Progress on 3-loop Feynman integrals for 4-point 1-mass processes

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based on 2112.14275(JHEP), in collaboration with Dhimiter Canko
and ongoing work with Thomas Gehrmann, Petr Jakubčík, Cesare Carlo Mella, Lorenzo Tancredi.

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Table of Contents

Table of Contents

Introduction

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Introduction

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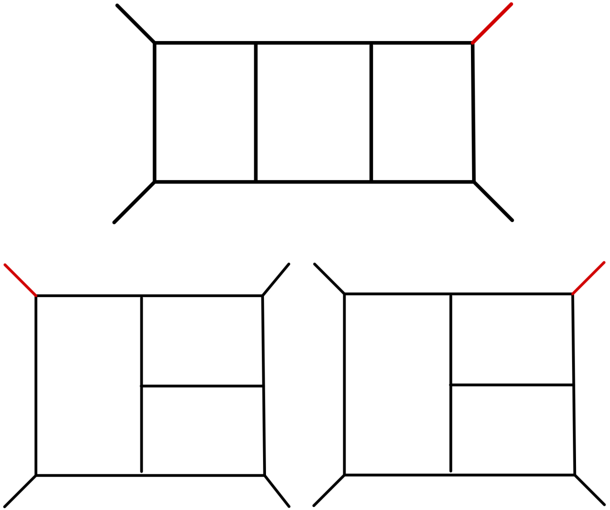
  - **Henn, Smirnov, Smirnov, JHEP07(2013)128.**
  - **Henn, Mistlberger, Smirnov, Wasser, JHEP04(2020)167.**
- Three-lopp planar master integrals for $2 \to 2$ scattering with one massive leg:
  - **Di Vita, Mastrolia, Schubert, Yundin, JHEP09(2014)148.**
  - **Canko and NS, JHEP02(2021)080.**
  - **Canko and NS, JHEP04(2022)134.**

Table of Contents

Planar top sectors



Master integrals and kinematics

- 83(Ladder), 117(TennisCourt1) and 166(TennisCourt2) master integrals (KIRA2, FIRE6).
- $\sum_{i=1}^{4} p_i = 0;\ p_4^2 = m^2;\ p_i^2 = 0$ for $i = 1;2;3$, $s_{ij} = (p_i + p_j)^2$.
- Euclidean region: $s_{13} < 0; \quad s_{23} < 0; \quad m^2 < s_{13} + s_{23}$.
- Scattering kinematics

$$\text{s-channel} : m^2 > 0; \quad s_{12} \quad m^2; \quad s_{23} \quad 0; \quad s_{13} \quad 0 \tag{1}$$

$$\text{t-channel} : m^2 > 0; \quad s_{12} \quad 0; \quad s_{23} \quad m^2; \quad s_{13} \quad 0 \tag{2}$$

$$\text{u-channel} : m^2 > 0; \quad s_{12} \quad 0; \quad s_{23} \quad 0; \quad s_{13} \quad m^2: \tag{3}$$

Main steps

- Construct a canonical basis[1] **g**.
- Apply the Simplified Differential Equations approach[2].
- Compute necessary boundary terms.
- Results in terms of MPLs and analytic continuation.

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \tag{4}$$

$$G(0, \ldots, 0; x) = \frac{1}{n!} \log^n(x) \tag{5}$$

[1]Henn, Phys. Rev. Lett. **110** (2013), 251601.
[2]Papadopoulos, JHEP **07** (2014), 088

Canonical basis

- Up to seven propagators: Magnus series expansions[3] (Federico Gasparotto, Luca Mattiazzi).
- Up to nine propagators: `Mathematica` package `DlogBasis`[4].
- Top sector: Analyse leading singularities in 4D loop-by-loop and use known 1-,2-loop results as building blocks[5].

[3] Argeri et al., JHEP **03** (2014), 082.
[4] Henn, Mistlberger, Smirnov, Wasser, JHEP04(2020)167.
[5] P. Wasser, PhD thesis.

Simplified Differential Equations (SDE)

- Parametrise the external momenta by introducing a dimensionless parameter x in the following manner

$$p_1 = q_3; \quad p_2 = \quad q_1 \quad q_2 \quad q_3; \quad p_3 = xq_1; \quad p_4 = q_1 + q_2 \quad xq_1 \tag{6}$$

with $\sum_{i=1}^{4} q_i = 0;\ q_i^2 = 0.$

- Mapping for the kinematic invariants between the two momentum configurations

$$s_{13} = \quad (S_{12} + S_{23})x; \quad s_{23} = S_{23}x; \quad m^2 = S_{12}(1 \quad x) \tag{7}$$

with $S_{12} = (q_1 + q_2)^2;\ S_{23} = (q_2 + q_3)^2.$

- Euclidean region: $0 < x < 1;\ S_{12} < 0;\ 0 < \tilde{y} < 1$, with $\tilde{y} = S_{23}$=S_{12}.

