^{34}+1931040885785102661976064n^{33}+\\
&25510503383281445462081536n^{32}+288418124175428279391485952n^{31}+2822442799033603081019326464n^{30}+\\
&24120717233320712351821332480n^{29}+181295944719289040999116701696n^{28}+1205246297785423925076555694080n^{27}+\\
&7119049557560114436136213413888n^{26}+37496933571993839665392189775872n^{25}+176616172467048982234270428880896n^{24}+\\
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&4074895386694182240941538222230233221n^{9}+2970477229398746689186622534784613554n^{8}+\\
&1845274131994015990683957902602775337n^{7}+962091291302144537393228847830431614n^{6}+\\
&412595107814836563208757757032740146n^{5}+141540723940232563767779647013785485n^{4}+37292931812630561528276365992452010n^{3}+\\
&7074865777225416725452872895397100n^{2}+858794112392644074221312049837000n+49997386738260112603615104780000\Big)f[n+3]
\end{aligned}
$$

$$+(n+5)^3(2n+5)(2n+7)(2n+9)^4\Big(12369505812480n^{38} + 1546355730284544n^{37} + 93441851805138944n^{36} +$$
$$3636063211393908736n^{35} + 102413434086873890816n^{34} + 2225107112182077718528n^{33} +$$
$$38808234188348931964928n^{32} + 558299807912629375074304n^{31} + 6755648626273815474733056n^{30} +$$
$$69769132238801205785001984n^{29} + 621900006220029229458259968n^{28} + 4826558182244413850688946176n^{27} +$$
$$32840774268722977511855751168n^{26} + 196981883700048989849717882880n^{25} +$$
$$1046061529031136798450810839040n^{24} + 4934888224954929426023144030208n^{23} +$$
$$20735286278224836075286873214976n^{22} + 77745549200390911029444008457216n^{21} +$$
$$260448286122609254214904458392064n^{20} + 780087654447729149285799146869248n^{19} +$$
$$2089276462852113795051294249728512n^{18} + 5001455921015163002705347586646080n^{17} +$$
$$10691068512696184477385875851523744n^{16} + 20374769440121072185247660725156544n^{15} +$$
$$34542976501702600883669655947085712n^{14} + 51947527795197316142253213880200764n^{13} +$$
$$69039779136078090572935768218052854n^{12} + 80712286124402599779679594199103258n^{11} +$$
$$82519759833385882007812859351392458n^{10} + 73248127158607338722648198918322201n^{9} +$$
$$55935262205790259307904762197107653n^{8} + 36322355479155199114489624391144238n^{7} +$$
$$19756597118002557191991191826327042n^{6} + 8822212911433711339358062994077203n^{5} +$$
$$3145597282374650512689680780380605n^{4} + 859907105684964990690798899478888n^{3} +$$
$$168963309995629650025632011492580n^{2} + 21205680751316222158938757272000n +$$
$$1274120732351744651125603886400\Big)P[n+4]$$

$$-(n+5)^2(n+6)^4(2n+5)(2n+7)(2n+9)^3(2n+11)^4\Big(309237645312n^{32} + 28361279668224n^{31} + 1249518729297920n^{30} + 35220794552352768n^{29} + 713726163159089152n^{28} + 11076866026783113216n^{27} + 136959486138712588288n^{26} + 1385658801437173350400n^{25} + 11691772665924577918976n^{24} + 83438339505976242995200n^{23} + 508989054278115477684224n^{22} + 2675508113418826174332928n^{21} + 12193213796145039633072128n^{20} + 48399020537651722726242304n^{19} + 167881257973769248139515904n^{18} + 510012482113388176546187776n^{17} + 1358662126092561923541267968n^{16} + 3174925021159974655053814528n^{15} + 6504205668151125355938798848n^{14} + 11663792381020901870157176128n^{13} + 18263581057905911985340656960n^{12} + 24881010123632244515458585528n^{11} + 29346856353503020415409305704n^{10} + 29775859546803351930591002266n^{9} + 25770328899499991754425455738n^{8} + 18817114309842270306167785140n^{7} + 11424980760825630752861027739n^{6} + 5656051955667821083952617134n^{5} + 2221448212382554437709999491n^{4} + 664859653803075491350122060n^{3} + 142190920852333874895041748n^{2} + 19313175036907229252501700n + 1248723341516324359641600\Big)\,P[n+5]==0$$

