Quasi-local energy and compactification

Raquel Santos García

INVISIBLES18 WORKSHOP

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Work in collaboration with
Enrique Alvarez, Jesus Anero and Guillermo Milans del Bosch

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Instituto de Física Teórica UAM-CSIC

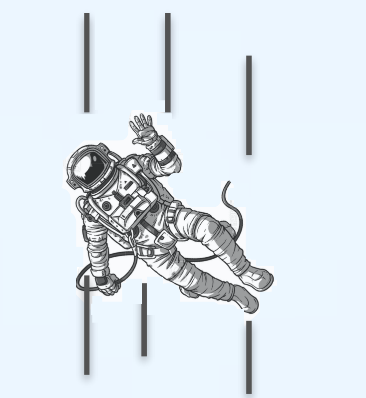
UNIVERSIDAD AUTONOMA DE MADRID

elusives

invisiblesPlus

Energy in General Relativity

The four-momentum cannot be a **four-vector** because it can always be made to **vanish locally** in a free falling frame.



Free falling observer

Equivalence principle →

Zero gravitational energy

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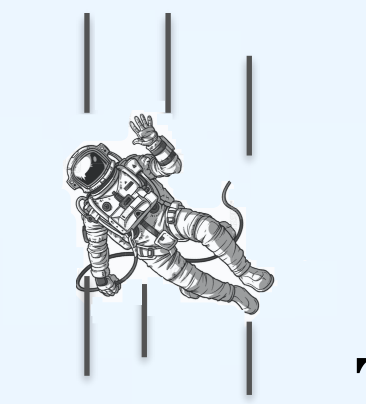

Free falling observer → (Equivalence principle) → Zero gravitational energy

The energy momentum tensor is covariantly conserved

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\underbrace{\partial_\mu T^{\mu\nu}}_{\text{Conservation in the usual sense}} = \underbrace{-\Gamma^\mu_{\mu\lambda} T^{\lambda\nu} - \Gamma^\nu_{\mu\lambda} T^{\lambda\mu}}_{\text{The purely gravitational contribution is not a tensor}}$$

Conservation in the usual sense

The purely gravitational contribution is not a tensor

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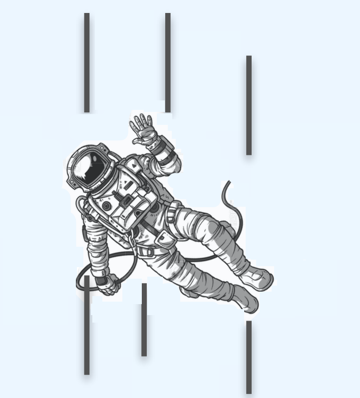

Free falling observer → (Equivalence principle) → Zero gravitational energy

Ground state of a Quantum theory of Gravity?

↓

We need a definition of energy for the gravitational field

Quasi-local gravitational energy

We want to compute a quantity which can be 'locally' sensitive to the strength of the gravitational field.

Quasi-local gravitational energy

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We associate to a given hypersurface of a spacetime, the **integral of the trace of the extrinsic curvature**.

The **quasi-local energy** (QLE) is defined as

$$Q(\Sigma) \equiv \int_\Sigma K$$

Dependent on the tangent and normal vectors of the hypersurface

[Brown and York 1993]

Quasi-local gravitational energy

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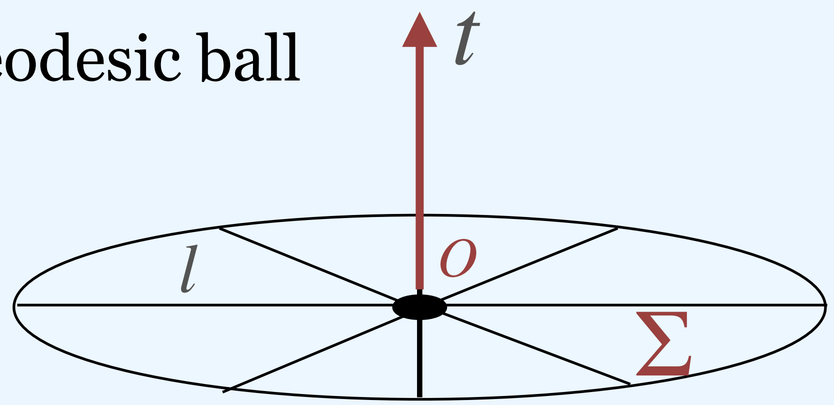
We can construct the surface as

Geodesic ball



We take a point and send geodesics of fixed length normal to the time direction.