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$$\partial_x \mathbf{g} = \left(\sum_{i=1}^{4} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g} \tag{8}$$

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- $\mathbf{M}_i$: residue matrices corresponding to each pole l_i, consisting of rational numbers.

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- Pole structure:

$$x;\ x-1;\ x-\frac{1}{1+\tilde{y}};\ x+\frac{1}{\tilde{y}}\ . \tag{9}$$

General solution up to weight six

$$
\begin{aligned}
\mathbf{g} = {}^{0}\mathbf{b}_0^{(0)} + \; & \left(\sum G_i \mathbf{M}_i \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) + {}^{2}\left(\sum G_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(0)} + \sum G_i \mathbf{M}_i \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right) + ::: \\
+ {}^{6}\Big(\mathbf{b}_0^{(6)} & + \sum G_{ijklmn} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{M}_n \mathbf{b}_0^{(0)} + \sum G_{ijklm} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{b}_0^{(1)} \\
+ \sum G_{ijkl} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{b}_0^{(2)} & + \sum G_{ijk} \mathbf{M}_i \mathbf{M}_j \mathrm{M}_k \mathbf{b}_0^{(3)} + \sum G_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(4)} + \sum G_i \mathbf{M}_i \mathbf{b}_0^{(5)}\Big)
\end{aligned}
\tag{10}
$$

- $G_{ab:::} := G(l_a; l_b; :::; x)$.
- $\mathbf{b}_0^{(i)}$: boundary terms involving rational numbers and $f \;\, (i);\; \log(\;\, S_{12});\; \log(\tilde{y}) g$.

Boundary terms

- Residue matrix for $l_1 = 0$! $\mathbf{M}_1 = \mathbf{SDS}^{-1}$.
- $\mathbf{R} = \mathbf{S}e^{\mathbf{D}\log(x)}\mathbf{S}^{-1}$.
- $\mathbf{b} = \sum_{i=0}^{6} {}^{i}\, b_0^{(i)}$.
- IBP reduction: $\mathbf{g} = \mathbf{TI}$.
- Expansion-by-regions using asy: $I_i \underset{x\to 0}{=} \sum_j x^{b_j+a_j}\, I_i^{(b_j+a_j)}$.
- Master equation:

$$\mathbf{Rb} = \lim_{x\to 0} \mathbf{TI}\Big|_{\mathcal{O}(x^{0+a_j\epsilon})} \tag{11}$$

Analytic continuation

- Tools: `HyperInt`, `PolyLogTools`, `GiNaC`.

Regions	**Indices**	**Argument**	**Indices**	**Argument**
Euclidean	$f0; 1; \quad 1{=}\tilde{y}; 1{=}(1+\tilde{y})g$	x		
s-channel	$f0; 1; \quad 1{=}\tilde{y}; 1{=}(1+\tilde{y})g$	x		
t-channel	$f0; 1; \quad \tilde{y}; 1+\tilde{y}g$	$1{=}x$	$f0; 1g$	$1{=}\tilde{y}$
u-channel	$f0; 1; \quad \tilde{y}; 1+\tilde{y}g$	$1{=}x$	$f0; \quad 1g$	$\tilde{y}$

Table: Structure of MPLs appearing in each of the 4 kinematic regions.

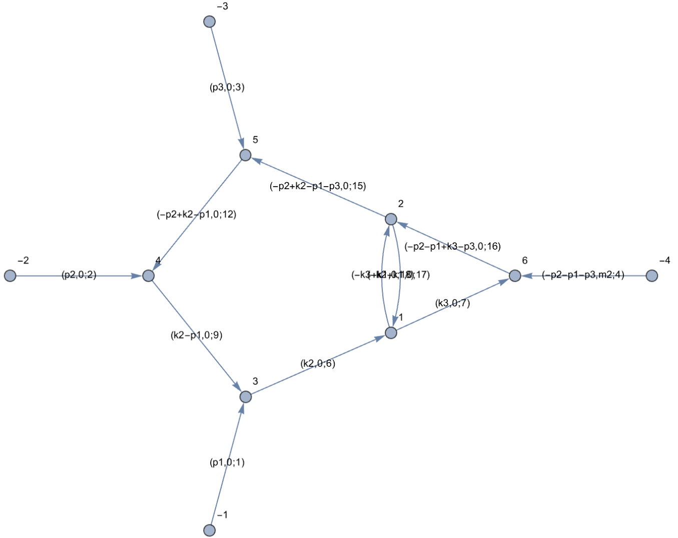
R	$W=1$	$W=2$	$W=3$	$W=4$	$W=5$	$W=6$	**Total**	**Timings (sec)**
E	4	14	50	124	367	734	1293	39.0225769
s	4	14	50	124	367	734	1293	39.2172529
t	6	18	58	155	419	603	1259	62.0567800
u	5	16	54	147	403	572	1197	55.1049640

