Calculating the Casimir effect using the boundary element method

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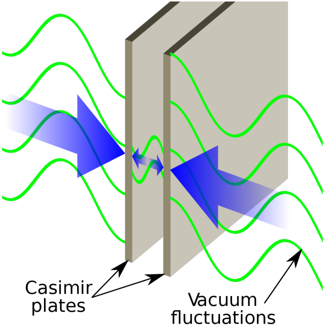
Goals

1. Give a brief introduction to the Casimir force
2. Show some examples of renormalization
3. Explain how to numerically solve integral equations



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  - Discretizes the whole space, which is computationally expensive
- Functional integral method
  - Expresses the force in terms of a matrix determinant derived from a functional integral
  - Cumbersome and theoretically shaky
  - Efficient, but we can't parallelize it

The boundary integral method

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- Gives the pressure at each point
- Requires solving a system of linear equations, which is trivially parallelizable
- Simplifications from symmetries can be explicitly implemented

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Underlying equations

The free massless scalar field is determined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$$

which obeys the Euler-Lagrange equation,

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

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This is equivalent to the wave equation,

$$\nabla^2\phi - \phi_{tt} = 0$$

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With boundary conditions, only a set of resonance frequencies $\{\omega_n\}$ are allowed. The energy is given by

$$E = \frac{1}{2} \sum_n \omega_n$$

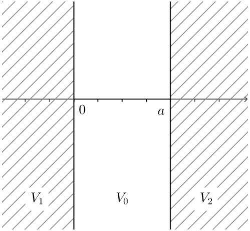
This is the Casimir energy.

Parallel plates

As an example, let's consider parallel plates in one-dimensional space with Dirichlet boundary conditions.

$$\phi_{xx} + \omega^2 \phi = 0$$

$$\phi(0) = \phi(a) = 0$$



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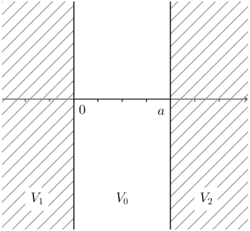

General solutions are given by

$$\phi(x) = A\cos\omega_n x + B\sin\omega_n x, \qquad \omega_n = \frac{n\pi}{a}$$

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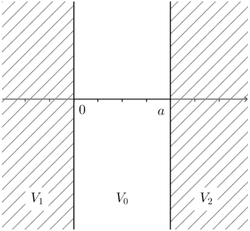

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This presents a problem, because

$$E = \frac{1}{2} \sum_n \omega_n = \sum_{n=1}^{\infty} \frac{n\pi}{2a} = \infty$$

Renormalization

Approaches to renormalization

Renormalization

Approaches to renormalization

- Separate the energy into two parts $E(\boldsymbol{x}, t) = E_\infty + \mathcal{E}(\boldsymbol{x}, t)$, where $\mathcal{E}$ is finite and E_∞ is constant. Note that $\nabla E = \nabla \mathcal{E}$.

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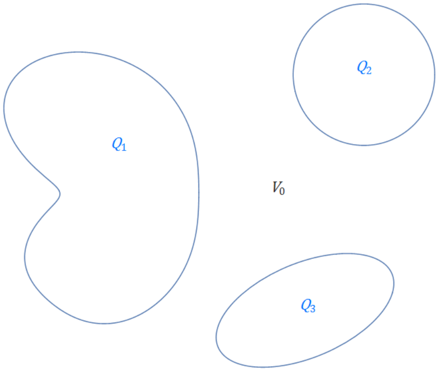
$$\mathfrak{R} \sum_{n=1}^{\infty} n = -\frac{1}{12}$$

Outline

Configuration

We consider a set of compact objects $V_1, \ldots, V_r$ with boundaries $Q_1, \ldots, Q_r$. Let V_0 be the exterior of all objects, and denote by Q the union of all surfaces.



