The BPHZ theorem: a decisive turn in the history of quantum field theory

Sergey Volkov

ITP KIT, Humboldt fellowship

KSETA, 2023

Quantum Field Theory

Quantum field theory is a theoretical framework that tries to combine quantum mechanics, classical field theory and special relativity.

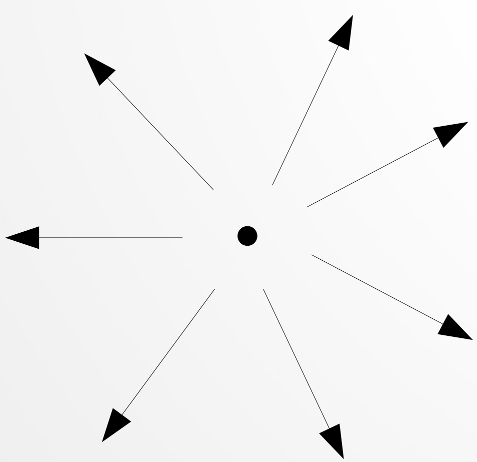
1920s – first developments.

A problem: all observables seemed *infinite*.

~1947-1950 – first solutions of the infinity problem, first meaningful results that agree with experiments, a formulation of quantum electrodynamics.
Electron's g-factor (magnetic moment / predicted by classical physics) ≈ $2+\frac{\alpha}{\pi}$
Hydrogen energy level shift.
The efforts of R. Feynman, J. Schwinger, H. Bethe, S. Tomonaga and others.

These lectures is about a *rigorous* mathematical proof of the ultraviolet divergences cancellation and related questions. **Understanding mathematical arguments is the most important!**

Infinities of ultraviolet type exist in classical electrodynamics:



A point charge and its electric field

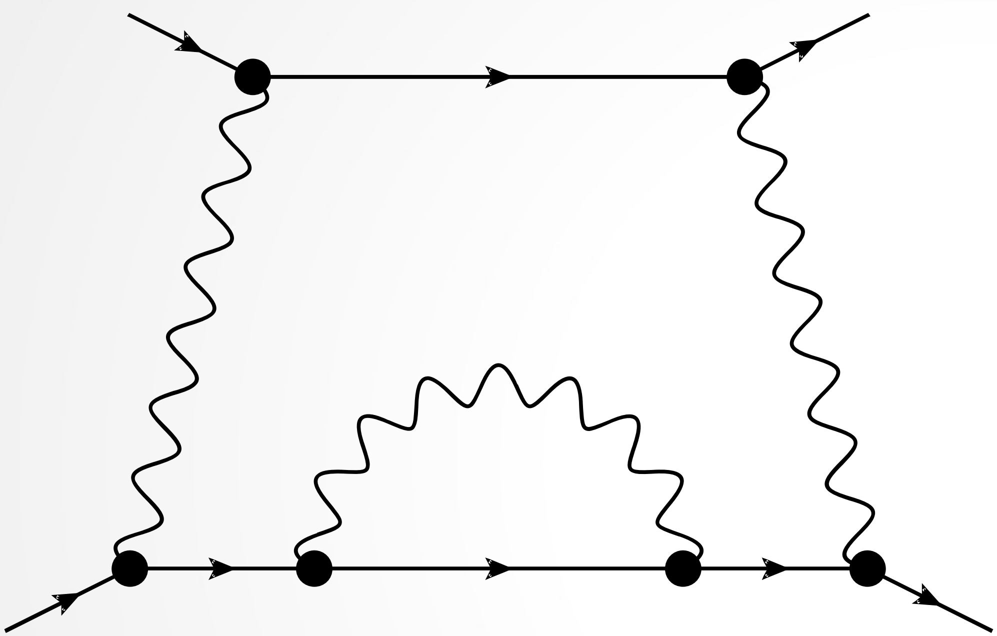
Electric field (Coulomb's law): $|E| = C/r^2$

Energy density: $u = C|E|^2 = C/r^4$

Total electromagnetic energy: $C\int_0^{+\infty} u(r)^2 r^2 dr = C\int_0^{+\infty} \frac{dr}{r^2} = \infty$

Feynman diagrams: an instrument of obtaining Lorentz-covariant results from quantum field theory

Particles are free (do not interact) at infinity ($t \to \pm\infty$)



An example from quantum electrodynamics

Internal and external lines.

Each line has a 4-momentum.

External momenta are on the mass shell:
$p^2=m^2$

Straight lines are electrons/positrons, wiggly lines are photons.

Arrow directions show the charge motion;
$p_0>0$ for electrons, $p_0<0$ for positrons.

4-momentum is conserved at each point.

It gives a contribution to the particle scattering amplitude as an expansion in the coupling constant (*with reservations*).

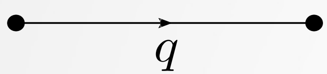
IMPORTANT NOTE!
The derivation of Feynman diagrams from first principles is *not completely rigorous*.
The procedure of obtaining results from Feynman diagrams is also *not fully justified*.
The theory of handling infinities is developed mostly for Feynman diagrams (*not for equations of motion* and so on).

 Sergey Volkov sergey.volkov@partner.kit.edu

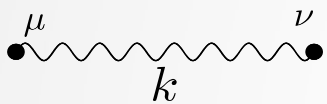
QED Feynman rules

QED Lagrangian density (in Feynman gauge):

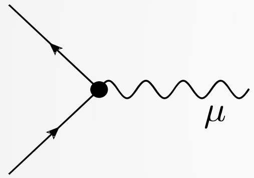
$$L = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 - e\overline{\psi}\gamma^\mu \psi A_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$



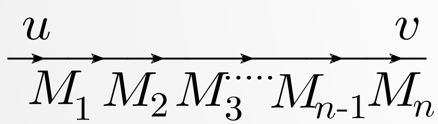
$$: \frac{i(\not{q} + m)}{q^2 - m^2 + i0}$$



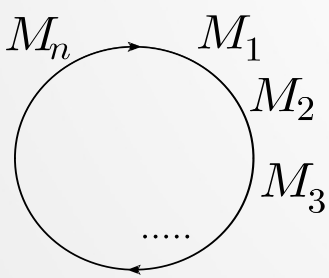
$$: -\frac{ig_{\mu\nu}}{k^2 + i0}$$



$$: -ie\gamma_\mu$$



$$: \overline{v} M_n M_{n-1} \ldots M_2 M_1 u$$



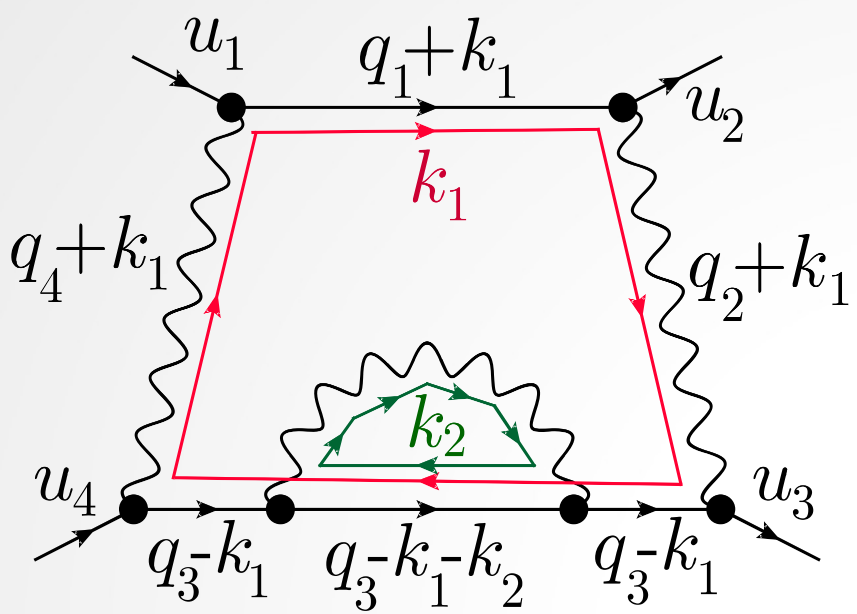
$$: -\mathrm{tr}(M_n M_{n-1} \ldots M_2 M_1)$$

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$$

$$\not{p} = p^\mu \gamma_\mu$$

The metric tensor $g_{\mu\nu}$ corresponds to (+1,-1,-1,-1)

Feynman diagrams: a basis of independent loops and integration



q_1, q_2, q_3, q_4 are linear combinations of external momenta (satisfying the momentum conservation law).

k_1, k_2 are loop momenta.

u_1, u_2, u_3, u_4 are Dirac spinors of external particles.

A basis of independent loops can be chosen differently.

$$\int \overline{u}_2\gamma_\mu \frac{i(\not{q}_1+\not{k}_1+m)}{(q_1+k_1)^2-m^2+i0}\gamma_\nu u_1$$

$$\times\overline{u}_3\gamma^\mu \frac{i(\not{q}_3-\not{k}_1+m)}{(q_3-k_1)^2-m^2+i0}\gamma_\xi \frac{i(\not{q}_3-\not{k}_1-\not{k}_2+m)}{(q_3-k_1-k_2)^2-m^2+i0}\gamma^\xi \frac{i(\not{q}_3-\not{k}_1+m)}{(q_3-k_1)^2-m^2+i0}\gamma^\nu u_4$$

$$\times\frac{-1}{k_2^2+i0}\frac{-1}{(q_2+k_1)^2+i0}\frac{-1}{(q_4+k_1)^2+i0}d^4k_1d^4k_2$$

Issues related to Feynman diagrams

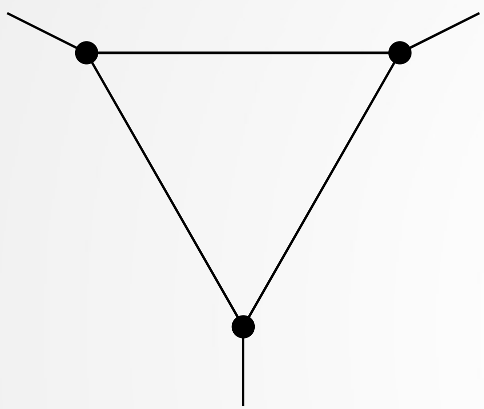
(an incomplete list)

- Integrals with Minkowsky space propagators don't exist.
- $+i0$ in the propagator denominators.
- Ultraviolet divergences.
- Infrared divergences.
- Mixed ultraviolet-infrared divergences.
- Overlapping divergences.
- Since the integrals written directly don't exist, a regularization is required.
- Physical parameters must be correctly defined (≠bare parameters).
- Diagrams must be amputated (to avoid division by zero).
- External line "wave function" renormalization (which is not connected directly with the renormalization of physical parameters).
- An emission of an infinite number of soft photons should be taken into account (otherwise the probability is zero).

Issues related to Feynman diagrams: Minkowsky-space integrals don't exist

The integral $\int f(x)dx$ exists if and only if $\int|f(x)|dx$ is finite.

The Riemann rearrangement theorem: if $f_n \to 0$ as $n \to \infty$ and $\Sigma|f_n|=\infty$ then ***any*** sum can be obtained by permutation of f_n.



A Feynman diagram of a scalar theory

The integral for zero external momenta $\int \frac{1}{(k^2+i\varepsilon)^3} d^4k$

does not exist, because in the area $|k_0 - |\mathbf{k}|| < \frac{\varepsilon}{|\mathbf{k}|+1}$

the function absolute value > $1/(3\varepsilon)^3$, but the area measure is of order $\int r dr = \infty$.

These integrals don't exist for any external momenta and masses.

The corresponding Euclidean integral

$$\int \frac{1}{(|k|^2_{\mathrm{eucl}} + i\varepsilon)^3} d^4k = C \int_0^{+\infty} \frac{r^3}{(r^2+i\varepsilon)^3} dr$$

is absolutely convergent.

 Sergey Volkov sergey.volkov@partner.kit.edu

Issues related to Feynman diagrams: a regularization of Minkowsky space integrals

Ways to regularize Minkowsky space integrals:

- Additional regularization parameter:

$$\frac{1}{k^2 - m^2 + i0} \longrightarrow \frac{1}{k^2 - m^2 + i(\varepsilon_{\mathrm{IR}} + \varepsilon_{\mathrm{Mink}} |k|^2_{\mathrm{Eucl}})}$$

  or simultaneously (W. Zimmermann's approach):

$$\frac{1}{k^2 - m^2 + i0} \longrightarrow \frac{1}{k^2 - m^2 + i\varepsilon(\mathbf{k}^2 + m^2)}$$

- Integrate first over k_0 and only then over **k**.
- Wick rotation: rotate the contour of the integration over k_0, put it to the imaginary axis (by Cauchy's theorem). All integrals becomes Euclidean. In some cases it makes the denominators separated from 0.
- Schwinger and Feynman parameters.