This defines a spacelike hypersurface Σ.

Quasi-local gravitational energy

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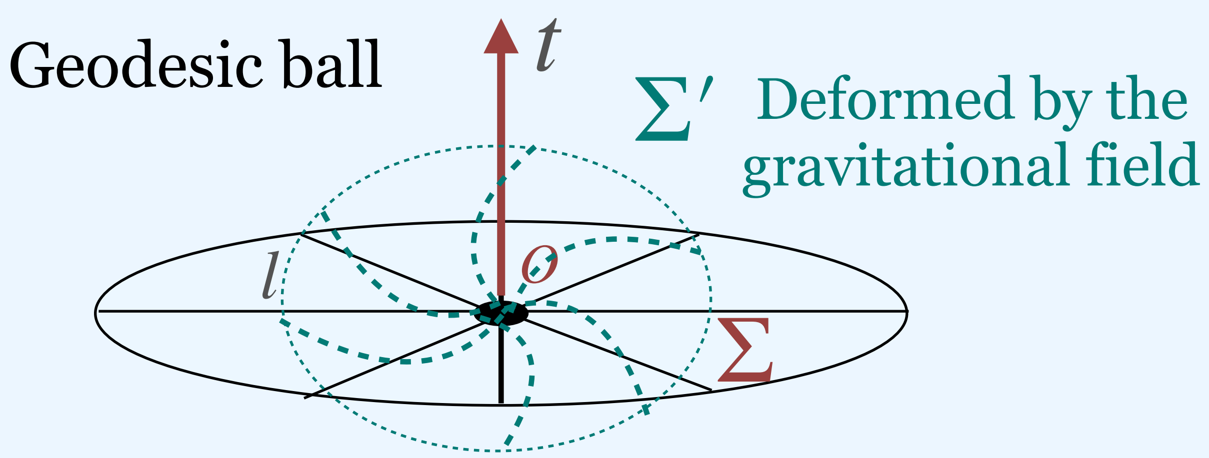

The extrinsic curvature of the hypersurface is sensitive to the strength of the gravitational field

QLE and compactification

Idea

The QLE definition of Brown and York is able to discriminate between the **uncompactified** Minkowski spacetime and the spacetimes with **compact dimensions**.

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The QLE definition of Brown and York is able to discriminate between the **uncompactified** Minkowski spacetime and the spacetimes with **compact dimensions**.