In[11]:= $\mathbf{recSol = SolveRecurrence[rec, P[n]]}$

In[11]:= $\mathbf{recSol = SolveRecurrence[rec, P[n]]}$

Out[11]= $\{\{0, \frac{(3+2n)(3+4n)}{(1+n)^2(1+2n)^2}\}$

$\{0, -\frac{(3+2n)(-8-9n+2n^2)}{(1+n)^2(1+2n)^2}\}$

$\{0, -\frac{(3+2n)(-5+8n^2)}{2(1+n)^2(1+2n)^2} + \frac{(3+2n)\sum_{i=1}^{n}\frac{1}{i}}{(1+n)(1+2n)} + \frac{2(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\}$

$\{0, \frac{(3+2n)(-513-2184n-2416n^2+768n^4)}{2(1+n)^3(1+2n)^3} + \frac{14(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)} + (-\frac{2(3+2n)(3+44n+48n^2)}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)})\sum_{i=1}^{n}\frac{1}{i} + \frac{12(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^2}{(1+n)(1+2n)} + \frac{56(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \frac{4(3+2n)(3+44n+48n^2)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{(1+n)(1+2n)}\}$

$$\Big\{0,\ \frac{1}{16(1+n)^4(1+2n)^4}\big(72519+572343n+1814716n^2+2918100n^3+2442240n^4+912896n^5+24576n^6-$$

$$49152n^7\big)+\frac{16(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{3(1+n)(1+2n)}+\Big(-\frac{(3+2n)\big(29+307n+322n^2\big)}{4(1+n)^2(1+2n)^2}+\frac{44(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i^2}+$$

$$\Big(\frac{(3+2n)\big(91+259n+974n^2+1784n^3+1024n^4\big)}{4(1+n)^3(1+2n)^3}+\frac{22(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)}+\frac{24(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}-$$

$$\frac{4(3+2n)\big(-13-4n+16n^2\big)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2}+\frac{16(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i}+\Big(-$$

$$\frac{(3+2n)\big(19+92n+80n^2\big)}{(1+n)^2(1+2n)^2}+\frac{40(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\Big(\sum_{i=1}^{n}\frac{1}{i}\Big)^2+\frac{20(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^3}{3(1+n)(1+2n)}+$$

$$\frac{64(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{3(1+n)(1+2n)}-\frac{3(3+2n)\big(63+209n+150n^2\big)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)^2(1+2n)^2}+$$

$$\Big(\frac{(3+2n)\big(347+1795n+4302n^2+4856n^3+2048n^4\big)}{2(1+n)^3(1+2n)^3}+\frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{-1+2i}-$$

$$\frac{4(3+2n)\big(19+92n+80n^2\big)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)^2(1+2n)^2}+\frac{32(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^3}{3(1+n)(1+2n)}-$$

$$\frac{8(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)^2}{i}}{(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)^2}{-1+2i}}{(1+n)(1+2n)}$$

$$-\frac{32(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{(1+n)(1+2n)}+$$

$$\frac{32(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{-1+2j}\big)^2}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{-1+2j}\big)^2}{-1+2i}}{(1+n)(1+2n)}\Big\},\{1,0\}\Big\}$$

In[12]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

In[12]:= **sol = FindLinearCombination[recSol, {0, initial}, n, 7, MinInitialValue → 1]**