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The stress-energy tensor:

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$

Conservation laws:

$$\partial_\nu T^{\mu\nu} = 0$$

$$\partial_t \boldsymbol{p} + \nabla\cdot S = 0$$

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$$\boldsymbol{F}_\alpha = \partial_t \int_{V_\alpha} dV\,\boldsymbol{p}(\boldsymbol{x},t) = -\oint_{Q_\alpha} d\boldsymbol{A}\cdot S(\boldsymbol{x},t)$$

Quantizing the stress tensor

The stress tensor is given by

$$S(\boldsymbol{x}, t) = -\nabla\phi\nabla\phi + \frac{1}{2}\mathsf{Tr}(\nabla\phi\nabla\phi)I - \frac{1}{2}\phi_t^2 I$$

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To be able to quantize this, we use point splitting

$$S(\boldsymbol{x}, t) = \lim_{\substack{\boldsymbol{x}'\to\boldsymbol{x}\\ t'\to t}} \left(-\nabla\nabla' + \frac{1}{2}\mathsf{Tr}(\nabla\nabla') - \frac{1}{2}\partial_t\partial_{t'}\right)\phi(\boldsymbol{x}, t)\phi(\boldsymbol{x}', t')$$

Then we can quantize the field and get the quantum stress tensor

$$\hat{S}(\boldsymbol{x}, t) = \lim_{\substack{\boldsymbol{x}'\to\boldsymbol{x}\\ t'\to t}} \left(-\nabla\nabla' + \frac{1}{2}\mathsf{Tr}(\nabla\nabla') - \frac{1}{2}\partial_t\partial_{t'}\right)\frac{1}{2}\{\hat{\phi}(\boldsymbol{x}, t)\hat{\phi}(\boldsymbol{x}', t')\},$$

where $\{\cdot,\cdot\}$ is the anti-commutator.

The Green's function

After a Wick rotation $s = it$, we can show that

$$\frac{1}{2}\left\langle \{\hat{\phi}(\boldsymbol{x}, s)\hat{\phi}(\boldsymbol{x}', s')\} \right\rangle = \left\langle \mathcal{T}[\hat{\phi}(\boldsymbol{x}, s)\hat{\phi}(\boldsymbol{x}', s')] \right\rangle \equiv D(\boldsymbol{x}, s, \boldsymbol{x}', s')$$

where $\mathcal{T}$ indicates time ordering.

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where $\mathcal{T}$ indicates time ordering. D depends only on the time difference $s - s'$, and it is periodic with period $\beta = 1/T$, where T is temperature. As $T \to 0$, we can take the Fourier transform, and the $D(\boldsymbol{x}, \boldsymbol{x}', \omega)$ turns out to be a Green's function for the operator $L = \nabla^2 - \omega^2$,

$$\nabla^2 D(\boldsymbol{x}, \boldsymbol{x}', \omega) - \omega^2 D(\boldsymbol{x}, \boldsymbol{x}', \omega) = \delta(\boldsymbol{x} - \boldsymbol{x}')$$

The Green's function

Now the quantum stress tensor can be written as

$$\left\langle \hat{S}(\boldsymbol{x},\omega) \right\rangle = \lim_{\boldsymbol{x}' \to \boldsymbol{x}} \left(-\nabla\nabla' + \frac{1}{2}\mathsf{Tr}(\nabla\nabla') + \frac{1}{2}\omega^2 \right) D(\boldsymbol{x},\boldsymbol{x}',\omega)$$

and the force integral becomes

$$\boldsymbol{F}_\alpha = -\int_{Q_\alpha} dA \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(-\partial_{\boldsymbol{n}}\nabla' + \frac{1}{2}\boldsymbol{n}\nabla\cdot\nabla' + \frac{1}{2}\boldsymbol{n}\omega^2 \right) D(\boldsymbol{x},\boldsymbol{x},\omega)$$

Summary of the force integral

The force on Q_α is

$$\boldsymbol{F}_\alpha = -\int_{Q_\alpha} dA \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(-\partial_{\boldsymbol{n}} \nabla' + \frac{1}{2} \boldsymbol{n} \nabla \cdot \nabla' + \frac{1}{2} \boldsymbol{n} \omega^2 \right) D(\boldsymbol{x}, \boldsymbol{x}, \omega)$$

and we can thus define pressure as

$$p = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(-\partial_{\boldsymbol{n}} \nabla' + \frac{1}{2} \boldsymbol{n} \nabla \cdot \nabla' + \frac{1}{2} \boldsymbol{n} \omega^2 \right) D(\boldsymbol{x}, \boldsymbol{x}, \omega)$$

where D is found by solving

$$\nabla^2 D(\boldsymbol{x}, \boldsymbol{x}', \omega) - \omega^2 D(\boldsymbol{x}, \boldsymbol{x}', \omega) = \delta(\boldsymbol{x} - \boldsymbol{x}')$$