Issues related to Feynman diagrams: +*i0* in the propagator denominators

- "Causal prescription".
- We should replace $+i0$ with $+i\varepsilon$ or a more complicated expression, $\varepsilon \to +0$.
- We have some freedom in ordinary cases. For example, we can write $+i\varepsilon$ in one propagator and $+2i\varepsilon$ in another one, without affecting the final result (provided that all other regularization issues are resolved).
- A deformation of the (multidimensional) contour of the loop momenta $k_1,\ldots,k_L$ in $4L$-dimensional complex plane helps to avoid ε (if there are no IR divergences).

  Not only the Wick rotation...

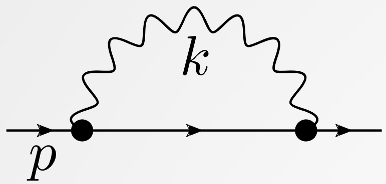
  The Cauchy-Poincaré theorem (a multidimensional generalization of the Cauchy theorem and a complex-variable generalization of the change of variables theorem)

  [B. V. Shabat, Introduction to Complex Analysis, P. II. Functions of several variables, Chapter II.5]
  [D. E. Soper, Phys. Rev. D 62, 014009 (2000), Appendix]
  [S. Borowka, J. Carter, G. Heinrich, Comp. Phys. Comm. 184, Is. 2, 396-408 (2013), Section 2.2]
- However, if we have infrared divergences, the freedom ceases to exist, one needs to be more careful. A non-zero photon mass is usually introduced in QED:

$$-\frac{g_{\mu\nu}}{k^2+i0} \quad \longrightarrow \quad -\frac{g_{\mu\nu}}{k^2-\lambda^2+i\varepsilon}, \quad \lambda>0$$

After the non-zero photon mass is introduced, the freedom with $+i\varepsilon$ appears; thus, the limit $\varepsilon \to +0$ should be taken first, and only after that we can take $\lambda \to +0$.

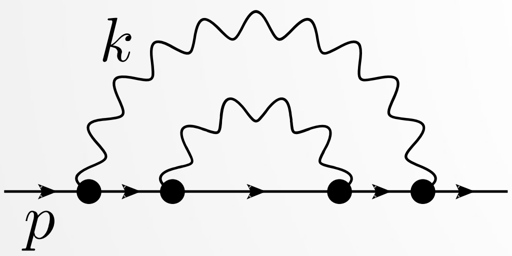
Issues related to Feynman diagrams: ultraviolet divergences



$$\Sigma_1(p) = \int \gamma_\mu \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2 + i0} \gamma^\mu \frac{1}{k^2 + i0} d^4k$$

Approximated UV analysis
(ignoring cancellations in the denominators): $\Sigma_1(p) \asymp \int \frac{d^4k}{k^3} + \not{p} \int \frac{d^4k}{k^4}$

Really, the situation is not so bad (if we regularize it properly), because *k* and -*k* cancel each other, but nevertheless **both terms are UV divergent**!



$$\Sigma_2(p) = \int \gamma_\mu \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2 + i0} \Sigma_1(p+k) \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2 + i0} \gamma^\mu \frac{1}{k^2 + i0} d^4k$$

Analogously, $\Sigma_2(p) \asymp \int \frac{\Sigma_1(p+k)}{k^4} d^4k$

The UV divergence of Σ_1 is *multiplied* by a new UV divergence. **UV divergences can be nested!**
Moreover, the proportional to $p+k$ UV-divergent term of Σ_1 increases the UV degree!

Issues related to Feynman diagrams: regularization of ultraviolet divergences

The reason of UV divergences: a point-like field interaction.
Really, *we don't know* how the interaction works at very small distances.

Regularization: a parameter $\varepsilon \to +0$ that summarizes our unknowledge of the interaction structure at small distances and high energies. As $\varepsilon \to +0$, observable physical parameters like particle masses tend to ∞. A renormalization is required (changing the bare parameters to make the physical parameters constant). If this growth is not too fast (like $\log \varepsilon$), we can suppose that the "*natural regularization*" leads to only a small shift of the physical parameters (relative to the bare parameters).

Ideal regularization: a finite radius of interaction, smoothness (gaussian-like or something like this).
Unfortunately, it does not work at all (it violates everything and gives catastrophic irremovable infinities).

One can say that different regularizations differ in intermediate values, but give the same final result. However, this statement is meaningless: there is no general definition of a regularization; and of course, there is no proof that all regularizations are equivalent.

Issues related to Feynman diagrams: regularization of UV divergences (examples)

- **Cut-off like regularizations.** Loop momenta space in integrals is restricted in some way.
  Good: the nearest to the ideal regularization.
  Bad: it violates the symmetries like the Lorentz covariance and gauge invariance (it depends on the realization).

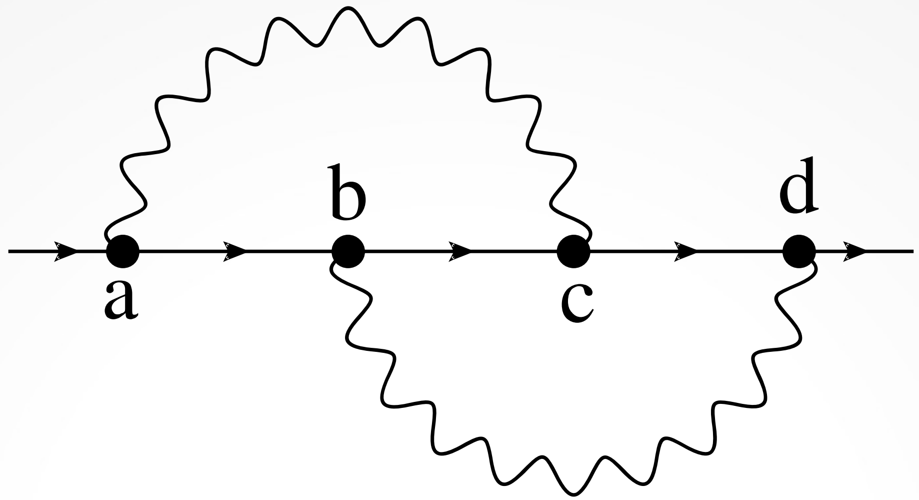
- **The Pauli-Villars regularization.**
  The propagator are replaced with linear combinations with different masses:
  $$\frac{1}{k^2-m^2+i0} \longrightarrow \frac{1}{k^2-m^2+i0}+\frac{C_1}{k^2-M_1^2+i0}+\ldots+\frac{C_n}{k^2-M_n^2+i0}$$
  $M_j\to\infty$. The coefficients and masses can be adjusted to remove all UV divergences.
  Good: it preserves the Lorentz covariance.
  Good: it is not too far from the ideal regularization.
  Bad: it violates the gauge invariance.

- **A modified Pauli-Villars regularization**, in which each term has the same mass M_j in all propagators of a fermion loop (in QED).
  Good: it preserves the gauge invariance (in QED, in the form of Ward-Takahashi identities) together with the Lorentz covariance.
  Bad: subdiagrams behave differently relative to the same graphs as whole diagrams (thus, its physical interpretation is very doubtful as well as the possibility of usage for proving properties).
  [N. N. Bogolyubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields, Chapter IV (Nauka, Moscow, 1984; Wiley, New York, 1980)]

- **Dimensional regularization** (UV divergences are regularized together with the IR ones).
  $D=4+\varepsilon$ is the space-time dimensionality.
  Good: it preserves the gauge invariance, Lorentz covariance.
  Good: In many cases it is possible to manipulate with intermediate infinite values as with finite values.
  Good: It is very useful for calculations.
  Bad: in most of cases, there is no D for which the corresponding integral exists. A secondary regularization, analytical continuation or other constructions are required to define it.
  Bad: in most of cases, IR and UV divergences are inseparable and look similarly.
  Bad: it has nothing in common with the ideal regularization. An analytical continuation in complex plane is inevitable. Its physical interpretation is very doubtful. A very serious grounds and a justification is required.

- **Dimensional regularization combined with a non-zero photon mass.**
  Good: all good properties of dimensional regularization.
  Good: UV and IR divergences can be studied separately.
  Bad: all bad properties of dimensional regularization (except the unexistence of the integral for all D).
  Bad: it is difficult to extend it beyond QED.

Issues related to Feynman diagrams: overlapping ultraviolet divergences



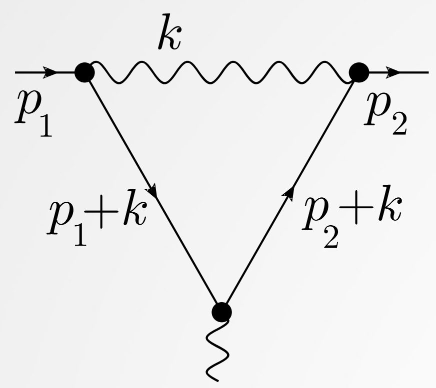
Direct UV power counting shows that the whole diagram **abcd** has an UV divergence, but also the subdiagrams **abc** and **bcd** have it.

For *nested* divergences one can imagine that the Feynman amplitudes can be correctly defined (without UV divergences) order by order; the lower-order correctly defined Feynman amplitudes are substituted to the higher-order integrals.

However, there is no hope that this is possible, if we have *overlapping* UV divergences.

Even if a divergence elimination procedure is developed, the question arises about the physical interpretation of this procedure. **However, this is solvable in a natural way!**

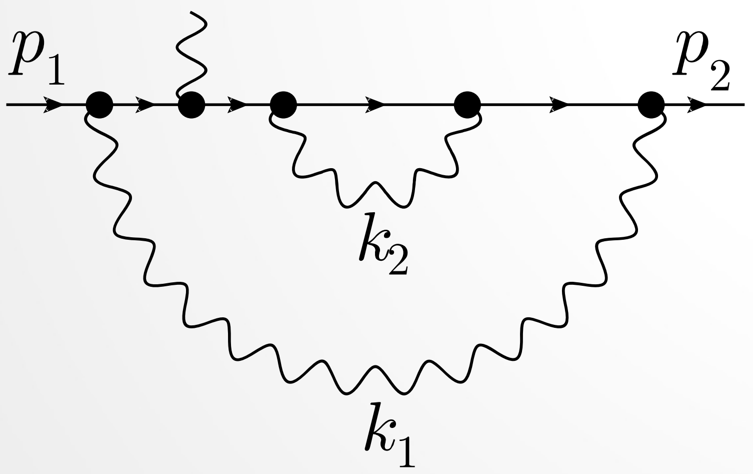
Issues related to Feynman diagrams: infrared divergences



$$\Gamma_\mu(p_1,p_2) = \int \gamma_\nu \frac{\not{p}_2 + \not{k} + m}{(p_2+k)^2 - m^2 + i0} \gamma_\mu \frac{\not{p}_1 + \not{k} + m}{(p_1+k)^2 - m^2 + i0} \gamma^\nu \frac{1}{k^2 + i0} d^4 k$$

A physical situation: $(p_1)^2=(p_2)^2=m^2$.
All propagator denominators tends to 0 as $k \to 0$.
An *accurate* power counting shows that we have a divergence when $k \to 0$.
A non-zero photon mass helps: k^2+i0 ---> $k^2-\lambda^2+i0$

A widespread misconception: all physical IR divergences are of logarithmic type



$(p_1)^2=(p_2)^2=m^2$
$k_2 \neq 0$ is fixed
$k_1 \to 0$
In this case, we have a power-type IR divergence!
To make it logarithmic, *a correct mass renormalization is required*!

Issues related to Feynman diagrams: handling infrared divergences

The probability that only a finite number of photons is emitted always equals 0. This leads to infinities in the perturbation series terms, like

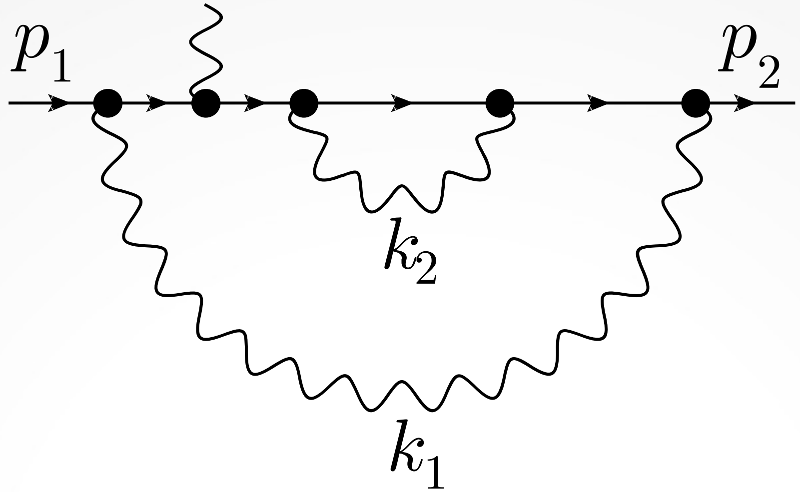
$$0 = e^{-\infty} = \sum_{n=0}^{+\infty} \frac{(-\infty)^n}{n!}$$

A *finite sensitivity* of the photon detector must be taken into account.

