Neutrino Cosmology Exercises

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Strategy: Doing exercises 1-4 should not be too time consuming and will give you a good idea of key numbers and calculations relevant for the cosmology of neutrinos. Exercises 5-10 are a bit more advanced, but can be done and will give you a flavor of how cosmological bounds are derived on sterile neutrinos as well as other types of light particles from cosmological observations. They will also give you a bit of familiarity with the late time evolution of the CNB.

1. **Number and energy density in the Cosmic Neutrino Background today**
   Estimated time: 15 mins, *difficulty*: 3/10, *result*: $n_\nu^{\rm today} = 346{\rm cm}^{-3}$, $\Omega_\nu h^2|^{\rm rel} = 1.75 \times 10^{-5}$, $\Omega_\nu h^2|^{\rm non-rel} = \sum m_\nu/(91.4\,{\rm eV})$.
   Taking the CNB temperature today to be $T_\nu = 2.75/1.4\,{\rm K}$, obtain: 1) the number density of neutrinos+antineutrinos today. 2) The fraction of the energy density of the Universe they represent. Note that today's critical energy density is $\rho_c = 1.054 \times 10^4 h^2 {\rm eV/cm}^3$. Do the calculation for neutrinos which are relativistic today, as well as for those which are non-relativistic today. Compare the two using $m_\nu = 0.05\,{\rm eV}$ and $m_\nu = 0$.

2. **Time of neutrinos becoming non-relativistic**
   Estimated time: 10 mins, *difficulty*: 3/10, *result*: $z_\nu \simeq 200 m_\nu/(0.1{\rm eV})$, at least 2.
   Taking $T_\nu = 2.75\,{\rm K}/1.4$, obtain the redshift at which neutrinos would become non-relativistic. Note that this happens when $T_\nu \lesssim m_\nu/3$. Given the observed mass squared differences in neutrino oscillation experiments, how many neutrinos are non-relativistic today?

3. **Calculation of the temperature of the Cosmic Neutrino Background**
   Estimated time: 10 mins, *difficulty*: 3/10, *result*: $T_\nu \simeq T_\gamma/1.4$.
   Using entropy conservation, calculate the temperature of the cosmic neutrino background after electrons and positrons have annihilated from the plasma. Take into account that neutrinos decoupled at a temperature of around 2 MeV, at which point electrons and positrons were still relativistic.

4. **Neutrino Oscillations in the Early Universe**
   Estimated time: 20 mins, *difficulty*: 4/10, *result*: $T_\nu^{\rm osc} \simeq \mathcal{O}(10)\,{\rm MeV}$.
   Using the 2-neutrino oscillation formula $P(\nu_\alpha \to \nu_\beta, t) = \sin^2(2\theta)\sin^2(\Delta m^2 t/(4E_\nu))$ estimate the time at which neutrino oscillations will take place. For the numerics, use $\theta = 0.1$, $\Delta m^2 = 10^{-3}\,{\rm eV}^2$, and $t = 1/(2H)$ with $H = 1.66\sqrt{g_\star}T^2/M_{\rm Pl}$, with $g_\star \simeq 10.75$ and $M_{\rm Pl} = 1.22 \times 10^{19}\,{\rm GeV}$.

5. **Calculation of the contribution to $\Delta N_{\rm eff}$ from a thermal Axion**
   Estimated time: 15 mins, *difficulty*: 6/10, *result*: $\Delta N_{\rm eff} \simeq 0.03$.
   Using entropy conservation, calculate the contribution to $\Delta N_{\rm eff}$ of an axion (scalar particle with one internal degree of freedom) which was present in thermal abundances in the early Universe but decoupled at a temperature of $T = 1\,{\rm TeV}$ at which point all the SM particles were present and relativistic. In particular, at the time $g_\star = 106.75$. *Remember the definition of $N_{\rm eff} \equiv 8/7(11/4)^{4/3}[\rho_{\rm rad} - \rho_\gamma]/\rho_\gamma$ at $T \ll m_e$.

6. **Distance a neutrino travels in the Universe**
Estimated time: 15 mins, *difficulty*: 4/10, *result*: $L \simeq 20\,\mathrm{Mpc}\,0.1\,\mathrm{eV}/m_\nu$
In order to estimate the length scales at which neutrinos cannot cluster estimate the distance they have traveled since the Big Bang. You can use as an estimate $L \simeq v_\nu\, t_U$, where $v_\nu = \langle p_\nu \rangle / m_\nu$ is the velocity today and where $\langle p_\nu \rangle \simeq 3T_\nu$. Take $t_U = 13.8\,\mathrm{Gyr}$ for numerical evaluation.