Boundary conditions

If $\mathcal{A}$ is a linear operator and $\hat{\phi}$ satisfies a boundary condition on the form

$$\mathcal{A}\hat{\phi}(\boldsymbol{x}, s) = 0, \qquad \boldsymbol{x} \in Q$$

then similar conditions apply to D,

$$\mathcal{A}D(\boldsymbol{x}, \boldsymbol{x}', \omega) = 0, \qquad \boldsymbol{x} \in Q$$
$$\mathcal{A}'D(\boldsymbol{x}, \boldsymbol{x}', \omega) = 0, \qquad \boldsymbol{x}' \in Q$$

For example, with Dirichlet conditions, $\hat{\phi}(\boldsymbol{x}, s) = 0$ so

$$D(\boldsymbol{x}, \boldsymbol{x}', \omega) = 0, \qquad \boldsymbol{x} \in Q \text{ or } \boldsymbol{x}' \in Q$$

Outline

Integral identity

Let $D_0(\boldsymbol{x}, \boldsymbol{x}'')$ be the free Green's function for L, satisfying

$$LD_0(\boldsymbol{x}, \boldsymbol{x}'') = \delta(\boldsymbol{x} - \boldsymbol{x}''), \qquad D_0 \to 0 \text{ when } |\boldsymbol{x}| \to \infty$$

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For $\boldsymbol{x}', \boldsymbol{x}'' \in V_0$, Green's second identity gives

$$\int_{V_0} d\boldsymbol{\xi} \left[D(\boldsymbol{\xi}, \boldsymbol{x}')LD_0(\boldsymbol{\xi}, \boldsymbol{x}'') - D_0(\boldsymbol{\xi}, \boldsymbol{x}'')LD(\boldsymbol{\xi}, \boldsymbol{x}')\right]$$
$$= -\int_{Q} d\boldsymbol{\xi} \left[D(\boldsymbol{\xi}, \boldsymbol{x}')\partial_{\boldsymbol{n}}D_0(\boldsymbol{\xi}, \boldsymbol{x}'') - D_0(\boldsymbol{\xi}, \boldsymbol{x}'')\partial_{\boldsymbol{n}}D(\boldsymbol{\xi}, \boldsymbol{x}')\right]$$

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For $\boldsymbol{x}', \boldsymbol{x}'' \in V_0$, Green's second identity gives

$$\int_{V_0} d\boldsymbol{\xi} \left[D(\boldsymbol{\xi}, \boldsymbol{x}') L D_0(\boldsymbol{\xi}, \boldsymbol{x}'') - D_0(\boldsymbol{\xi}, \boldsymbol{x}'') L D(\boldsymbol{\xi}, \boldsymbol{x}') \right]$$
$$= - \int_Q d\boldsymbol{\xi} \left[D(\boldsymbol{\xi}, \boldsymbol{x}') \partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}'') - D_0(\boldsymbol{\xi}, \boldsymbol{x}'') \partial_{\boldsymbol{n}} D(\boldsymbol{\xi}, \boldsymbol{x}') \right]$$

which implies that D satisfies the integral identity

$$D(\boldsymbol{x}'', \boldsymbol{x}') = D_0(\boldsymbol{x}', \boldsymbol{x}'') + \int_Q d\boldsymbol{\xi} \big[D_0(\boldsymbol{\xi}, \boldsymbol{x}'') \partial_{\boldsymbol{n}} D(\boldsymbol{\xi}, \boldsymbol{x}') - D(\boldsymbol{\xi}, \boldsymbol{x}') \partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}'') \big]$$

Dirichlet boundary conditions

$$D(\boldsymbol{x}'', \boldsymbol{x}') = D_0(\boldsymbol{x}', \boldsymbol{x}'') + \int_Q d\boldsymbol{\xi} \big[D_0(\boldsymbol{\xi}, \boldsymbol{x}'') \partial_{\boldsymbol{n}} D(\boldsymbol{\xi}, \boldsymbol{x}') - D(\boldsymbol{\xi}, \boldsymbol{x}') \partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}'') \big]$$

Consider Dirichlet conditions, $D(\boldsymbol{x}', \boldsymbol{x}'') = 0$ when $\boldsymbol{x}' \in Q$ or $\boldsymbol{x}'' \in Q$. With these conditions, the force integral becomes

$$\boldsymbol{F}_\alpha = \frac{1}{2} \int_{Q_\alpha} dA \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \partial_{\boldsymbol{n}\boldsymbol{n}'} D(\boldsymbol{x}, \boldsymbol{x}, \omega)$$

and the integral identity simplifies to

$$D(\boldsymbol{x}'', \boldsymbol{x}') = D_0(\boldsymbol{x}', \boldsymbol{x}'') + \int_Q d\boldsymbol{\xi}\, D_0(\boldsymbol{\xi}, \boldsymbol{x}'') \partial_{\boldsymbol{n}} D(\boldsymbol{\xi}, \boldsymbol{x}')$$