Two regularization parameters: λ is the photon mass, Λ is the photon detector sensitivity.

The renormalization also plays a role in the cancellation of IR divergences. For example, there are physical observables that don't depend on the photon detector sensitivity (like the magnetic moment); in these cases, the elimination of IR divergences is directly connected to the renormalization, as well as it is for the UV divergences.

Issues related to Feynman diagrams: mixed infrared-ultraviolet divergences



$$\Gamma_\mu(p_1,p_2) = \int \gamma_\nu \frac{\not{p}_2 + \not{k}_1 + m}{(p_2+k_1)^2 - m^2 + i0} \Sigma(p_2+k_1) \frac{\not{p}_2 + \not{k}_1 + m}{(p_2+k_1)^2 - m^2 + i0} \gamma_\mu \frac{\not{p}_1 + \not{k}_1 + m}{(p_1+k_1)^2 - m^2 + i0} \gamma^\nu \frac{1}{(k_1)^2 + i0} d^4 k_1$$

$$\Sigma(p) = \int \gamma_\nu \frac{\not{p} + \not{k}_2 + m}{(p+k_2)^2 - m^2 + i0} \gamma^\nu \frac{1}{(k_2)^2 + i0} d^4 k_2$$

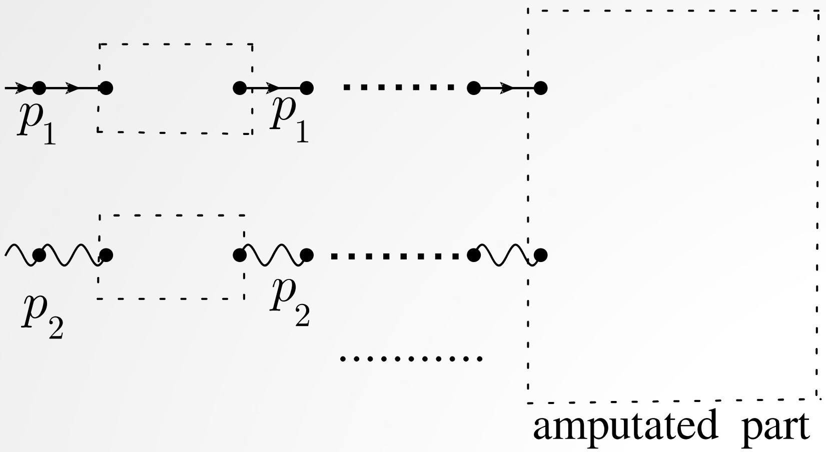
Σ(p) is UV-divergent.

If $(p_1)^2=(p_2)^2=m^2$, all denominators in the formula for $\Gamma_\mu(p_1,p_2)$ tends to *0*; an IR divergence *enhances* the UV divergence of *Σ(p)*.

IR and UV divergences can't be separated!

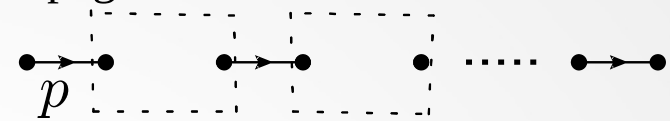
 sergey.volkov@partner.kit.edu

Issues related to Feynman diagrams: amputation and external leg renormalization

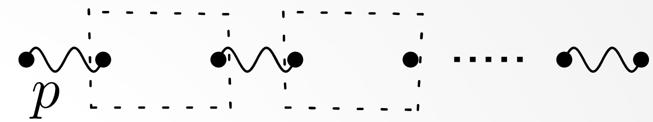


not amputated Feynman diagram

Full electron propagator:

$S(p)=$ 

Full photon propagator:

$D_{\mu\nu}(p)=$ 

Renormalization constants are extracted from full propagators:

$$S(p) \sim Z_2 \frac{i(\not{p}+m)}{p^2-m^2+i0}, \quad D_{\mu\nu}(p) \sim Z_3 \frac{-ig_{\mu\nu}}{p^2+i0} + f(p^2)p_\mu p_\nu$$

If external momenta are on the mass shell ($(p_1)^2=m^2$, $(p_2)^2=0$), a division by zero occurs in not amputated diagrams.

A diagram is amputated if it does not have an internal line that separates one external line from the others.

LSZ prescription: the amputated Feynman amplitude should be multiplied by $(Z_2)^{a/2}(Z_3)^{b/2}$, where a and b are the numbers of external fermion and photon lines (in QED). LSZ is supposed to be explained by *particle dressing*.
Amputated diagrams are useful for calculations.

Not amputated diagrams have sense only if *all external lines are the continuations of internal lines*. In this case, one should consider *slightly off-shell* external momenta, take the corresponding residue and multiply by $(Z_2)^{-a/2}(Z_3)^{-b/2}$.
Not amputated diagrams are useful for proving gauge invariance (especially in non-abelian theories).

All kinds of diagrams require external leg renormalization!
Gauge invariance exists only after the external leg renormalization!
Z_2 and Z_3 are UV-divergent in most cases. Z_2 is also IR-divergent!

In C- or P-violating theories, Z_2 are different for left and right parts. In CP-violating theories, Z_2 can be independent for adjoint fields.

UV divergences: the status

- The cancellation of UV divergences has to do with the renormalization of the Largangian constants.
- Feynman diagrams – is a *hard-won* instrument of obtaining Lorentz-covariant results for particle scattering problems from quantum field theory.
- Dealing with Feynman diagrams has a lot of issues. New definitions are required to make the corresponding integrals existing; these definitions have no physical grounds; they are based on an experience. Moreover, the derivation of Feynman diagrams from the first principles is also not fully justified.
- Another way is to abandon Feynman diagrams. However, in this case, the situation becomes much more complicated. One has to deal with an infinite-dimensional space. Currently all approaches that are not based on Feynman diagrams (like lattice calculations) work only in those cases when they are demonstrably equivalent to Feynman diagrams.
- Feynman diagrams work only for particle scattering problems. Sometimes it can be generalized to some physical observables beyond scattering problems, but only with serious reservations. Currently there is no general theory of renormalization.
- However, an experience shows that Feynman diagrams are suitable for obtaining very precise physical observable values.

Dealing with UV divergences and BPHZ: the history

- A procedure of dealing with UV divergences in Feynman diagrams was in general terms formulated by F. J. Dyson and A. Salam in **1949-1951**.
  [F. J. Dyson, Phys. Rev. 75, 1736 (1949)]
  [A. Salam, Phys. Rev. 84, 426 (1951)]
  A technique of working with overlapping divergences was developed.
  A connection with the physical renormalization was explained.
  However, these results were too far from correct mathematical formulations.

- This procedure was formulated as a mathematical theorem by N. N. Bogoliubov and O. Parasiuk in **1957**. Bogoliubov's ***R***-operation is a generalization of the construction of F. J. Dyson and A. Salam.
  [N. N. Bogoliubov and O. S. Parasiuk, *On the Multiplication of the causal function in the quantum theory of fields*, Acta Math. 97, 227 (1957)]
  The ideas:
  1) to use the Schwinger parameters: $\frac{1}{k^2-m^2+i0}=\frac{1}{i}\int_0^{+\infty} e^{i\alpha(k^2-m^2)}d\alpha$
  2) to subtract all UV divergences directly in Schwinger parametric space;
  3) to prove the integrals finiteness by giving an upper bound on the absolute value of the integrands.
  The proof had a serious mistake.
  Moreover, the title of the paper was misleading…
  However, this consideration raised the hope that quantum field theory has a meaning and can be rigorously examined. Moreover, the way was showed…

- The mistakes were corrected by K. Hepp in **1966**.
  [K. Hepp, Commun. Math. Phys. 2, 301 (1966)]
  The general three ideas are the same.

- W. Zimmermann demonstrated in **1969** that the UV divergences can be subtracted directly in momentum space.
  [W. Zimmermann, Commun. Math. Phys. 15, 208 (1969)]

A misleading title of N. N. Bogoliubov's and O. S. Parasiuk's paper

N. N. Bogoliubov and O. S. Parasiuk, *On the Multiplication of the causal function in the quantum theory of fields*, Acta Math. 97, 227 (1957)

In German: *Über die Multiplikation der Kausalfunktionen in der Quantentheorie der Felder.*

The idea comes from Feynman diagrams in coordinate representation. In this representation, the Feynman amplitudes are obtained as integrals of the form

$$\int D_1(x_{i_1} - x_{j_1}) D_2(x_{i_2} - x_{j_2}) \ldots d^4x_1 d^4x_2 \ldots d^4x_n$$

The points $x_1, x_2, \ldots, x_n$ correspond to the vertexes of the Feynman diagram.
$D_j(x)$ are the propagators in coordinate space.

The idea of N. N. Bogoliubov and O. S. Parasiuk: the propagators $D_j(x)$ are not well defined around $x=0$ as well as their products around the vectors $(x_1, x_2, \ldots, x_n)$ in which some of x_j coincide.

However, it is very important that this "redefinition" works only for scattering matrixes.

For example, if we introduce a smooth function $g(x)$ = *the intensity of switching on the interaction*, the integral turns into

$$\int g(x_1) g(x_2) \ldots g(x_n) D_1(x_{i_1} - x_{j_1}) D_2(x_{i_2} - x_{j_2}) \ldots d^4x_1 d^4x_2 \ldots d^4x_n$$

In this case, the whole theory crashes down! (it works only when $g \to 1$)
However, the propagators and their multiplications remain the same.

The BPHZ theorem: two parts

1. [not fully correct] Reduction of a physical observable (with some reservations) to the sum of integrals like

$$\int_0^{+\infty} F(\alpha_1, \alpha_2, \ldots, \alpha_n) d\alpha_1 d\alpha_2 \ldots d\alpha_n$$

   Each step of the reduction has a logic, but it is not fully correct (and *can't be correct*).
   This includes the proof that this reduction is equivalent to a renormalization of the Lagrangian constants.

2. [rigorous] The proof that each of these integrals is finite.
   (rigorous theorems is a great rarity in quantum field theory)

Quantum field theory works very good!

Despite all its flaws in logic, quantum field theory works with a very high precision!

Electron's g-factor:

2.00231930436321(46)

It includes:

QED corrections up to the 10-th order

Electroweak corrections

Hadronic corrections

Motivation: understanding the mathematical reasoning is very important!

These theorems themselves are incomplete (from the physical point of view) and useless in physics. However, understanding the underlying reasoning is very useful. It gives a strategic advantage!

- [FOUNDATIONS] The foundations of quantum field theory have serious problems with logic. A situation is possible that it should be completely remade basing on entirely different principles. The mechanism of how the cancellation of divergences works in current theories can serve as a good hint!
- [CALCULATIONS] Understanding the structure of divergences and their cancellation gives a freedom in the development of calculation procedures. Higher precisions, more complicated processes are needed… Existing methods often fail on current computers due to different reasons. A scientist not clamped in *dimensional regularization* has a great advantage!

The most important discovery in physics in the beginning of the 20th century

The most important discovery in physics in the beginning of the 20th century

Relativity?

The most important discovery in physics in the beginning of the 20th century

Relativity?
Wave-particle duality?

 Sergey Volkov sergey.volkov@partner.kit.edu

The most important discovery in physics in the beginning of the 20th century

Relativity?

Wave-particle duality?

Quantization?

The most important discovery in physics in the beginning of the 20th century

Relativity?

Wave-particle duality?

Quantization?

Quantum indeterminism?

The most important discovery in physics in the beginning of the 20th century

Relativity?
Wave-particle duality?
Quantization?
Quantum indeterminism?
The Heisenberg uncertainty principle?

The most important discovery in physics in the beginning of the 20th century

Relativity?

Wave-particle duality?

Quantization?

Quantum indeterminism?

The Heisenberg uncertainty principle?

The quantum observer effect?

The most important discovery in physics in the beginning of the 20th century

Relativity?
Wave-particle duality?
Quantization?
Quantum indeterminism?
The Heisenberg uncertainty principle?
The quantum observer effect?
An absence of trajectories of microobjects?

The most important discovery in physics in the beginning of the 20^{th} century

Relativity?
Wave-particle duality?
Quantization?
Quantum indeterminism?
The Heisenberg uncertainty principle?
The quantum observer effect?
An absence of trajectories of microobjects?
Quantum entanglement?

The most important discovery in physics in the beginning of the 20th century

Relativity?
Wave-particle duality?
Quantization?
Quantum indeterminism?
The Heisenberg uncertainty principle?
The quantum observer effect?
An absence of trajectories of microobjects?
Quantum entanglement?
Einstein-Podolsky-Rosen paradoxes?

The most important discovery in physics in the beginning of the 20th century

Relativity?
Wave-particle duality?
Quantization?
Quantum indeterminism?
The Heisenberg uncertainty principle?
The quantum observer effect?
An absence of trajectories of microobjects?
Quantum entanglement?
Einstein-Podolsky-Rosen paradoxes?
Space-time curvature?