7. **Understanding correlations of the neutrino mass with other cosmological parameters and the CMB**
Estimated time: 20 mins, *difficulty*: 5/10
The angular diameter distance to the CMB is written as:

$$D_A = \int_0^{z_{\rm CMB}} \frac{1}{H(z')} dz' \tag{1}$$

where $H(z)$ is the Hubble expansion rate, namely $H(z) = (8\pi G/3)\rho(z)$ in a flat Universe and where $z_{\rm CMB} \simeq 1100$. Considering a flat Universe with only matter and a cosmological constant yields:

$$H(z)^2 = H_0^2 \left[\Omega_\Lambda + \Omega_m (1+z)^3\right] \tag{2}$$

where H_0 is the Hubble expansion rate today. Given that the Universe should be flat, $\Omega_\Lambda = 1 - \Omega_m$ and this means that the only free parameters are H_0 and Ω_m. Calculate numerically:

$$\frac{D_A(\Omega_m = 0.3, H_0 = 70\,\mathrm{km/s/Mpc})}{D_A(\Omega_m = 0.4, H_0 = 70\,\mathrm{km/s/Mpc})} \tag{3}$$

The result is $\simeq 1.18$ which shows that if H_0 is being fixed and the matter density decreases the distance to the CMB increases. This is because Ω_Λ will increase as a result.

Calculate now

$$\frac{D_A(\Omega_m = 0.3, H_0 = 70\,\mathrm{km/s/Mpc})}{D_A(\Omega_m = 0.4, H_0 = 0.57\,\mathrm{km/s/Mpc})} \tag{4}$$

the result is $\simeq 1$. What the exercise shows is that the keeping constant the angular diameter distance to the CMB effectively requires the H_0 constant to be lower if the matter density is higher. Enhancing the neutrino mass effectively means enhancing Ω_m and therefore if the Hubble constant is ~ 73 this leads to a smaller inferred Ω_m and therefore to a smaller inference of the neutrino mass value. A direct measurement of H_0 has implications for the inference on $\sum m_\nu$.

8. **Estimate of the cosmological Sterile Neutrino bound from $N_{\rm eff}$**
Estimated time: 50 mins, *difficulty*: 8/10, *results*: (a) $T_{\rm max} = 20\,\mathrm{MeV}(\Delta m^2/\mathrm{eV})^{1/6}$, (b) $\theta^2 < 10^{-3} G_F M_{\rm Pl} \sqrt{\Delta m^2}$.
We have seen that a key criterium to see if a particle was produced in the early Universe is to compare its particle production rate Γ and the expansion rate of the Universe H. If $\Gamma > H$ then this particle was produced in a thermal abundance in the early Universe. Under certain

approximations, the interaction rate of a sterile neutrino can be written as [see Abazajian astro-ph/0511630]:

$$\Gamma \simeq \Gamma_\nu \frac{1}{1 + (100\,\Gamma_\nu/\Gamma_{\rm osc})^2} \sin^2 2\theta \quad (5)$$

where $\Gamma_\nu \simeq G_F^2 T^5$ is the typical active neutrino interaction rate and $\Gamma_{\rm osc} \simeq \Delta m^2/T$ is the typical oscillation rate, and the factor of 100 appears effectively due to the fact that neutrino oscillations are suppressed in the early Universe due to the large matter densities after the Big Bang. θ here is the mixing angle between the active and the sterile neutrino state.

(a) Obtain the temperature at which sterile neutrinos are maximally produced.

(b) For a sterile neutrino of $\Delta m^2 = 1\,{\rm eV}^2$ obtain the value of the vacuum mixing angle for which $\Gamma \simeq H(T)$, T being the temperature at which the rate is maximal. Compare this with Figure 9 of Gariazzo, de Salas & Pastor [1905.11290].

9. **Cosmological impact of $N_{\rm eff}$ at the time of BBN**

Estimated time: 60 mins, *difficulty*: 9/10. *solution:* (a) $t_U = 1/(2H)$, (b) $Y_p \equiv 4n_{^4{\rm He}}/(4n_{^4{\rm He}} + n_p) = 2n/p/(1+2n/p)$ (c) $T_F \simeq 0.52\,{\rm MeV}(5.5 + 1.75N_{\rm eff})^{1/6}$, (d) $Y_{\rm P} \simeq 0.244(N_{\rm eff}/3)^{0.16}$, (e) 5%, and therefore it will be constrained.

– Protons and neutrons were interacting in the early Universe via interactions of the type $n + \nu_e \leftrightarrow p + e^-$, $n + e^+ \leftrightarrow p + \bar\nu_e$ and $n \leftrightarrow p + e^- + \bar\nu_e$. These interactions (but for neutron decay) freeze-out at a temperature of around $T_F \simeq 0.77\,{\rm MeV}$. The number density of neutrons with respect to protons at the time is given by the thermal equilibrium relation:

$$\frac{n}{p} \simeq \exp(-Q/T) \quad (6)$$

where $Q = m_n - m_p \simeq 1.3\,{\rm MeV}$. Using $T_F \simeq 0.7\,{\rm MeV}$ one gets $n/p \simeq 1/6.5$. From then onwards, the free neutrons in the plasma were simply decaying until $T_{\rm BBN} = 0.075\,{\rm MeV}$ at which point the Universe is cool enough such that deuterium could start to form and in a rapid time almost all the free neutrons in the plasma formed stable ^{4}He. This effectively means that:

$$\left.\frac{n}{p}\right|_{\rm BBN} \simeq \exp(-Q/T_F) \times e^{-t_{\rm BBN}/\tau_n} \quad (7)$$

where the latter factor just comes from neutron decay.