Dirichlet boundary conditions - Renormalization

$$D(\boldsymbol{x}'', \boldsymbol{x}') = D_0(\boldsymbol{x}', \boldsymbol{x}'') + \int_Q d\boldsymbol{\xi}\, D_0(\boldsymbol{\xi}, \boldsymbol{x}'')\partial_{\boldsymbol{n}} D(\boldsymbol{\xi}, \boldsymbol{x}')$$

Let $\boldsymbol{x}_\alpha$ be a point on Q_α, and let $\boldsymbol{x}'' \to \boldsymbol{x}_\alpha$. We get

$$-D_0(\boldsymbol{x}', \boldsymbol{x}_\alpha) = \int_Q d\boldsymbol{\xi}\, D_0(\boldsymbol{\xi}, \boldsymbol{x}_\alpha)\partial_{\boldsymbol{n}} D(\boldsymbol{\xi}, \boldsymbol{x}')$$

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$$D(\boldsymbol{x}'',\boldsymbol{x}') = D_0(\boldsymbol{x}',\boldsymbol{x}'') + \int_Q d\boldsymbol{\xi}\, D_0(\boldsymbol{\xi},\boldsymbol{x}'')\partial_{\boldsymbol{n}} D(\boldsymbol{\xi},\boldsymbol{x}')$$

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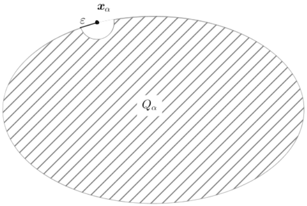

$$\int_Q d\boldsymbol{\xi} = \int_{C_\varepsilon} d\boldsymbol{\xi} + PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}$$

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Next we let $\boldsymbol{x}' \to \boldsymbol{x}_\beta \in Q_\beta$. This causes a problem if $\boldsymbol{x}_\alpha = \boldsymbol{x}_\beta$.

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Next we let $\boldsymbol{x}' \to \boldsymbol{x}_\beta \in Q_\beta$. This causes a problem if $\boldsymbol{x}_\alpha = \boldsymbol{x}_\beta$.

To solve this, introduce the self-pressure D_β, corresponding to the pressure we would get if Q_β had been the only object. That is, it satisfies the equation

$$-D_0(\boldsymbol{x}', \boldsymbol{x}_\alpha) = PV_{\boldsymbol{x}_\alpha} \int_{Q_\beta} d\boldsymbol{\xi}\, D_0(\boldsymbol{\xi}, \boldsymbol{x}_\alpha)\partial_{\boldsymbol{n}} D_\beta(\boldsymbol{\xi}, \boldsymbol{x}')$$

Dirichlet boundary conditions - The self pressure

Now let

$$\mathcal{D}(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) = \begin{cases} D(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - D_\beta(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta), & \alpha = \beta, \\ D(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta), & \alpha \neq \beta \end{cases}$$

be the regularized pressure. When subtracting the self-pressure, the D_0 terms cancel, so taking the limit $\boldsymbol{x}' \to \boldsymbol{x}_\beta$ is no problem.

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be the regularized pressure. When subtracting the self-pressure, the D_0 terms cancel, so taking the limit $\boldsymbol{x}' \to \boldsymbol{x}_\beta$ is no problem.

By the way, what we eventually are interested in is $\partial_{\boldsymbol{n}\boldsymbol{n}'}\mathcal{D}$, not $\partial_{\boldsymbol{n}}\mathcal{D}$. In order to acquire the normal derivative, we must take the gradient $\nabla_{\boldsymbol{x}'}$ before letting $\boldsymbol{x}'$ go to the boundary.