The most important discovery in physics in the beginning of the 20^{th} century

Relativity?
Wave-particle duality?
Quantization?
Quantum indeterminism?
The Heisenberg uncertainty principle?
The quantum observer effect?
An absence of trajectories of microobjects?
Quantum entanglement?
Einstein-Podolsky-Rosen paradoxes?
Space-time curvature?
Topological space-time paradoxes?

The most important discovery in physics in the beginning of the 20th century

Relativity?
Wave-particle duality?
Quantization?
Quantum indeterminism?
The Heisenberg uncertainty principle?
The quantum observer effect?
An absence of trajectories of microobjects?
Quantum entanglement?
Einstein-Podolsky-Rosen paradoxes?
Space-time curvature?
Topological space-time paradoxes?

NO.

The most important discovery in physics in the beginning of the 20th century

Relativity?
Wave-particle duality?
Quantization?
Quantum indeterminism?
The Heisenberg uncertainty principle?
The quantum observer effect?
An absence of trajectories of microobjects?
Quantum entanglement?
Einstein-Podolsky-Rosen paradoxes?
Space-time curvature?
Topological space-time paradoxes?

NO.

The most important is: **COMPUTATIONAL COMPLEXITY**.

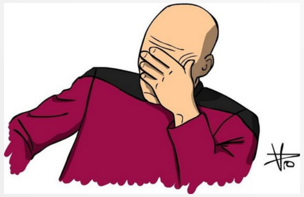
Motivation: the computation complexity of real world processes at the foundamental level is huge

- Full analysis of theories and equations is impossible.
- Scientists try to invent more simple theories with a “*partial logic*” and to compare them with experiments.
- Some of these “partially logical” theories become successful due to different reasons: there is a natural selection of scientific ideas, theories and scientists.
- Since the beginning of the 20th century, theoretical physics evolves through a *direct “Darwinian” natural selection*; it gives a lot of byproducts...

Motivation and jokes: the byproducts of the "Darwinian" natural selection in theoretical physics

- Ignoring mathematical consistency saves time and therefore leads to a win! Theories with only an imitation of logic survive!
- On the other hand, any attempts to put things in order with mathematical consistency require a colossal amount of time and resources and therefore lead to catastrophic loss.
- After a few generations the situation with logical consistency becomes catastrophic and absurd.
- A situation is possible that the theory will be completely revised basing on entirely different principles. In this case, all the *success* of the current theories should be transferred to these principles. A partitipation in this process requires a very deep understanding of the foundations!

Levels of mathematical rigour:

0. Correct.
1. Correct in general, but some accurate proofs are required like the theorems of existence and uniqueness.
   Example: the differential equation $df/dx = 2\sqrt{f(x)},\ \ x \geq 0,\ \ f(0) = 0$
   has a solution $f(x)=0$, but also another solution $f(x)=x^2$.
2. A serious reconstruction of definitions and proofs is required.
   Examples: *1-1+1-1+1-1+…. = 1/2*
   1+2+3+4+… = -1/12
3.  , but works! ← **quantum field theory**
4. Incorrect.

 Sergey Volkov sergey.volkov@partner.kit.edu

Motivation and jokes: tactical and strategic win in theoretical physics

To obtain a tactical gain:

- Ignore all issues related to mathematical consistency.
- Always use ‘’mass destruction’’ techniques like dimensional regularization.
- Always take on the most frontier and complicated problems (no one will notice that you fail).
- If you don’t understand something, say that this is well-understood.
- Be pragmatic: if some question can’t lead to a publication, don’t waste your time on discussions about it.

To obtain a strategic success:

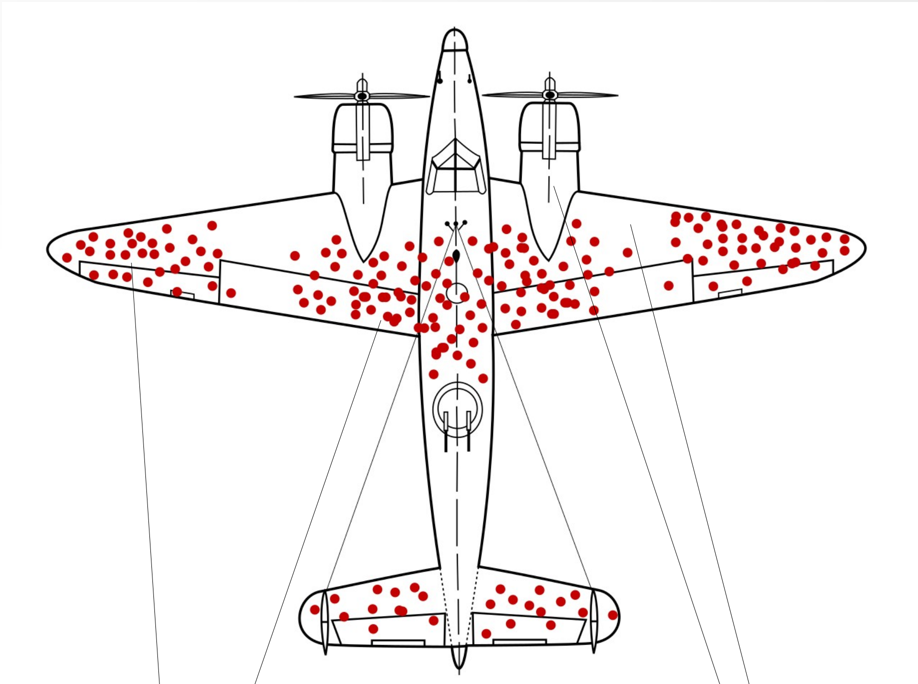
- Understand the foundations, how the infinities work and cancel.

Motivation and jokes: survival bias in theoretical physics

The literature in theoretical physics makes an *illusion* that the problems related to mathematical consistency are not interesting, solved a long time ago, or this apparent inconsistency is a part of nature.

However, all the literature is created by survivors!

One who take on too difficilt tasks and tries to solve them *honestly* does not survive: he/she is unable to make publications, to demonstrate success, to create a scientific school and to have followers.



The problems scientists work on and have difficulties

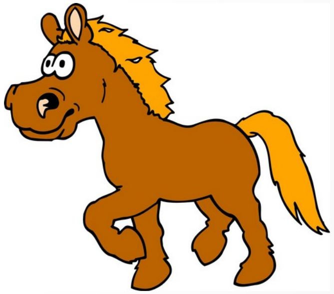
Mathematical consistency, the structure of infinities ("not interesting")

Motivation and jokes: the qualities that a scientist need for building a career in theoretical physics

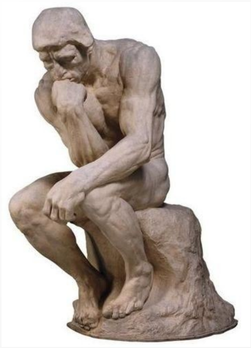


Good:
- an ability not to notice gaps in logic (sincerely)
- arrogance (“it is trivial”, “it is well understood”)
- an ability to stick to difficult “frontier” problems
- a slippery character (as a protection against criticism)
- sociability, teamwork
- friendliness
- rapidity
- stamina, performance
- shallowness of thinking
- nonverbal communication expertise
- cunningness, deceitfulness
- akting skills
- an ability to manipulate people and weave intrigues



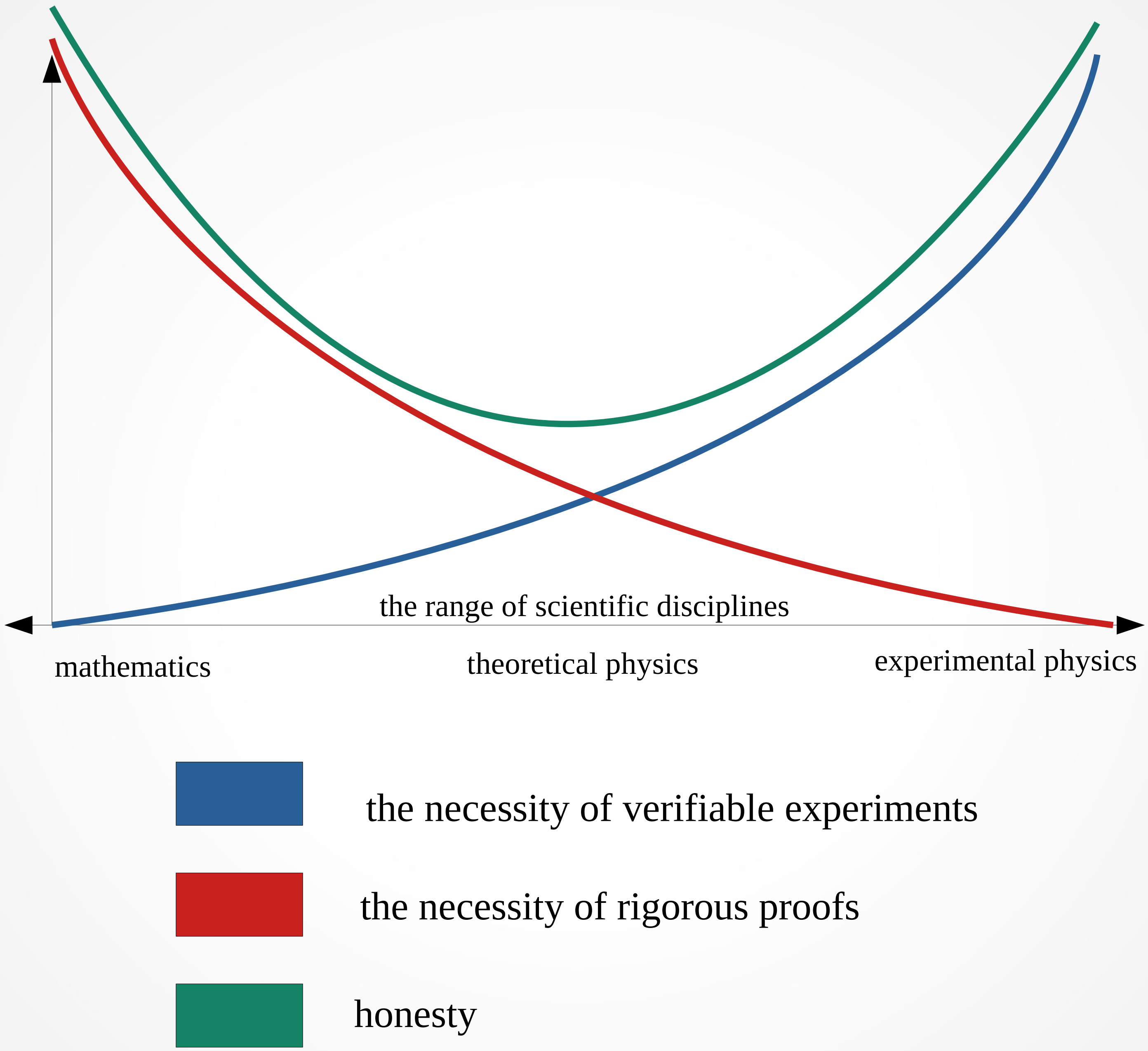




Bad:
- deep thinking
- honesty
- an ability to distinguish correct and incorrect reasoning
- determination to bring things to an end



These are exactly the qualities for the strategic win!