Table: Number of MPLs per weight and region, and timings for the numerical evaluation of the total MPLs.

Are these enough for an amplitude calculation?

- E.g. $q\bar{q}$! $Z + g$ at 3 loops (leading colour).
- 20 planar top sectors, 3 irreducible.
- Reducible top sectors have subsectors that contribute additional master integrals.
- Define a single family for all top sectors ! $291(= 235 + 56)$ master integrals.
- Only one genuinely new master integral.

New master integral



Progress on planar master integrals

- Canonical basis for all 291 masters.
- Differential equations in $y = s_{13}$=m^2; $z = s_{23}$=m^2.
- Alphabet

$$fy;z;1 \quad y;1 \quad z;1 \quad y \quad z;y+zg: \qquad (12)$$

Table of Contents

Progress on non-planar master integrals

- 15 irreducible top sectors!
- Sectors with many master integrals.

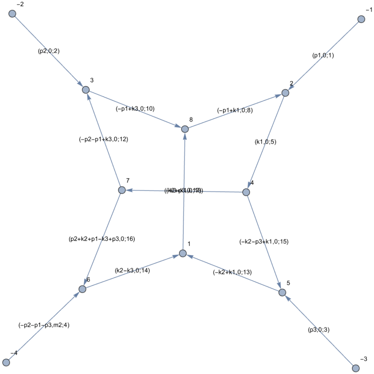
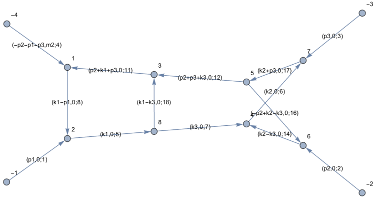
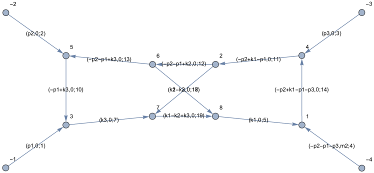


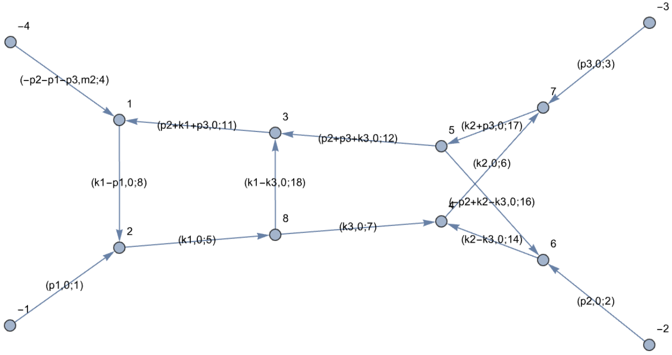
Figure: Top sector with 19 master integrals.

Ladder-like top sectors





Preliminary results



- Canonical basis of 114 master integrals.
- Pole structure (SDE):

$$x;\ x \quad 1;\ x \quad \frac{1}{1+\tilde{y}};\ x+\frac{1}{\tilde{y}} \quad : \tag{13}$$

Preliminary results

- Canonical basis of 150 master integrals.
- Pole structure (SDE) *!* 2 new entries:

$$x;\ x \quad 1;\ x \quad \frac{1}{1+\tilde{y}};\ x+\frac{1}{\tilde{y}};x \quad \frac{1+\tilde{y}}{\tilde{y}};x \quad \frac{\tilde{y}}{1+\tilde{y}} \quad : \tag{14}$$

Preliminary results

- 121 master integrals.
- Ongoing work to obtain a canonical basis.
- At least one square root $\sqrt{m^2 s_{12} s_{23} s_{13}}$.

Table of Contents

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Thank you for your attention!