Dirichlet boundary conditions - The boundary integral problem

The boundary integral problem for $\mathcal{D}$ becomes

$$V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) + PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}\, D_0(\boldsymbol{\xi}, \boldsymbol{x}_\alpha)\partial_{\boldsymbol{n}\boldsymbol{n}'}\mathcal{D}(\boldsymbol{\xi}, \boldsymbol{x}_\beta) = 0$$

where

$$\begin{aligned} V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) = &- \partial_{\boldsymbol{n}} D_0(\boldsymbol{x}_\beta, \boldsymbol{x}_\alpha) \\ &- PV_{\boldsymbol{x}_\alpha} \int_{Q_\beta} d\boldsymbol{\xi}\, D_0(\boldsymbol{\xi}, \boldsymbol{x}_\alpha)\partial_{\boldsymbol{n}\boldsymbol{n}'} D_\beta(\boldsymbol{\xi}, \boldsymbol{x}_\beta) \end{aligned}$$

when $\boldsymbol{x}_\alpha$ and $\boldsymbol{x}_\beta$ are on different surfaces, and it is zero otherwise.

Neumann boundary conditions in 2D

$$D(\boldsymbol{x}'', \boldsymbol{x}') = D_0(\boldsymbol{x}', \boldsymbol{x}'') + \int_Q d\boldsymbol{\xi} \big[D_0(\boldsymbol{\xi}, \boldsymbol{x}'') \partial_{\boldsymbol{n}} D(\boldsymbol{\xi}, \boldsymbol{x}') \\ - D(\boldsymbol{\xi}, \boldsymbol{x}') \partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}'') \big]$$

Next consider Neumann conditions, $\partial_{\boldsymbol{n}'} D(\boldsymbol{x}', \boldsymbol{x}'') = 0$ when $\boldsymbol{x}' \in Q$ and $\partial_{\boldsymbol{n}''} D(\boldsymbol{x}', \boldsymbol{x}'') = 0$ when $\boldsymbol{x}'' \in Q$. In two dimensions, the force integral becomes

$$\boldsymbol{F}_\alpha = -\frac{1}{2} \int_{Q_\alpha} dA \, \boldsymbol{n} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\partial_{\boldsymbol{t}\boldsymbol{t}'} + \omega^2 \right) D(\boldsymbol{x}, \boldsymbol{x}, \omega)$$

and the boundary identity becomes

$$D(\boldsymbol{x}'', \boldsymbol{x}') = D_0(\boldsymbol{x}', \boldsymbol{x}'') - \int_Q d\boldsymbol{\xi} \, D(\boldsymbol{\xi}, \boldsymbol{x}') \partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}'')$$

Neumann boundary conditions in 2D - Renormalization

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Like before, we first let $\boldsymbol{x}'' \to \boldsymbol{x}_\alpha \in Q_\alpha$.

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Like before, we first let $\boldsymbol{x}'' \to \boldsymbol{x}_\alpha \in Q_\alpha$. Now the integral over C_ε gives a contribution

$$\begin{aligned} &- \int_{C_\varepsilon} d\boldsymbol{\xi}\, D(\boldsymbol{\xi}, \boldsymbol{x}')\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}'') \\ \approx &- D(\boldsymbol{x}_\alpha, \boldsymbol{x}') \int_{C_\varepsilon} d\boldsymbol{\xi}\, \partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}'') \to \frac{1}{2} D(\boldsymbol{x}_\alpha, \boldsymbol{x}') \end{aligned}$$

The integral over the remainder of the curve becomes a principal value integral,

$$\frac{1}{2} D(\boldsymbol{x}_\alpha, \boldsymbol{x}') = D_0(\boldsymbol{x}', \boldsymbol{x}_\alpha) - PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}\, D(\boldsymbol{\xi}, \boldsymbol{x}')\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}_\alpha)$$

Neumann boundary conditions in 2D - Renormalization

$$\frac{1}{2}D(\boldsymbol{x}_\alpha, \boldsymbol{x}') = D_0(\boldsymbol{x}', \boldsymbol{x}_\alpha) - PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}\, D(\boldsymbol{\xi}, \boldsymbol{x}')\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}_\alpha)$$

When letting $\boldsymbol{x}' \to \boldsymbol{x}_\beta \in Q_\beta$, we get a problem if $\boldsymbol{x}_\alpha = \boldsymbol{x}_\beta$, because D_0 is singular. As before, we introduce the self-pressure D_β and the regularized pressure $\mathcal{D}$.

$$\frac{1}{2}D_\beta(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) = D_0(\boldsymbol{x}_\beta, \boldsymbol{x}_\alpha) - PV_{\boldsymbol{x}_\alpha} \int_{Q_\beta} d\boldsymbol{\xi}\, D_\beta(\boldsymbol{\xi}, \boldsymbol{x}_\beta)\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}_\alpha)$$

Neumann boundary conditions in 2D - The boundary integral problem

With this, we get

$$\frac{1}{2}\mathcal{D}(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) = V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}\, \mathcal{D}(\boldsymbol{\xi}, \boldsymbol{x}_\beta)\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}_\alpha)$$

where

$$V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) = D_0(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - \int_{Q_\beta} d\boldsymbol{\xi}\, D_\beta(\boldsymbol{\xi}, \boldsymbol{x}_\beta)\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}_\alpha)$$

when $\boldsymbol{x}_\alpha$ and $\boldsymbol{x}_\beta$ are on different objects, and V is zero otherwise.