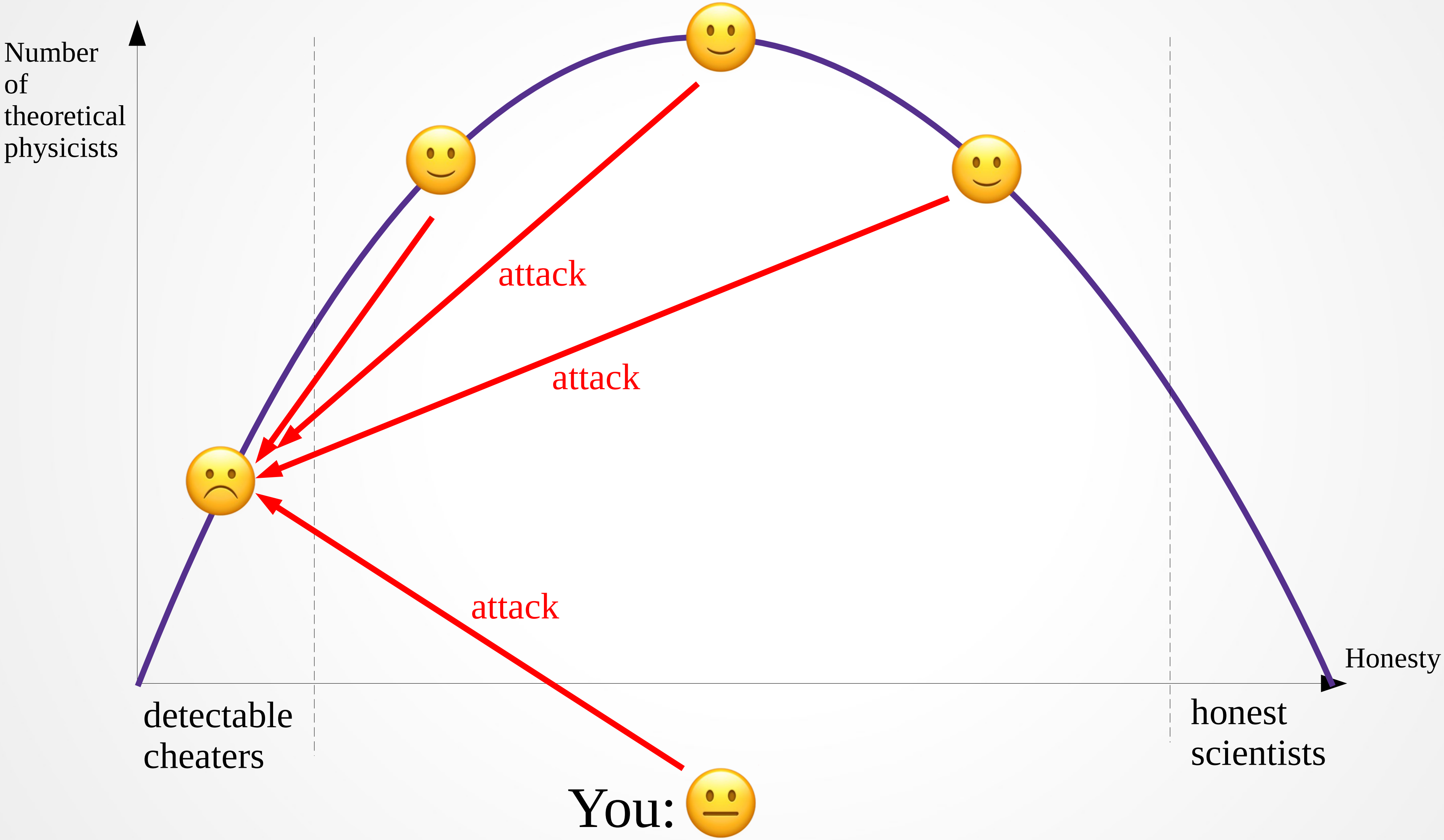
Motivation and jokes:
mathematics, experimental and theoretical physics



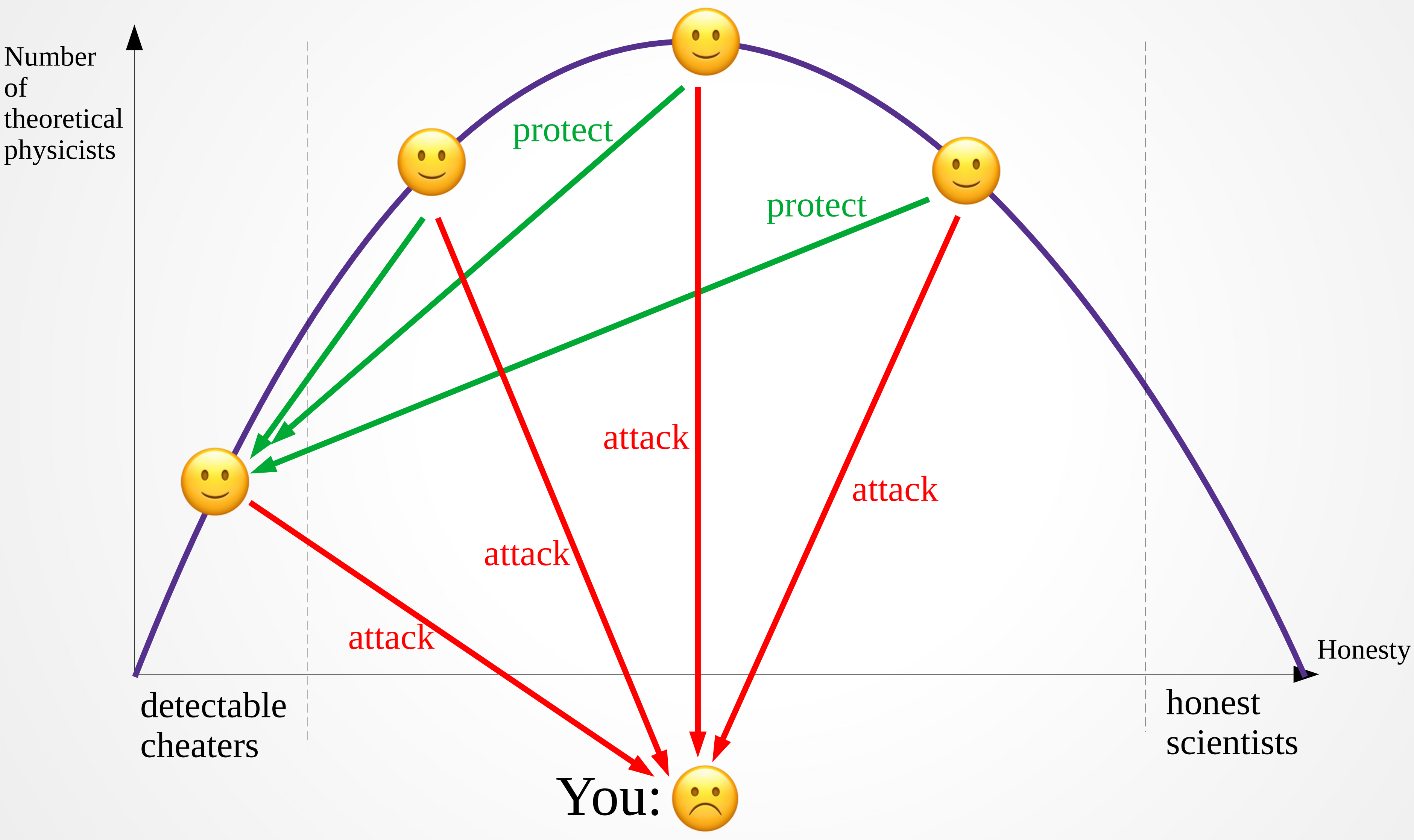
Motivation and jokes: the distribution of the honesty of theoretical physicists

Motivation and jokes:
the attempts to make theoretical physics clear

Motivation and jokes: the attempts to make theoretical physics clear (situation #1)



Motivation and jokes: the attempts to make theoretical physics clear (situation #2)



Motivation and jokes: a better way

Outline

- Introduction
- General ideas of handling UV divergences
- Application to the renormalization of quantum electrodynamics
- Formulations in terms of finite integrals
- The proof of the BPHZ theorem
- Conclusions

General ideas of handling UV divergences: the outline

- Introduction
- **General ideas of handling UV divergences**

  - recognition of divergences
  - subtraction of divergences
  - the relationship between divergence elimination and renormalization
- Application to the renormalization of quantum electrodynamics
- Formulations in terms of finite integrals
- The proof of the BPHZ theorem
- Conclusions

General ideas of handling UV divergences: UV degree of divergence

Γ is a Feynman diagram.
Vertex(Γ) = the set of all vertices in Γ.
Int(Γ) = the set of all internal lines of Γ.
Ext(Γ) = the set of all external lines of Γ.
Loop(Γ) = |Int(Γ)|-|Vertex(Γ)|+1 = the number of independent loops in Γ.
Each vertex v has its polynomial P_v.
Each line l (internal or external) also has its polynomial P_l (for convenience we assume that the external line polynomial is the same as the polynomials of the internal line of the same type, although it is never used in Feynman amplitudes).
Lines(v) = the set of all lines (internal and external *incident* to the vertex v).

The UV degree of divergence:

$$\omega(\Gamma)=4\mathrm{Loop}(\Gamma)-2|\mathrm{Int}(\Gamma)|+\sum_{v\in\mathrm{Vertex}(\Gamma)}\deg(P_v)+\sum_{l\in\mathrm{Int}(\Gamma)}\deg(P_l)$$
$$=4+2|\mathrm{Int}(\Gamma)|-4|\mathrm{Vertex}(\Gamma)|+\sum_{v\in\mathrm{Vertex}(\Gamma)}\deg(P_v)+\sum_{l\in\mathrm{Int}(\Gamma)}\deg(P_l)$$

The euclidean Feynman integral written directly behaves at ∞ as $\int r^{\omega(\Gamma)-1}dr$. It diverges if $\omega(\Gamma)\geq 0$.

For each vertex we define:

$$\omega_v=|\mathrm{Lines}(v)|-4+\deg(P_v)+\frac{1}{2}\sum_{l\in\mathrm{Lines}(v)}\deg(P_l)$$

The UV of divergence can be expressed through ω_v and the properties of the external lines:

$$\sum_{v\in\mathrm{Vertex}(\Gamma)}\omega_v=\omega(\Gamma)-4+|\mathrm{Ext}(\Gamma)|+\frac{1}{2}\sum_{l\in\mathrm{Ext}(\Gamma)}\deg(P_l)$$

General ideas of handling UV divergences: renormalizable and not renormalizable theories

$$\omega_v = |\mathrm{Lines}(v)| - 4 + \deg(P_v) + \frac{1}{2} \sum_{l\in \mathrm{Lines}(v)} \deg(P_l)$$

$$\omega(\Gamma) = \sum_{v\in \mathrm{Vertex}(\Gamma)} \omega_v + 4 - |\mathrm{Ext}(\Gamma)| - \frac{1}{2} \sum_{l\in \mathrm{Ext}(\Gamma)} \deg(P_l)$$

The diagram Γ is divergent if the UV degree of divergence $\omega(\Gamma)\geq 0$.

A theory is called *renormalizable*, if $\omega_v \leq 0$ for all possible vertexes (in this case, only a finite number of the external line configurations may lead to a divergence). A formal renormalizability does not mean that the theory is renormalizable from the physical point of view (the counterterms can violate the needed symmetries, for example).

In quantum electrodynamics, chromodynamics: $\omega_v=0$ (if we take a good gauge).
In Standard Model: $\omega_v\leq 0$ (but not always = 0).

Divergent diagrams in QED (N_f and N_γ are the numbers of external fermions and photons):

- $N_f=0$, $N_\gamma=1$ (does not exist due to the Furry theorem: we can just ignore these diagrams)
- $N_f=0$, $N_\gamma=2$ (exists, requires a renormalization) [$\omega=2$]
- $N_f=0$, $N_\gamma=3$ (does not exist due to the Furry theorem: we can just ignore these diagrams)
- $N_f=0$, $N_\gamma=4$ (exists in diagrams, but is cancelled in the final result) [$\omega=0$]
- $N_f=2$, $N_\gamma=0$ (exists, requires a renormalization) [$\omega=1$]
- $N_f=2$, $N_\gamma=1$ (exists, requires a renormalization) [$\omega=0$]

General ideas of handling UV divergences: the divergence subtraction

An example: a hypothetical scalar 1-loop self-energy Feynman integral in Euclidean space:

$$\Sigma(p) = \int \frac{1}{(p+k)^2+m^2} \frac{1}{k^2+m^2} d^4k$$

The UV degree of divergence $\omega=0$. We have a *logarithmic divergence*.

An observation: p does not play a role when k is large. Thus, $\Sigma(0)$ contains all the information about the UV behavior of $\Sigma(p)$.

This can be demonstated by the *direct subtraction in momentum space*:

$$\Sigma(p) - \Sigma(0) = \int \frac{k^2-(p+k)^2}{((p+k)^2+m^2)(k^2+m^2)^2} d^4k.$$

Since $k^2-(p+k)^2=-p^2-2kp$, the integral is finite.

The putting of the subtraction under the integral sign is incorrect from the mathematical point of view. It should be considered as a definition!

The replacement of $\Sigma(p)$ with $\Sigma(p)-\Sigma(0)$ for each subdiagram of this type everywhere in Feynman diagrams is equivalent to the situation when we allow to use a special vertex instead of subdiagrams of this type ==> a counterterm in the Lagrangian.

General ideas of handling UV divergences: the subtraction of stronger and nested divergences

If the UV degree of divergence $\omega>0$, the *Taylor expansion* around *0* should be subtracted:

$$\Sigma(p) \quad\longrightarrow\quad \Sigma(p) - \sum_{j=0}^{\omega} \frac{1}{j!} \left. \frac{\partial^j \Sigma(p)}{\partial p_{\mu_1} \ldots \partial p_{\mu_j}} \right|_{p=0} p_{\mu_1} \ldots p_{\mu_j}$$

It introduces also counterterm vertices with momenta polynomials into Feynman diagrams.

Note. If $\omega=1$, the UV divergence remains logarithmic (because *k* and -*k* cancel each other in the most divergent term). However, the linear expansion around *0* also needs to be subtracted.

Nested divergences can also be subtracted in the same way: the replacement should be applied sequentially from smaller to larger subdiagrams.

General ideas of handling UV divergences: a note about the momentum conservation law

If we have a Feynman amplitude

$$\Gamma(p_1,\ldots,p_n)$$

and want to take the Taylor expansion of it, we have to take into account that p_1, …,p_n satisfy the energy conservation law and therefore choose a basis of *n-1* elements.