Strategy

Compute the QLE for 3-spheres in M_5

Compute the QLE for '3-spheres' in $M_4 \times S_1$

One compact dimension

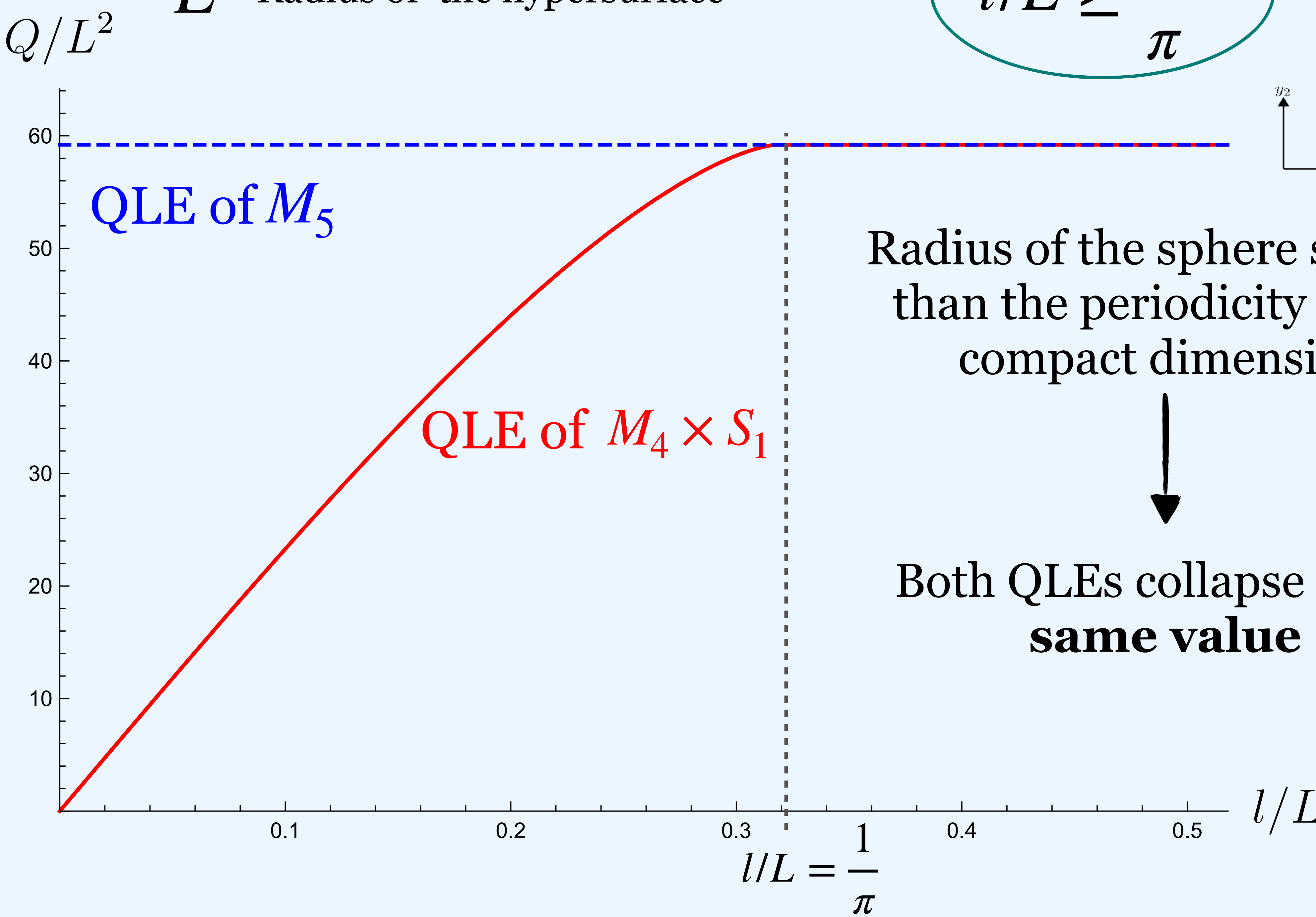
$$-\pi l \leq y \leq \pi l$$

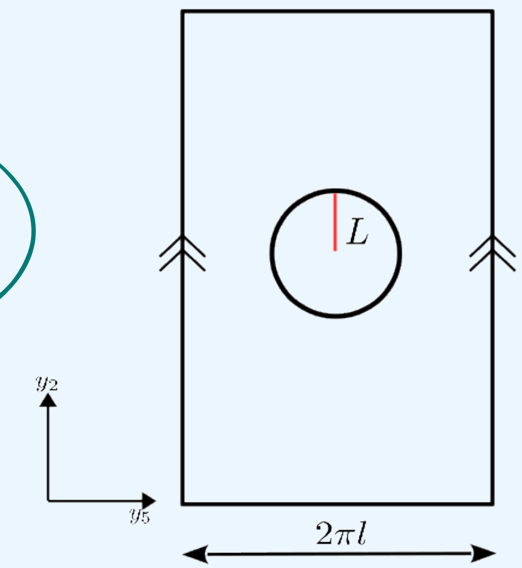
QLE and compactification

l Radius of the compact dimension

L Radius of the hypersurface

$$l/L \geq \frac{1}{\pi}$$





Radius of the sphere smaller than the periodicity of the compact dimension