Out[12]=

$$\frac{1}{3(1+n)^4(1+2n)^4}(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7)+\frac{32(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{9(1+n)(1+2n)}-$$

$$\frac{(3+2n)(-3+101n+126n^2)\sum_{i=1}^{n}\frac{1}{i^2}}{3(1+n)^2(1+2n)^2}-\frac{(3+2n)(115+921n+1967n^2+1524n^3+340n^4)\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)^3(1+2n)^3}+$$

$$\frac{44(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)(1+2n)}-\frac{(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{i})^2}{3(1+n)^2(1+2n)^2}+\frac{40(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^3}{9(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{9(1+n)(1+2n)}-\frac{4(3+2n)(77+261n+190n^2)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{3(1+n)^2(1+2n)^2}+\frac{16(3+2n)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}+$$

$$\frac{2(3+2n)(13-153n-303n^2+12n^3+172n^4)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^3(1+2n)^3}+\frac{88(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}-$$

$$\frac{4(3+2n)(-41-53n+2n^2)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^2(1+2n)^2}+\frac{80(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^2\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}+$$

$$\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{(-1+2i)^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}-\frac{4(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)^2(1+2n)^2}+$$

$$\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{i})(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)(1+2n)}+\frac{64(3+2n)(\sum_{i=1}^{n}\frac{1}{-1+2i})^3}{9(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{i}}{3(1+n)(1+2n)}-$$

$$\frac{32(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{-1+2i}}{3(1+n)(1+2n)}-\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{3(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{3(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{i}}{3(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{-1+2i}}{3(1+n)(1+2n)}$$

In[13]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[14]:= $\mathbf{sol = TransformToSSums[sol];}$

In[15]:= $\mathbf{sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;}$

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In[14]:= $\mathbf{sol = TransformToSSums[sol];}$

In[15]:= $\mathbf{sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;}$

Out[15]=
$$\frac{1}{3(1+n)^4(1+2n)^4}\big(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7\big)+\frac{64(3+2n)^2\mathtt{S}[1,n]}{3(1+n)(1+2n)^2}+\frac{64(3+2n)(2+3n)\mathtt{S}[1,n]^2}{3(1+n)(1+2n)^2}+\Big(-\frac{2(3+2n)\big(147+985n+1871n^2+1268n^3+212n^4\big)}{3(1+n)^3(1+2n)^3}+\frac{224(3+2n)\mathtt{S}[2,2n]}{3(1+n)(1+2n)}+\frac{128(3+2n)\mathtt{S}[-2,2n]}{3(1+n)(1+2n)}\Big)\mathtt{S}[1,2n]-\frac{4(3+2n)\big(23+123n+114n^2\big)\mathtt{S}[1,2n]^2}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)\mathtt{S}[1,2n]^3}{3(1+n)(1+2n)}+\frac{64(3+2n)\mathtt{S}[2,n]}{3(1+n)(1+2n)}-\frac{4(3+2n)\big(53+229n+190n^2\big)\mathtt{S}[2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)\mathtt{S}[3,2n]}{3(1+n)(1+2n)}+\Big(-\frac{64(3+2n)^2}{3(1+n)(1+2n)^2}-\frac{128(3+2n)(2+3n)\mathtt{S}[1,2n]}{3(1+n)(1+2n)^2}\Big)\mathtt{S}[-1,2n]-\frac{64(3+2n)(2+3n)\mathtt{S}[-1,2n]^2}{3(1+n)(1+2n)^2}-\frac{32(3+2n)\big(1+8n+8n^2\big)\mathtt{S}[-2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)\mathtt{S}[-3,2n]}{3(1+n)(1+2n)}-\frac{128(3+2n)\mathtt{S}[-2,1,2n]}{3(1+n)(1+2n)}$$

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In[14]:= **sol = TransformToSSums[sol];**

In[15]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[16]:= **SExpansion[sol, n, 2]**