Outline

Discretization - Dirichlet boundary conditions in 2D

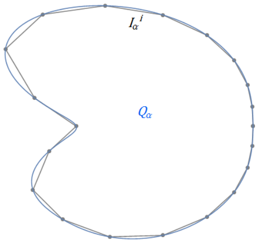
Considering Dirichlet boundary conditions, we have the equation

$$PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}\, D_0(\boldsymbol{\xi}, \boldsymbol{x}_\alpha) \partial_{\boldsymbol{n}\boldsymbol{n}'} \mathcal{D}(\boldsymbol{\xi}, \boldsymbol{x}_\beta) + V(\boldsymbol{x}_\beta, \boldsymbol{x}_\alpha) = 0$$

So now the question is, how do we apply numerical methods to solve it?

Discretization

Partition each object Q_γ into small pieces I_γ^k, for $k = 1, \ldots, N$.



Discretization

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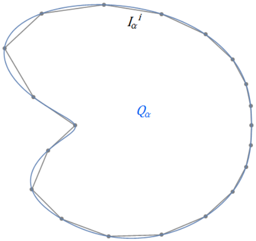


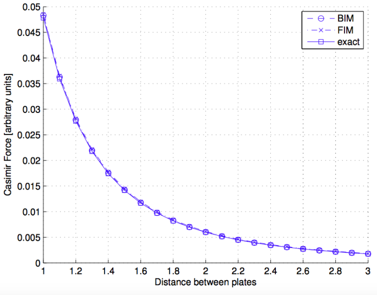
The equation becomes

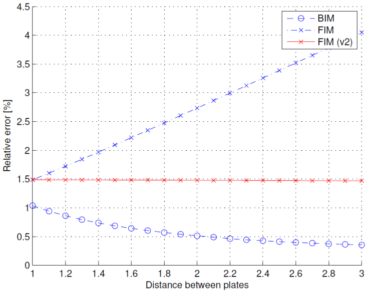
$$-V(\boldsymbol{x}_\beta^j, \boldsymbol{x}_\alpha^i) = \sum_{\gamma=1}^{r} \sum_{k=1}^{N} PV_{\boldsymbol{x}_\alpha^i} \int_{I_\gamma^k} d\boldsymbol{\xi}\, D_0(\boldsymbol{\xi}, \boldsymbol{x}_\alpha^i) \partial_{\boldsymbol{n}\boldsymbol{n}'} \mathcal{D}(\boldsymbol{\xi}, \boldsymbol{x}_\beta^j)$$

$$\approx \sum_{\gamma=1}^{r} \sum_{k=1}^{N} |I_\gamma^k| D_0(\boldsymbol{x}_\gamma^k, \boldsymbol{x}_\alpha^i) \partial_{\boldsymbol{n}\boldsymbol{n}'} \mathcal{D}(\boldsymbol{x}_\gamma^k, \boldsymbol{x}_\beta^j)$$

Results - Parallel plates in 2D

Results - Parallel plates in 2D





Results - Concentric circles in 2D

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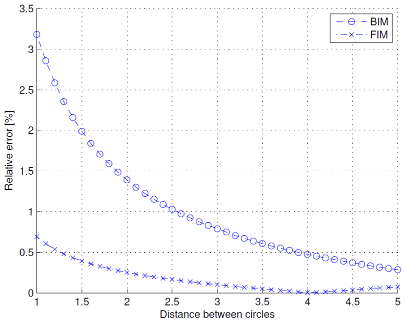
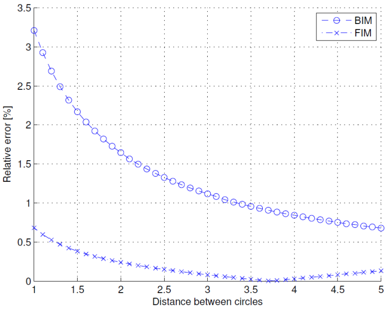