↓

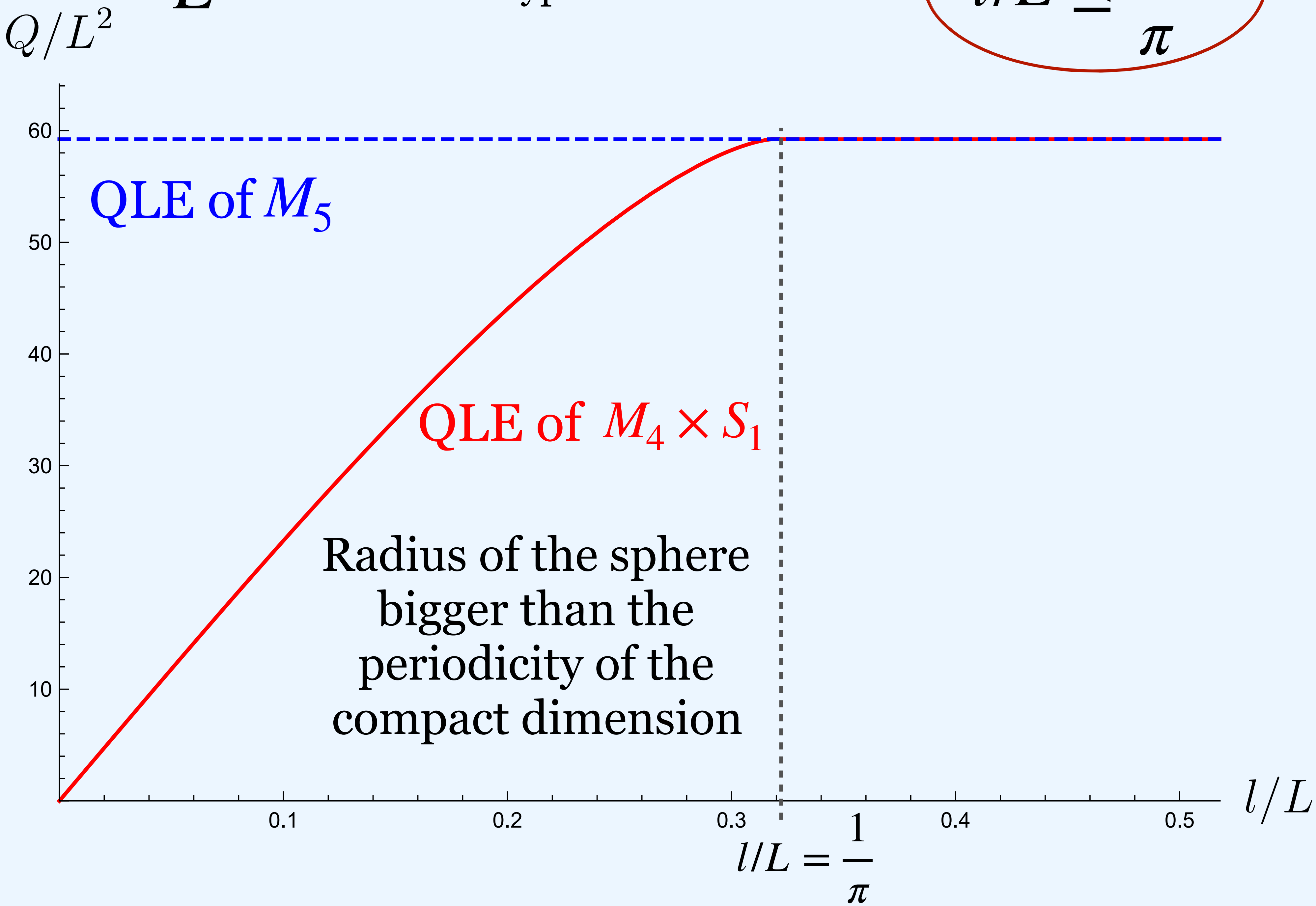
Both QLEs collapse to the **same value**

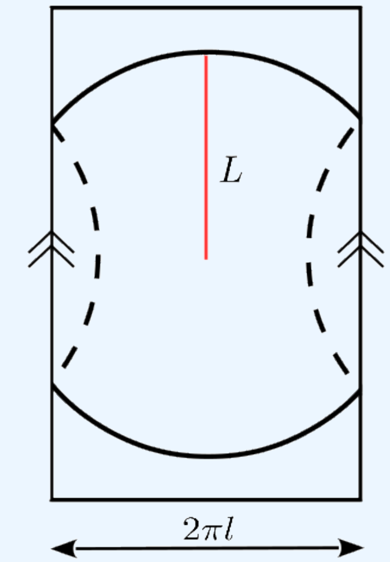
QLE and compactification

l Radius of the compact dimension

L Radius of the hypersurface

$$l/L \leq \frac{1}{\pi}$$





Summary and Outlook

The quasi-local energy could provide an **energetical argument** in favour of compactified or uncompactified spacetimes.