Out[16]=
$$\ln 2^2\left(\frac{64\mathrm{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\mathrm{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\mathrm{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

Calculations based on Tactic 4:

- J. Blümlein, CS. The Method of Arbitrarily Large Moments to Calculate Single Scale Processes in Quantum Field Theory. Physics Letters B 771, pp. 31-36. 2017.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The Three-Loop Splitting Functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,\mathrm{N_F})}$. Nucl. Phys. B. 922, pp. 1-40. 2017.
- J. Blümlein, P. Marquard, N. Rana, CS. The Heavy Fermion Contributions to the Massive Three Loop Form Factors. Nuclear Physics B 949(114751), pp. 1-97. 2019.
- A. Behring, J. Blümlein, A. De Freitas, A. Goedicke, S. Klein, A. von Manteuffel, CS, K. Schönwald. The Polarized Three-Loop Anomalous Dimensions from On-Shell Massive Operator Matrix Elements. Nuclear Physics B 948(114753), pp. 1-41. 2019.
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- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS, K. Schönwald. The three-loop single mass polarized pure singlet operator matrix element. Nuclear Physics B 953(114945), pp. 1-25. 2020.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. The Two-mass Contribution to the Three-Loop Polarized Operator Matrix Element $A_{gg,Q}^{(3)}$. Nuclear Physics B 955, pp. 1-70. 2020.
- A. Behring, J. Blm̈lein, A. De Freitas, A. von Manteuffel, K. Schönwald, CS. The Polarized Transition Matrix Element $A_{g,q}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 964, pp. 115331-115356, 2021.
- J. Blümlein, A. De Freitas, M. Saragnese, K. Schönwald, CS. The Logarithmic Contributions to the Polarized $O(\alpha_s^3)$ Asymptotic Massive Wilson Coefficients and Operator Matrix Elements in Deeply Inelastic Scattering. Physical Review D 104(3), pp. 1-73. 2021.
- J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements. Nucl. Phys. B 971, pp. 1-44. 2021.
- J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The three-loop polarized singlet anomalous dimensions from off-shell operator matrix elements. Journal of High Energy Physics 2022(193), pp. 0-32. 2022.
- J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The Two-Loop Massless Off-Shell QCD Operator Matrix Elements to Finite Terms. Nuclear Physics B 980(115794), pp. 1-131. 2022.
- J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The massless three-loop Wilson coefficients for the deep-inelastic structure functions F_2, F_L, xF_3 and g_1. Journal of High Energy Physics. 1-83. 2022.

Symbolic tools for special functions

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Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . .)

$$\boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

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Integral representation:

$$=\int_0^1 \frac{x^n - 1}{1 - x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx, \qquad \zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

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Asymptotic expansion:

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n)$$
$$- \tfrac{1}{100n^5} - \tfrac{1}{6n^4} + \tfrac{13}{36n^3} - \tfrac{1}{4n^2} - \tfrac{1}{n} + 2\zeta(3) + O(\tfrac{\ln(n)}{n^6}).$$

limit computations **numerical evaluation**

- Generalized algorithms for generalized harmonic sums

$$\sum_{k=1}^{n}\frac{2^k\sum_{i=1}^{k}\frac{2^{-i}\sum_{j=1}^{i}\frac{H_j}{j}}{i}}{k}=-\frac{21\zeta(2)^2}{20n}+\frac{1}{8n^2}+\frac{295}{216n^3}-\frac{1115}{96n^4}+O(n^{-5})$$
$$+\Big(\frac{1}{2n}-\frac{3}{4n^2}+\frac{19}{12n^3}-\frac{5}{n^4}+O(n^{-5})\Big)\zeta(2)$$
$$+2^n\Big(\frac{3}{2n}+\frac{3}{2n^2}+\frac{9}{2n^3}+\frac{39}{2n^4}+O(n^{-5})\Big)\zeta(3)$$
$$+\Big(\frac{1}{n}+\frac{3}{4n^2}-\frac{157}{36n^3}+\frac{19}{n^4}+O(n^{-5})\Big)(\log(n)+\gamma)$$
$$+\Big(\frac{1}{2n}-\frac{3}{4n^2}+\frac{19}{12n^3}-\frac{5}{n^4}+O(n^{-5})\Big)(\log(n)+\gamma)^2)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for cyclotomic harmonic sums