An example of an expansion up to the 1-th order:

$$\Gamma(p_1,p_2,p_3) \quad\longrightarrow\quad p_1=p-\tfrac{q}{2},\ p_2=p+\tfrac{q}{2},\ p_3=q \quad\longrightarrow\quad \Gamma(p,q)$$

$$\longrightarrow\quad \Gamma(p,q)-\Gamma(0,0)-\left.\frac{\partial\Gamma}{\partial p}\right|_{p,q=0}p-\left.\frac{\partial\Gamma}{\partial q}\right|_{p,q=0}q$$

For obtaining a counterterm vertex, one should express it through p_1, p_2, p_3 . It is ambiguous. In this case, it would be better to express the Feynman rules for counterterm vertices through symmetric multilinear forms on $p_1+\ldots+p_n=0$. It should be taken into account that different Lagrangian densities can be equivalent at the level of Lagrangians. It is better to choose first a basis at the level of Lagrangians.

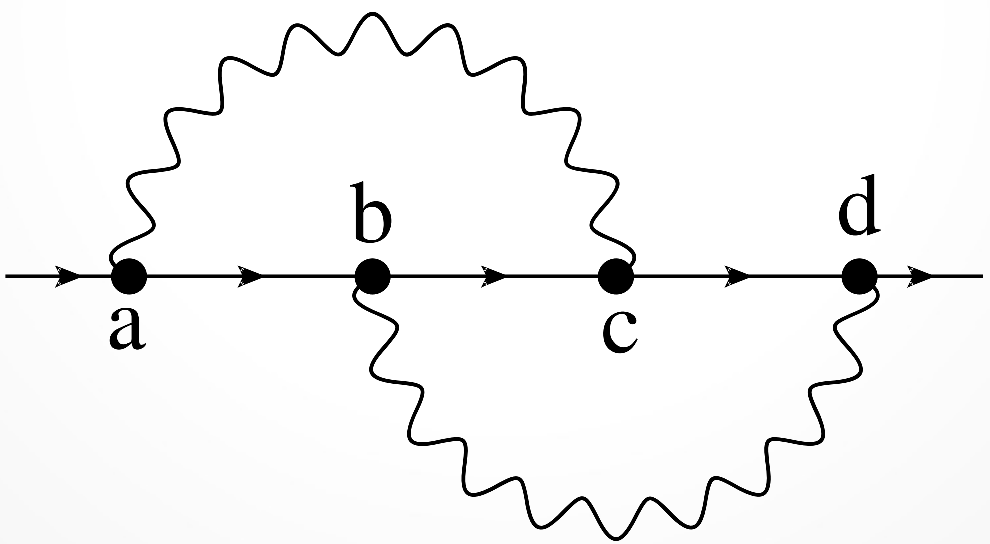
General ideas of handling UV divergences: overlapping divergences

The replacements like

$$\Sigma(p) \quad \longrightarrow \quad \Sigma(p) - \sum_{j=0}^{\omega} \frac{1}{j!} \left. \frac{\partial^j \Sigma(p)}{\partial p_{\mu_1} \ldots \partial p_{\mu_j}} \right|_{p=0} p_{\mu_1} \ldots p_{\mu_j}$$

in each UV-divergent subdiagram work if we have non-intersecting or nested UV-divergent subgraphs.

But what if the subdiagrams are overlapping?



Zimmermann's forest formula helps!

A ***forest*** is a set of subdiagrams of a diagram, each of them are non-intersecting (as sets of vertices) or nested.

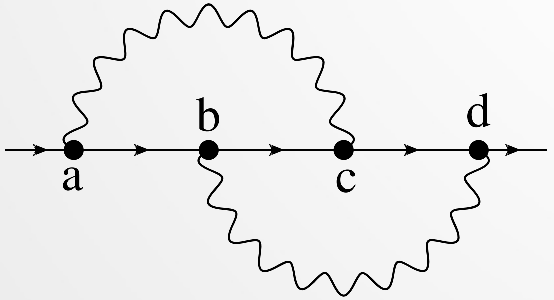
A subdiagram includes all lines connecting its vertexes (having both ends on its vertexes).

We always consider sets containing at least one line.

The operation that removes all UV divergences: $$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all forests of UV-divergent *1-particle irreducible* subdiagrams of the diagram.
1-particle irreducible = removing each line does not break the connectivity.

M_G replaces the Feynman amplitude of *G* with its Taylor expansion around *0* up to the degree *ω(G)*, where *ω(G)* is the UV degree of divergence of *G* (only extracts, not subtracts).
The replacements are performed *from smaller to larger subgraphs*.



$$1 - M_{abc} - M_{bcd} - M_{abcd} + M_{abcd}M_{abc} + M_{abcd}M_{bcd}$$

Similar to the inclusion-exclusion principle...

General ideas of handling UV divergences: overlapping divergences and Zimmermann's forest formula

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all forests of UV-divergent *1-particle irreducible* subdiagrams of the diagram.

M_G replaces the Feynman amplitude of *G* with its Taylor expansion around *0* up to the degree *ω(G)*, where *ω(G)* is the UV degree of divergence of *G* (only extracts, not subtracts).
The replacements are performed *from smaller to larger subgraphs*.

If there are no overlapping subgraphs, it is equivalent to

$$(1-M_{G_1})(1-M_{G_2})\ldots(1-M_{G_n})$$

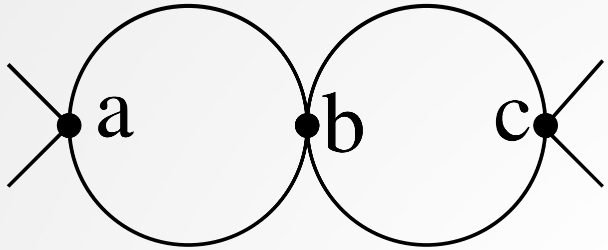
where $G_1, G_2, \ldots, G_n$ are *all* 1-particle irreducible UV-divergent subdiagrams of the diagram (it works as the subtraction of the Taylor expansion for each of these subdiagrams).

In the general case, the terms with overlapping subgraphs should be excluded.

 sergey.volkov@partner.kit.edu

General ideas of handling UV divergences: a misunderstanding about overlapping divergences

An example from φ^4 theory ($L = (1/2)[\partial^\mu\varphi\partial_\mu\varphi - m^2\varphi^2] - (\lambda/24)\varphi^4$):



G_1 and G_2 overlap means that

$$\mathrm{Vertex}(G_1) \cap \mathrm{Vertex}(G_2) \neq \emptyset, \quad \mathrm{Vertex}(G_1) \subsetneq \mathrm{Vertex}(G_2), \quad \mathrm{Vertex}(G_2) \subsetneq \mathrm{Vertex}(G_1)$$

The sets of vertexes, not lines, are considered.
For example, the sets **ab** and **bc** overlap, but their sets of lines don't intersect.

The forest formula is:

$$1 - M_{ab} - M_{bc} - M_{abc} + M_{abc}M_{ab} + M_{abc}M_{bc}$$

Note. The set **abc** is 1-particle irreducible, although it looks reducible.

General ideas of handling UV divergences: Zimmermann's forest formula and BPHZ

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all forests of UV-divergent *1-particle irreducible* subdiagrams of the diagram.

M_G replaces the Feynman amplitude of G with its Taylor expansion around *0* up to the degree $\omega(G)$, where $\omega(G)$ is the UV degree of divergence of G (only extracts, not subtracts).
The replacements are performed *from smaller to larger subgraphs*.

- Firstly the procedure was formulated as a recurrence relation.
  [A. Salam, Phys. Rev. 84, 426 (1951)]
  R-operation:
  [N. N. Bogoliubov and O. S. Parasiuk, Acta Math. 97, 227 (1957)]
- The forest formula is a solution of the recurrence relations. It is not so difficult, and it was obtained by different authors. *Traditionally*, it is called “Zimmermann’s forest formula”.
  [O. I. Zavialov and B. M. Stepanov, Yad. Fys. 1, 922 (1965), in Russian]
  [W. Zimmermann, Commun. Math. Phys. 15, 208 (1969)]
- The forest formula is more convenient for proving the divergence cancellation. The recurrence relations are more useful for some applications.
- We will use the forest formula as a formulation of the BPHZ procedure.

General ideas of handling UV divergences: questions about Zimmermann's forest formula

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all forests of UV-divergent *1-particle irreducible* subdiagrams of the diagram.

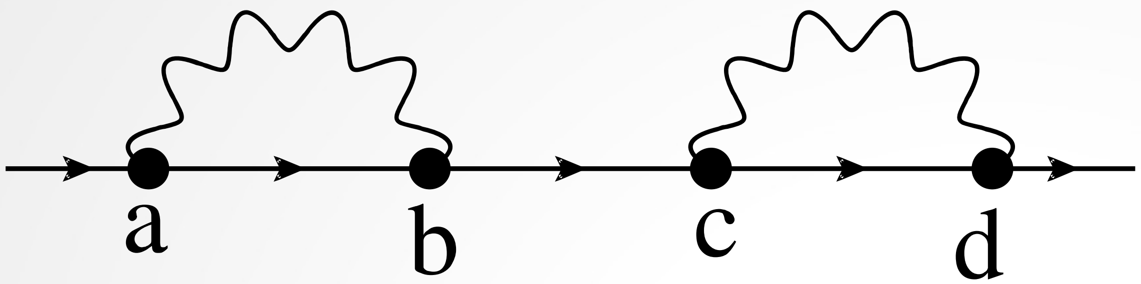
M_G replaces the Feynman amplitude of *G* with its Taylor expansion around *0* up to the degree *ω(G)*, where *ω(G)* is the UV degree of divergence of *G* (only extracts, not subtracts).
The replacements are performed *from smaller to larger subgraphs*.

Questions:

- Why only 1-particle irreducible subdiagrams?
- How to define the subtraction correctly taking into account Minkowsky-space propagators and so on?
- How to prove that it removes all UV divergences?
- What has this to do with **physics**?

General ideas of handling UV divergences: why are only 1-particle irreducible subdiagrams in the forest formula?

1-particle irreducible = removing each line does not break the connectivity

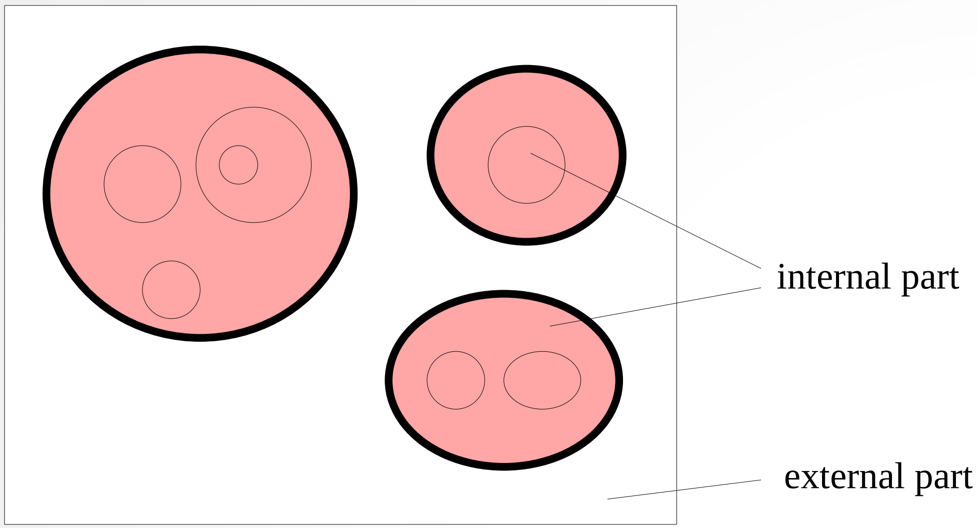


ab and **cd** are UV-divergent and 1-particle irreducible.
abcd, **abc**, **bcd** are UV-divergent, but not 1-particle irreducible (bc is a *bridge*).

- In principle, the inclusion of connected not 1-particle irreducible subdiagrams to the forest formula is correct.
- However, this inclusion is superfluous:
  1) it is not needed for the divergence elimination, because the bridge momenta are *uniquely* determined by the external momenta (no integration);
  2) it *subtracts nothing* due to the same reason (after the 1-particle irreducible components had already been subtracted).
- If the forest formula is modified for the physical renormalization, the division-by-zero problem occurs (as well as in the case of not amputated diagrams with on-shell external momenta).
- Also, a problem can occur with a modified subtraction that the replacement of a subdiagram with the corresponding counterterm vertex changes the amputatedness: the forest formula is not equivalent to the introduction of counterterms for amputated diagrams.

General ideas of handling UV divergences: Zimmermann's forest formula and **physics**

The application of the forest formula to each Feynman diagram is equivalent to the introduction of counterterms into the Lagrangian (but only after summation over all Feynman diagrams).



a hierarchy of subdiagrams in a forest

Let us consider one forest $f=\{G_1,\ldots,G_n\}$ in one diagram.