This is a proof of concept, but more general setups need to be studied, such as the introduction of fluxes in order to stabilise the compact dimension.

Work is ongoing regarding the use of the quasi-local energy as a tool to compute the **total energy** of full spacetimes in a **more covariant way**.

BACKUP

QLE and compactification

Codim-2 spheres M_5

Codim-2 spheres $M_4 \times S_1$

For both, the embedding

$$y_1 = T, \qquad \sum_{i=2}^{i=5} (y_i)^2 \equiv L^2 .$$

The two **normal** vectors and the three **tangent** vectors read

$$n_A \equiv \left(\frac{\partial}{\partial t}, \frac{y^i}{L} \frac{\partial}{\partial y^i} \right) \qquad A = 1,2$$

$$t_{\theta_1} = L\left(0, c\,\theta_1\, s\,\theta_2\, s\,\theta_3, c\,\theta_1\, s\,\theta_2\, c\theta_3, c\,\theta_1\, c\theta_2, -s\theta_1\right)$$

$$t_{\theta_2} = L\left(0, s\,\theta_1\, c\,\theta_2\, s\,\theta_3, s\,\theta_1\, c\,\theta_2\, c\theta_3, -s\theta_1\, s\theta_2, 0\right)$$

$$t_{\theta_3} = L\left(0, s\,\theta_1\, s\,\theta_2\, c\,\theta_3, -s\,\theta_1\, s\,\theta_2\, s\theta_3, 0, 0\right)$$

(Spherical coordinates)

QLE and compactification

Codim-2 spheres M_5

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Hence, the **extrinsic curvature tensor** yields $K^A_{ab} = \left(0, -\frac{1}{L}\,\delta_{\alpha\beta}t^\alpha_a t^\beta_b\right)$

QLE and compactification

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The trace of the extrinsic curvature is given by $K^\alpha = K^A n^\alpha_A = h^{ab}\ K^A_{ab} n^\alpha_A$

And the integration measure $\sqrt{h}\ n_\alpha\ dS$

Induced metric

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So we are interested in the integral

$$Q = \int_\Sigma \sqrt{h}\ K^\alpha n_\alpha dS$$

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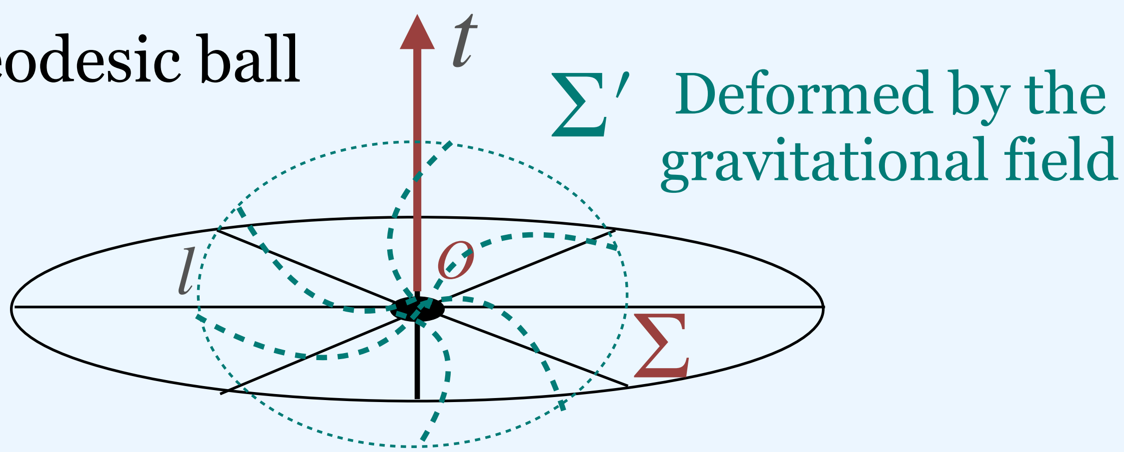
Difference between the two spacetimes