The goal of this exercise is to estimate the Helium abundance in the SM as well as in a Universe with $N_{\rm eff} = 4$.

(a) Knowing that $H = (1/a)da/dt$ calculate the age of the Universe as a function of temperature T and number of internal relativistic degrees of freedom contributing to radiation ($g_\star$). Remember that $H \simeq 1.66\sqrt{g_\star}T^2/M_{\rm pl}$ with $M_{\rm Pl} \simeq 1.22 \times 10^{19}$ GeV. Write $g_\star$ as a function of $N_{\rm eff}$.

(b) Knowing that all neutrons present at the time of BBN will form helium-4, calculate the helium mass fraction as a function of the neutron-to-proton ratio (n/p). This is, the fraction of total mass in the form of helium-4 (the rest are just protons).

(c) The proton-to-neutron interaction rate can be roughly be written as $\Gamma_{pn} \simeq 7\, G_F^2 T_\gamma^5$. Calculate the temperature of freeze-out of these interactions ($\Gamma \simeq H$) allowing for $N_{\rm eff}$ to be a free parameter. Namely, take $\rho_\nu = N_{\rm eff} \times 2\,(7/8)(\pi^2/30)T_\nu^4$.

(d) After proton-neutron freeze-out neutrons are simply decaying. Calculate the mass fraction of helium-4 knowing that the neutron lifetime is $\tau_n \simeq 878\,$s. In order to simplify the calculation, you can assume that by $T_\gamma = 0.075\,$MeV only photons and neutrinos are present in the plasma, the latter being parametrized with a contribution to $N_{\rm eff}$ (but with a smaller temperature than photons). Plot the final result as a function of $N_{\rm eff}$ and compare it with $Y_{\rm P} = 0.244(N_{\rm eff}/3)^{0.16}$.

(e) If there is an additional fourth sterile neutrino the Universe will expand faster and more helium will be formed. Estimate how much helium formed if $N_{\rm eff} = 4$. We have now measured the primordial helium abundance with 1% precision. Would $N_{\rm eff} = 4$ be excluded from this exercise?

10. **Thermalization of right handed neutrinos?**
Estimated time: 20 mins, *difficulty*: 4/10, *result*: (a) no, (b) no, (c) Max$\frac{\Gamma}{\rm H}$(T) $\times\, T^3$
Consider the case of Dirac neutrinos in the Standard Model. The coupling is $\mathcal{L} \in y_D \bar{\nu}_R \tilde{H} L$ where ν_R is the right-handed neutrino, $\tilde{H}$ is the Higgs charge conjugated field, and L is the lepton doublet. The coupling can be written as $y_D = \sqrt{2} m_\nu / v_H$, with $v_H = 246\,$GeV and for numerical purposes we can take $m_\nu \simeq 0.1\,$eV.

(a) Calculate the Higgs decay rate into this channel, $H \to \nu_L \bar{\nu}_R$. Is it larger than $H(T)$ at $T = m_H$? Are right handed neutrinos produced in the early Universe?

(b) Consider now the case $T < m_H$ and interactions of the type $\nu_R \bar{\nu}_R \leftrightarrow e_L \bar{e}_R$. How fast are they? Do they thermalize?

(c) The number density of this particles for standard Dirac neutrinos is tiny, but can you estimate their number density? *Tip: it will be proportional to* Γ/H.

Useful formulae

Massless Thermodynamics, $m = 0\,,\ \mu = 0$			
Quantity	Fermi-Dirac	Bose-Einstein	Maxwell-Boltzmann
n	$g\frac{3}{4}\frac{\zeta(3)}{\pi^2}\,T^3$	$g\frac{\zeta(3)}{\pi^2}\,T^3$	$g\frac{1}{\pi^2}\,T^3$
ρ	$g\frac{7}{8}\frac{\pi^2}{30}\,T^4$	$g\frac{\pi^2}{30}\,T^4$	$g\frac{3}{\pi^2}\,T^4$
p	$\rho/3$	$\rho/3$	$\rho/3$

(8)

where $\zeta(3) \simeq 1.20206$.

$$H = 1.66\sqrt{g_\star}T^2/M_{\rm pl} \quad \text{[Hubble parameter in a radiation dominated Universe]} \tag{9}$$
$$M_{\rm pl} = 1.22 \times 10^{19}\,\text{GeV} \quad \text{[Planck Mass]} \tag{10}$$
$$\rho_c = 1.05 \times 10^4 h^2\,\text{eV/cm}^3 \quad \text{[Critical Energy Density today]} \tag{11}$$