$G_1,\ldots,G_k$ are *maximal* (with respect to inclusion) elements of f.

External part – the part of the Feynman diagram outside $G_1,\ldots,G_k$ ($G_1,\ldots,G_k$ are replaced with the special vertices).

Internal part – the structure inside G_j, $j=1,\ldots,k$ (the corresponding subdiagram and the inner elements of *f*). Each internal part gives a momentum *polynomial*.

The internal part of *one* G_j (j=1,…,k) forms the contribution to the counterterm. The set of all these contributions (with coefficients) depends only on the subgraph to which it is placed (as well as the possibility to place it there).

If we draw a diagram with counterterm vertexes, each of this vertexes can be expanded to a forest part. After summation over all possible expansions, we obtain the coefficient $C_1C_2\ldots C_k$, where C_j corresponds to the counterterm vertex *j* and is determined by the type of this vertex.

To be more accurate with symmetry coefficients…

ERROR: it works only **without** a requirement that a subdiagram contains all lines with both ends in its set of vertices. These definitions are equivalent in each diagram, see in the BPHZ proof part.

One have to prove that the forest formula is equivalent to the introduction of counterterms taking into account symmetry coefficients.

It is convenient to suppose that all vertexes of the diagram are enumerated: $1,2,\ldots,N$. External lines of the same type are ordered; for each vertex v, the lines of same type incident to v are ordered. The coefficient of the diagram contribution is $(1/N!)$.

If we take one term of the forest formula, the largest (with respect to inclusion) vertex subsets to which operators is applied are $V_1,\ldots,V_k$. To make a correspondence with counterterm diagrams and a diagram with counterterm vertices, we will consider the diagram with subtractions together with vertices $v_1 \in V_1,\ldots,v_k \in V_k$ called the main vertices. After the introduction, each term exists in $|V_1|\times\ldots\times|V_k|$ copies. Thus, the coefficient is $1/(|V_1|\ldots|V_k|N!)$.

The counterterm diagrams are extracted (with all subdiagram subtractions) from V_j keeping the vertex order and line orders corresponding to vertexes. The diagram with counterterm vertices is obtained by removing all vertexes that are in V_j but not equal v_j (keeping the vertex order and by moving all the lines to v_j).

The <u>object obtained from the diagram</u> contains:

- The diagram with counterterm (special) vertices (without subtractions); special vertices don't have orders of lines.
- The counterterm diagrams (with subtractions); the diagrams don't have orders of external lines.
- The correspondence between special vertices and counterterm diagrams.
- The correspondence between special vertex lines and external lines of the counterterm diagrams.

Each object is calculated $N!/[(|V_1|-1)!\ldots(|V_k|-1)!(N-|V_1|-\ldots-|V_k|+k)]$ times. Thus, the coefficient of the object is $1/[|V_1|!\ldots|V_k|!(N-|V_1|-\ldots-|V_k|+k)!]$. It equals the product of the diagram coefficients. Thus, everything works.

Fermion and sign issues should also be resolved.

 Sergey Volkov sergey.volkov@partner.kit.edu

General ideas of handling UV divergences: a recurrence relation for obtaining counterterms

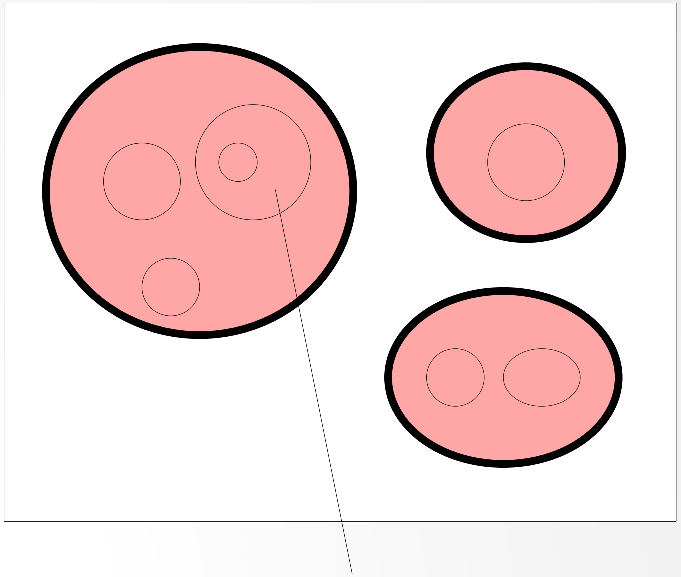
Each counterterm coefficient is treated *perturbatively*:

$C_j = C_{j1}\alpha^1 + C_{j2}\alpha^2 + C_{j3}\alpha^3 + \ldots,$

α is the coupling constant.

The coefficients C_{jk} are obtained *sequentially* (ordered by k):

- Draw all Feynman diagrams of the needed type and degree (in α), with counterterm vertices; the degrees (in α) of the used counterterms *is also calculated*; use only the counterterms of degree <k.
- Perform the summation.
- Apply the operator -*M*.
- Enjoy!



The counterterms for obtaining counterterms are determined with the same procedure (but at earlier steps)

No complicated combinatorial constructions like the forest formula are required!

However, this procedure requires a regularization allowing to manipulate infinite intermediate values.

Adding finite values to the counterterm coefficients is also allowed at each step!

Note. In general, arbitrary adding finite values to the counterterms is not allowed (it can lead to a divergence).
The adding is possible only step-by-step, when the previously modified counterterms are taken into account in the Feynman diagrams.

Outline

- Introduction
- General ideas of handling UV divergences
- **Application to the renormalization of quantum electrodynamics**
  - subtraction and counterterms
  - physical conditions
  - the in-place renormalization
  - the relationship between forests and physics
  - a freedom in the subtraction procedure
- Formulations in terms of finite integrals
- The proof of the BPHZ theorem
- Conclusions

Renormalization of quantum electrodynamics: general ideas of using the forest formula

Lagrangian density with a gauge fixing term:

$$L = L_0 + L_1, \quad L_0 = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2, \quad L_1 = -e\overline{\psi}\gamma^\mu \psi A_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

ξ is a parameter of gauge fixing.

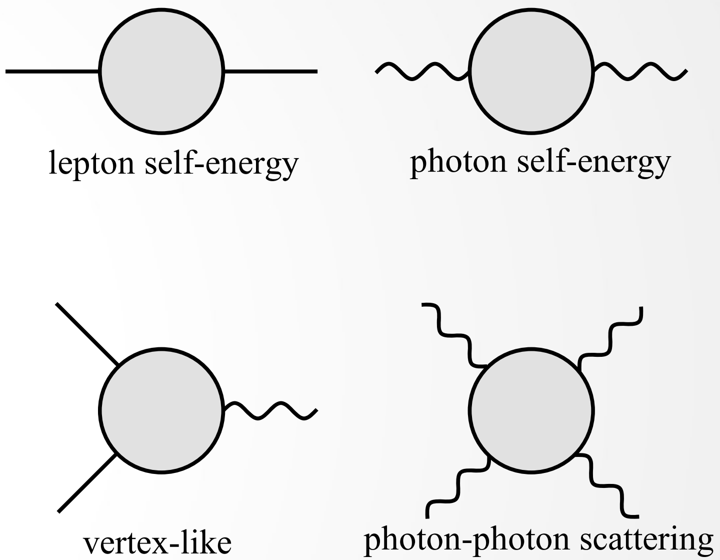
The algorithm:

- Draw all Feynman diagrams for L (the propagators come from L_0, the vertexes from L_1).
- Perform the divergence subtraction, it should be equivalent to the introduction of counterterms: $L_{\text{bare}}=L+L_{\text{ct}}$.
  The bare Lagrangian must be reducible to the same form by the change of variables $\psi\to a\psi$, $A\to bA$ (may be with different $m_{\text{bare}}\neq m$, $e_{\text{bare}}\neq e$, $\xi_{\text{bare}}\neq \xi$ that can be infinite).
- The physical parameters m_{phys} and e_{phys} can be extracted from the renormalized Feynman amplitudes (using the *on-shell conditions*).
- It must be $m_{\text{phys}}=m$ (otherwise the perturbation theory crashes down).
- It is possible that $e_{\text{phys}}\neq e$, but both parameters are finite.
- The Feynman amplitudes have a physical meaning only with the external line renormalization.

Renormalization of quantum electrodynamics: counterterms

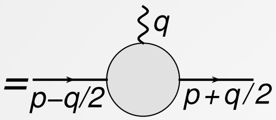
$$L = L_0 + L_1, \quad L_0 = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2, \quad L_1 = -e\overline{\psi}\gamma^\mu\psi A_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

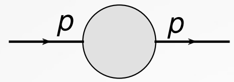
- <u>lepton self-energy</u>
  ω=1
  the counterterms are proportional to $\overline{\psi}\psi, \ \overline{\psi}\gamma^\mu\partial_\mu\psi$
- <u>photon self-energy</u>
  ω=2
  $(\partial_\mu A^\mu)^2, \quad A^\mu\partial_\nu\partial^\nu A_\mu$ are OK.
  $A^\mu A_\mu$ is not good, but the subtraction at zero momenta cancels this term (only after summation over Feynman diagrams)
- <u>vertex-like</u>
  ω=0
  the counterterms like $\overline{\psi}\gamma^\mu\partial_\mu\psi$
- <u>photon-photon scattering</u>
  ω=0
  It gives *AAAA*-like bad counterterms, but they are cancelled if we subtract at zero momenta (only after summation over Feynman diagrams)

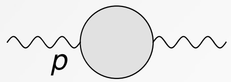


Renormalization of quantum electrodynamics: the on-shell conditions

Amputated vertex and self-energies:

$\Gamma_\mu(p,q)=$ 

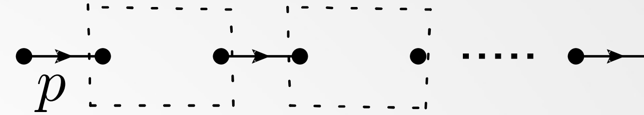
$\Sigma(p)=$ 

$\Pi_{\mu\nu}(p)=$ 

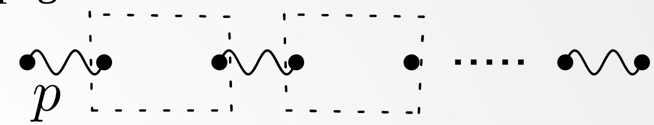
$$\Gamma_\mu(p,0)=a(p^2)\gamma_\mu+b(p^2)p_\mu+c(p^2)\not{p}p_\mu+d(p^2)(\not{p}\gamma_\mu-\gamma_\mu\not{p})$$

$$\Sigma(p)=r(p^2)+s(p^2)\not{p} \qquad \Pi_{\mu\nu}(p)=\Pi(p^2)g_{\mu\nu}+h(p^2)p_\mu p_\nu$$

Full electron propagator:

$S(p)=$ 

Full photon propagator:

$D_{\mu\nu}(p)=$ 

$$S(p)=\frac{ig_{\mu\nu}}{\not{p}-m-i\Sigma(p)+i0},\quad D_{\mu\nu}(p)=\frac{-i}{p^2+i\Pi(p^2)+i0}+f_1(p^2)p_\mu p_\nu$$

- The mass m should be physical: $S(p)$ has a pole at $p^2=m^2$.
  A linear condition: $r(m^2)+s(m^2)m=0$.
- The external line renormalization constants are extracted from the full propagators:

  $$S(p)\sim Z_2\frac{i(\not{p}+m)}{p^2-m^2+i0},\quad D_{\mu\nu}(p)\sim Z_3\frac{-ig_{\mu\nu}}{p^2+i0}+f(p^2)p_\mu p_\nu$$

  Linear conditions:

  $(Z_3)^{-1}-1=i\Pi'(0)$, $\quad (Z_2)^{-1}-1=-i[s(m^2)+2mr'(m^2)+2m^2s'(m^2)]$.

  Z_2, Z_3 relate to the LSZ conditions. They **DO NOT** describe the change of variables for reduction of the Lagrangian with counterterms to the counterterm-free form!
- The physical electric charge is extracted from the vertex-like diagrams:

  $e_{\text{phys}}=iZ_2(Z_3)^{1/2}[a(m^2)+b(m^2)m+c(m^2)m^2]$.

For convenience:

$(Z_1)^{-1}=i[a(m^2)+b(m^2)m+c(m^2)m^2]/e$.

The *Ward identity*:

$Z_1=Z_2$ (if the subtraction procedure preserves the Ward identity).

Thus, $e_{\text{phys}}=e(Z_3)^{1/2}$ (NOT e_{bare}!!!)