Inner circle



Outer circle

Results - Parallel plates in 3D

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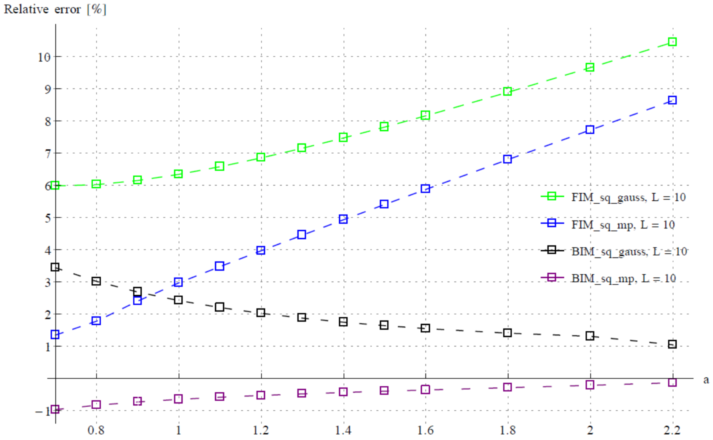

Results - Concentric spheres in 3D

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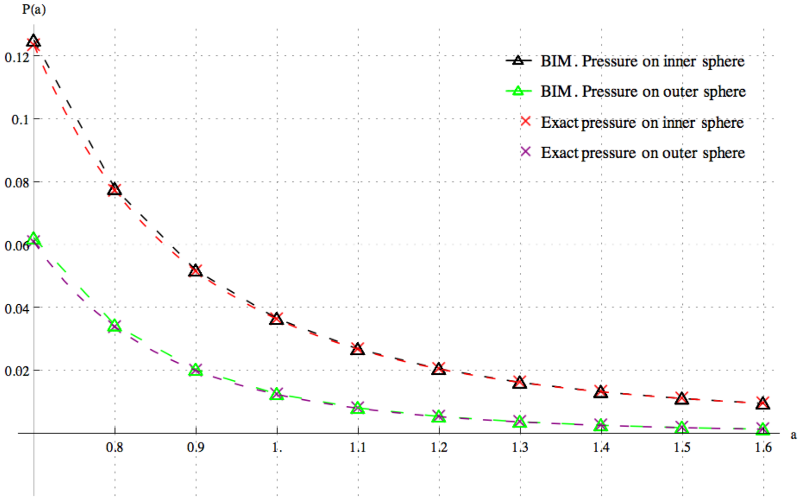
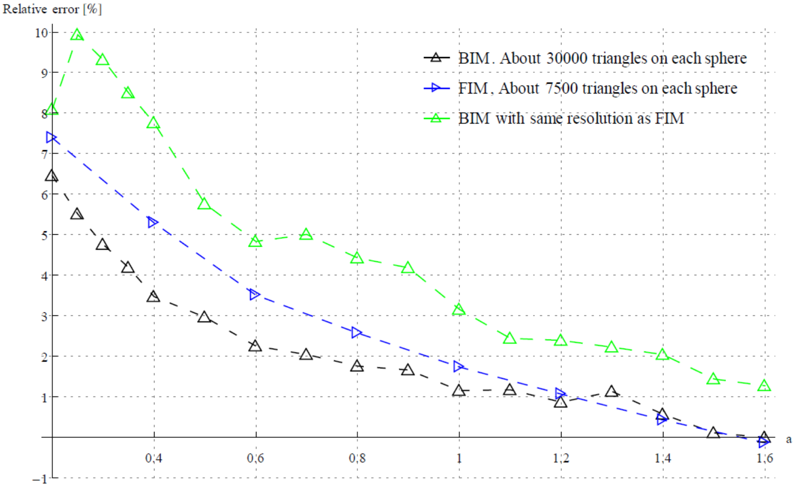