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{1+2i}}{j^2}}{(1+2k)^2} = \Big(-3+\frac{35\zeta(3)}{16}\Big)\zeta(2)-\frac{31\zeta(5)}{8}$$
$$+\frac{1}{n}-\frac{33}{32n^2}+\frac{17}{16n^3}-\frac{4795}{4608n^4}+O(n^{-5})$$
$$+\log(2)\Big(6\zeta(2)-\frac{1}{n}+\frac{9}{8n^2}-\frac{7}{6n^3}+\frac{209}{192n^4}+O(n^{-5})\Big)$$
$$+\Big(-\frac{7}{4}-\frac{7}{16n}+\frac{7}{16n^2}-\frac{77}{192n^3}+\frac{21}{64n^4}+O(n^{-5})\Big)\zeta(3)$$
$$+\Big(\frac{1}{16n^2}-\frac{1}{8n^3}+\frac{65}{384n^4}+O(n^{-5})\Big)(\log(n)+\gamma)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for nested binomial sums

$$\sum_{j=1}^{n}\frac{4^j H_{j-1}}{\binom{2j}{j}j^2}=7\zeta(3)+\sqrt{\pi}\sqrt{n}\Bigg\{\Bigg[-\frac{2}{n}+\frac{5}{12n^2}-\frac{21}{320n^3}-\frac{223}{10752n^4}+\frac{671}{49152n^5}$$
$$+\frac{11635}{1441792n^6}-\frac{1196757}{136314880n^7}-\frac{376193}{50331648n^8}+\frac{201980317}{18253611008n^9}$$
$$+O(n^{-10})\Bigg]\ln(\bar{n})-\frac{4}{n}+\frac{5}{18n^2}-\frac{263}{2400n^3}+\frac{579}{12544n^4}+\frac{10123}{1105920n^5}$$
$$-\frac{1705445}{71368704n^6}-\frac{27135463}{11164188672n^7}+\frac{197432563}{7927234560n^8}+\frac{405757489}{775778467840n^9}$$
$$+O(n^{-10})\Bigg\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]
Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]

Conclusion

Our calculations rely on

1. symbolic summation and integration methods to derive recurrences
2. flexible recurrence and DE solver
3. coupled systems solver
4. the large moment method

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5. special function algorithms
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    - to simplify the results further
    - to explore their mathematical structures and properties
6. stable and efficient software packages

Main CA-packages

In[17]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[18]:= << **MultiIntegrate.m**

MultiIntegrate by Jakob Ablinger © RISC-Linz

In[19]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[20]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[21]:= << **SumProduction.m**

SumProduction by Carsten Schneider © RISC-Linz

In[22]:= << **OreSys.m**

OreSys by Stefan Gerhold (optimized by Carsten Schneider) © RISC-Linz

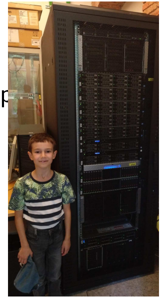
In[23]:= << **SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

Conclusion

Our calculations rely on

1. symbolic summation and integration methods to derive recurrences
2. flexible recurrence and DE solver
3. coupled systems solver
4. the large moment method
5. special function algorithms
   - to support the above calculations
   - to simplify the results further
   - to explore their mathematical structures and prop
6. stable and efficient software packages
7. 11 strong servers borrowed from DESY (10TB memory, 168 kernels)



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Within the RISC-DESY cooperation we expect that we will discover and explore many

new algorithms in CA and results in QFT!