Quasi-local gravitational energy

We can define the surface as



The extrinsic curvature of the hypersurface is sensitive to the strength of the gravitational field

The **extrinsic curvature tensor** is given by

$$K^A_{ab} \equiv -t^\lambda_b t^\alpha_a \nabla_\lambda n^A_\alpha$$

t^α_a Tangent vectors $a = 1,\ldots,m$

n^A_α Normal vectors $A = 1,\ldots,p$

$p \equiv (n - m)$ Codimension of the hypersurface

QLE and compactification

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QLE and compactification

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No restrictions in the range of the variables appearing in the integral

$$Q_{M_5} = \int_\Sigma \sqrt{h}\ K^\alpha n_\alpha dS = 6\pi^2 L^2$$

QLE and compactification

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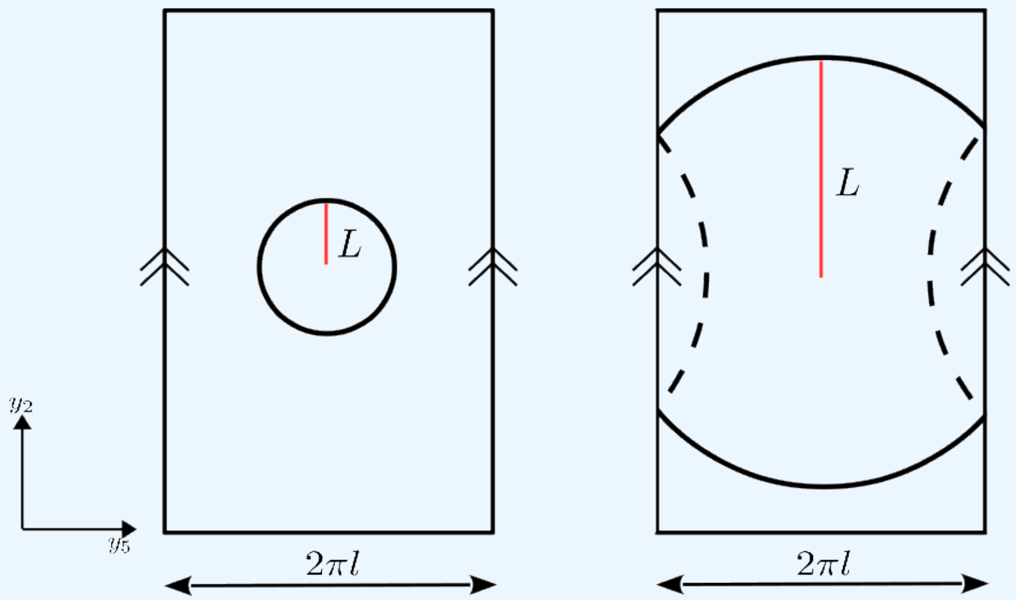
Codim-2 spheres $M_4 \times S_1$

For **small 3-spheres** that completly lie within the compact dimension,

$$-\frac{L}{2} \leq y_5 \leq \frac{L}{2}$$

When $L < 2\pi l$, the **range** is **restricted**

$$-l\pi \leq y_5 \leq l\pi$$



QLE and compactification

Surfaces of constant time and radius

$$y_1 = T, \qquad \sum_{i=2}^{i=5} (y_i)^2 \equiv L^2 .$$

The two **normal** vectors and the three **tangent** vectors are the same for the two spacetimes

→ Same value for the extrinsic curvature K

But for the QLE the integration range is different

$$Q = \int_{\Sigma} K$$

Difference between the two spacetimes

QLE and compactification

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$$Q_{M_4 \times S_1} = 6\pi^2 L^2 \quad for \quad L \leq l\pi$$

$$Q_{M_4 \times S_1} = 12\pi^2 l \sqrt{L^2 - \pi^2 l^2} + 12\pi L^2 \tan^{-1}\left(\frac{\pi l}{\sqrt{L^2 - \pi^2 l^2}}\right) \quad for \quad L > l\pi$$