Outer sphere

Results - Concentric spheres in 3D

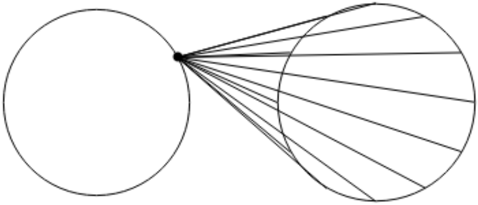


Outer sphere

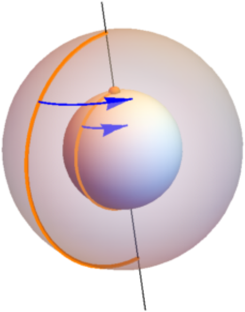
Outline

Interactions

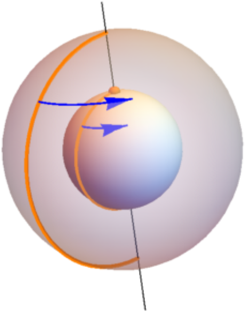
In $D(\boldsymbol{x}, \boldsymbol{x}')$, we can think of $\boldsymbol{x}$ as the observation point and $\boldsymbol{x}'$ as the source location. Then $D(\boldsymbol{x}, \boldsymbol{x}')$ tells us about the contribution of point $\boldsymbol{x}'$ to the Casimir effect at point $\boldsymbol{x}$.



Source location symmetries



Source location symmetries

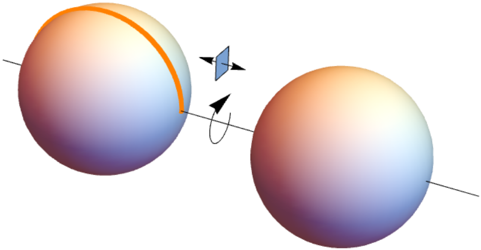


If g is an isometry that leaves the surfaces invariant and for which $\boldsymbol{x}$ is a fixed point, i.e. $g(\boldsymbol{x}) = \boldsymbol{x}$, it can be shown that

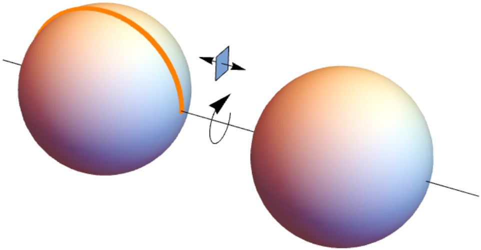
$$D(\boldsymbol{x}, g(\boldsymbol{x}')) = D(\boldsymbol{x}, \boldsymbol{x}'),$$

then this symmetry can be utilized when calculating $D(\boldsymbol{x}, \boldsymbol{x}')$.

Observation point symmetries



Observation point symmetries



If h is an isometry that leave the surfaces invariant, then we can show that

$$D(h(\boldsymbol{x}), \boldsymbol{x}') = D(\boldsymbol{x}, h^{-1}(\boldsymbol{x}')),$$

and we can utilize these kind of symmetries.

Outline

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- One problem is that the method gives $1/2$ the answer predicted by the exact solutions or the functional integral method, both for 2D and 3D
- Source test shows that this is not a problem with the renormalization, so it seems a factor 2 is missing in the force integral

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- Source tests proves that the renormalization is correct, except for the self-pressure
- Possibly a problem with how we are handling the $\partial_{\boldsymbol{tt'}} D$ part of the force integrand
- Other methods also have problems with Neumann conditions

Future work

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- Understand why the Dirichlet case gives a factor $1/2$

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- Show that the method gives correct results under Neumann boundary conditions

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- Understand why the Dirichlet case gives a factor $1/2$
- Show that the method gives correct results under Neumann boundary conditions
- Develop the method for vector fields and Maxwell's equations

Thank you for your attention!

References and further reading I

H. B. G. Casimir.
On the Attraction Between Two Perfectly Conducting Plates.
Kon. Ned. Akad. Wetensch., 10:261–263, 1948.

I. Kilen.
Numerical calculation of Casimir forces.
Master's thesis, University of Tromsø, June 2012.

I. Kilen and P. Jakobsen.
Computing the Casimir force using regularized boundary integral equations.
Physica Scripta, 89(11):115–205, 2014.

K. Ø. Mikalsen.
Deforming the vacuum. On the physical origin and numerical calculation of the Casimir effect.
Master's thesis, University of Tromsø, May 2014.

References and further reading II

M. Utheim.
Insulating the Vacuum. Calculating the Casimir force using the boundary integral method with von Neumann boundary conditions.
Master's thesis, UiT - The Arctic University of Norway, August 2016.

T. Emig, N. Graham, R. L. Jaffe, and M. Kardar.
Casimir forces between compact objects: The scalar case.
Phys. Rev. D, 77(2):025005, 2008.

S. M. Christensen.
Vacuum expectation value of the stress tensor in an arbitrary curved background: the covariant point-separation method.
Phys. Rev. D, 14(10):2490, 1976.