Each *amputated* Feynman amplitude with >=3 external lines is multiplied by

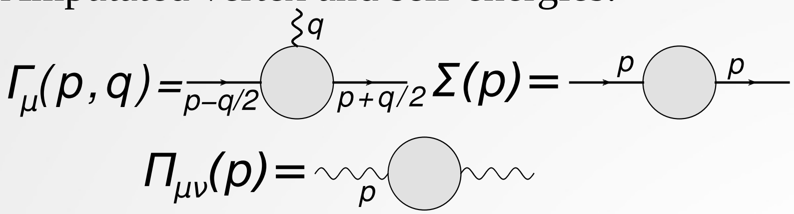
$(Z_2)^{a/2}(Z_3)^{b/2}$,

where a, b are the numbers of external fermion, photon lines.

 Sergey Volkov sergey.volkov@partner.kit.edu

Renormalization of quantum electrodynamics: the in-place on-shell renormalization making *Z*=1

Amputated vertex and self-energies:



$$\Gamma_\mu(p,0) = a(p^2)\gamma_\mu + b(p^2)p_\mu + c(p^2)\not{p}p_\mu + d(p^2)(\not{p}\gamma_\mu - \gamma_\mu\not{p})$$

$$\Sigma(p) = r(p^2) + s(p^2)\not{p} \qquad \Pi_{\mu\nu}(p) = \Pi(p^2)g_{\mu\nu} + h(p^2)p_\mu p_\nu$$

The linear on-shell conditions:

$r(m^2)+s(m^2)m=0,$

$(Z_1)^{-1}=i[a(m^2)+b(m^2)m+c(m^2)m^2]/e,$ $(Z_2)^{-1}-1=-i[s(m^2)+2mr'(m^2)+2m^2s'(m^2)],$ $(Z_3)^{-1}-1=i\Pi'(0),$

$e_{\mathrm{phys}} = (Z_1)^{-1}Z_2(Z_3)^{1/2}e$

Zimmermann's forest formula that removes all UV divergences:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

A ***forest*** is a set of subdiagrams of a diagram, each of them are non-intersecting or nested.
F is the set of all forests of UV-divergent 1-particle irreducible subdiagrams of the diagram.

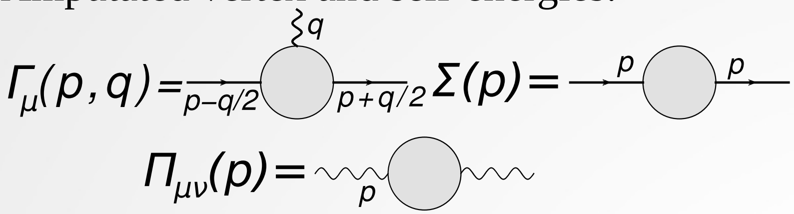
Usually, M_G extracts the Taylor expansion at zero momenta up to the degree *ω(G)*. However, one can modify the definition in order to perform the on-shell renormalization **in-place**:

- For photon self-energy and photon-photon scattering subgraphs M_G remains the same. It gives $Z_3=1$.
- For fermion self-energy subgraphs: $M\Sigma(p) = r(m^2) + s(m^2)\not{p} + 2m(\not{p}-m)[r'(m^2)+ms'(m^2)]$
  First 2 terms extract the overall UV divergence. The remaining term does not have it.
  This subtraction guarantees the mass condition and simultaneously $Z_2=1$.
- For vertex-like diagrams: $M\Gamma_\mu(p,q) = [a(m^2)+b(m^2)m+c(m^2)m^2]\gamma_\mu$ It gives $Z_1=1$.
  The *a(m²)*-term extracts the UV divergence, the remaining ones do not have an overall UV divergence.

 Sergey Volkov sergey.volkov@partner.kit.edu

Renormalization of quantum electrodynamics: the in-place renormalization forest formula and **physics**

Amputated vertex and self-energies:



$$\Gamma_\mu(p,0) = a(p^2)\gamma_\mu + b(p^2)p_\mu + c(p^2)\not{p}p_\mu + d(p^2)(\not{p}\gamma_\mu - \gamma_\mu\not{p})$$

$$\Sigma(p) = r(p^2) + s(p^2)\not{p} \qquad \Pi_{\mu\nu}(p) = \Pi(p^2)g_{\mu\nu} + h(p^2)p_\mu p_\nu$$

$$\sum_{\{G_1,\dots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \dots M_{G_n}$$

F is the set of all forests of 1-particle irreducible UV-divergent subgraphs.

- For photon self-energy and photon-photon scattering subgraphs M_G extracts the Taylor expansion at zero momenta up to the degree $\omega(G)$.
- For fermion self-energy subgraphs:
  $$M\Sigma(p) = r(m^2) + s(m^2)\not{p} + 2m(\not{p} - m)[r'(m^2) + ms'(m^2)]$$
- For vertex-like subgraphs:
  $$M\Gamma_\mu(p,q) = [a(m^2) + b(m^2)m + c(m^2)m^2]\gamma_\mu$$

But wait...

Yes, the definition of M_G is based on the on-shell conditions.

But what about the forest formula itself?

What about this combinatorics of non-overlapping subdiagrams?

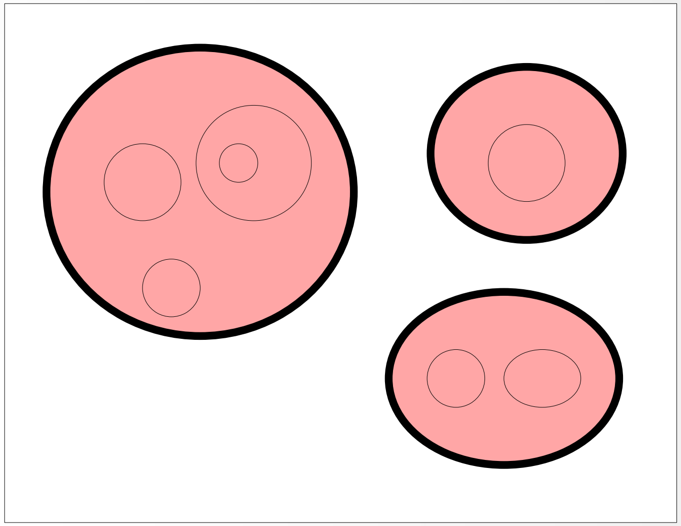
What has this to do with **physics**?

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all forests of 1-particle irreducible UV-divergent subgraphs.

What do we need?

- Equivalence to the introduction of counterterms.
We have already proved this statement:
if we draw a Feynman diagram with counterterm vertexes, each of the vertexes can be expanded to a part of the forest (a **fat** circle on a picture). After summation over all possible expansions we obtain the coefficient $C_1C_2\ldots C_k$, where C_j corresponds to the counterterm vertex *j* and depends only on the type of this vertex (it does not depend on the Feynman diagram and on the place of the vertex in this diagram).
Since we consider *only 1-particle irreducible subdiagrams*, the “countertermness” is valid for amputated diagrams.
- The physical conditions are satisfied.
Suppose we have a vertex-like or a self-energy diagram *G*, for which the linear condition is formulated.
Since *G* does not overlap with another subdiagrams, the multiplier $(1-M_G)$ is factorized.



Forests are natural!

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all forests of 1-particle irreducible UV-divergent subgraphs.

Yes, the forest formula with on-shell renormalization operators leads to the physical renormalization.

But why so complicated?

Is there a simpler solution for the in-place renormalization?

Possible simpler solution № 1:

just take $(1-M_G)$, where G is the Feynman diagram

What is wrong?

Possible simpler solution № 1:

just take $(1-M_G)$, where G is the Feynman diagram

What is wrong?

UV divergences cancellation?

Possible simpler solution № 1:

just take $(1-M_G)$, where G is the Feynman diagram

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics).
Suppose we have an ideal regularization for working with infinities.

Possible simpler solution № 1:

just take $(1-M_G)$, where G is the Feynman diagram

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics).
Suppose we have an ideal regularization for working with infinities.

Physical conditions?

Possible simpler solution № 1:

just take $(1-M_G)$, where G is the Feynman diagram

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics).
Suppose we have an ideal regularization for working with infinities.

Physical conditions?

No problem. The linear conditions are satisfied.

Possible simpler solution № 1:

just take $(1-M_G)$, where G is the Feynman diagram

What is wrong?

UV divergences cancellation?

Does not work, but we don’t care about this (we are interested only in physics).
Suppose we have an ideal regularization for working with infinities.

Physical conditions?

No problem. The linear conditions are satisfied.

Equivalence to the introduction of counterterms?

Possible simpler solution № 1:

just take $(1\text{-}M_G)$, where G is the Feynman diagram

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics).
Suppose we have an ideal regularization for working with infinities.

Physical conditions?

No problem. The linear conditions are satisfied.

Equivalence to the introduction of counterterms?

FAIL,
because the counterterm vertex can be at any place in the diagram based on Feynman rules with counterterms.

Possible simpler solution № 2:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

What is wrong?

F is the set of all **sets of non-intersecting** 1-particle irreducible UV-divergent subgraphs.

Nested subdiagrams are forbidden!

Possible simpler solution № 2:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

What is wrong?

UV divergences cancellation?

F is the set of all **sets of non-intersecting** 1-particle irreducible UV-divergent subgraphs.

Nested subdiagrams are forbidden!

Renormalization of quantum electrodynamics: the in-place renormalization forest formula and **physics**

Possible simpler solution № 2:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all **sets of non-intersecting** 1-particle irreducible UV-divergent subgraphs.

Nested subdiagrams are forbidden!

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics). Suppose we have an ideal regularization for working with infinities.

Possible simpler solution № 2:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all **sets of non-intersecting** 1-particle irreducible UV-divergent subgraphs.

Nested subdiagrams are forbidden!

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics).
Suppose we have an ideal regularization for working with infinities.

Equivalence to the introduction of counterterms?

Renormalization of quantum electrodynamics: the in-place renormalization forest formula and **physics**

Possible simpler solution № 2:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all **sets of non-intersecting** 1-particle irreducible UV-divergent subgraphs.

Nested subdiagrams are forbidden!

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics). Suppose we have an ideal regularization for working with infinities.

Equivalence to the introduction of counterterms?

No problem. The internal-external part factorization idea works. Nested diagrams are not obligatory for this.

Renormalization of quantum electrodynamics: the in-place renormalization forest formula and **physics**

Possible simpler solution № 2:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all **sets of non-intersecting** 1-particle irreducible UV-divergent subgraphs.

Nested subdiagrams are forbidden!

What is wrong?

UV divergences cancellation?

Does not work, but we don’t care about this (we are interested only in physics). Suppose we have an ideal regularization for working with infinities.

Equivalence to the introduction of counterterms?

No problem. The internal-external part factorization idea works. Nested diagrams are not obligatory for this.

Physical conditions?

Possible simpler solution № 2:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all **sets of non-intersecting** 1-particle irreducible UV-divergent subgraphs.

Nested subdiagrams are forbidden!

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics). Suppose we have an ideal regularization for working with infinities.

Equivalence to the introduction of counterterms?

No problem. The internal-external part factorization idea works. Nested diagrams are not obligatory for this.

Physical conditions?

FAIL,
because $(1-M_G)$, where *G* is the whole diagram, is not factorized.

Possible simpler solution № 3:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

What is wrong?

F is the set of all **sets of nested** 1-particle irreducible UV-divergent subgraphs.

Not intersecting subdiagrams are forbidden!

Possible simpler solution № 3:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all **sets of nested** 1-particle irreducible UV-divergent subgraphs.

Not intersecting subdiagrams are forbidden!

What is wrong?

UV divergences cancellation?

Possible simpler solution № 3:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all **sets of nested** 1-particle irreducible UV-divergent subgraphs.

Not intersecting subdiagrams are forbidden!

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics).
Suppose we have an ideal regularization for working with infinities.

Possible simpler solution № 3:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all **sets of nested** 1-particle irreducible UV-divergent subgraphs.

Not intersecting subdiagrams are forbidden!

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics).
Suppose we have an ideal regularization for working with infinities.

Physical conditions?

Possible simpler solution № 3:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all **sets of nested** 1-particle irreducible UV-divergent subgraphs.

Not intersecting subdiagrams are forbidden!

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics).
Suppose we have an ideal regularization for working with infinities.

Physical conditions?

No problem. The linear conditions are satisfied, because $(1\text{-}M_G)$ is factorized.

Possible simpler solution № 3:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all **sets of nested** 1-particle irreducible UV-divergent subgraphs.

Not intersecting subdiagrams are forbidden!

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics).
Suppose we have an ideal regularization for working with infinities.

Physical conditions?

No problem. The linear conditions are satisfied, because *(1-M_G)* is factorized.

Equivalence to the introduction of counterterms?

Renormalization of quantum electrodynamics: the in-place renormalization forest formula and **physics**

Possible simpler solution № 3:

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2} \ldots M_{G_n}$$

F is the set of all **sets of nested** 1-particle irreducible UV-divergent subgraphs.

Not intersecting subdiagrams are forbidden!

What is wrong?

UV divergences cancellation?

Does not work, but we don't care about this (we are interested only in physics). Suppose we have an ideal regularization for working with infinities.

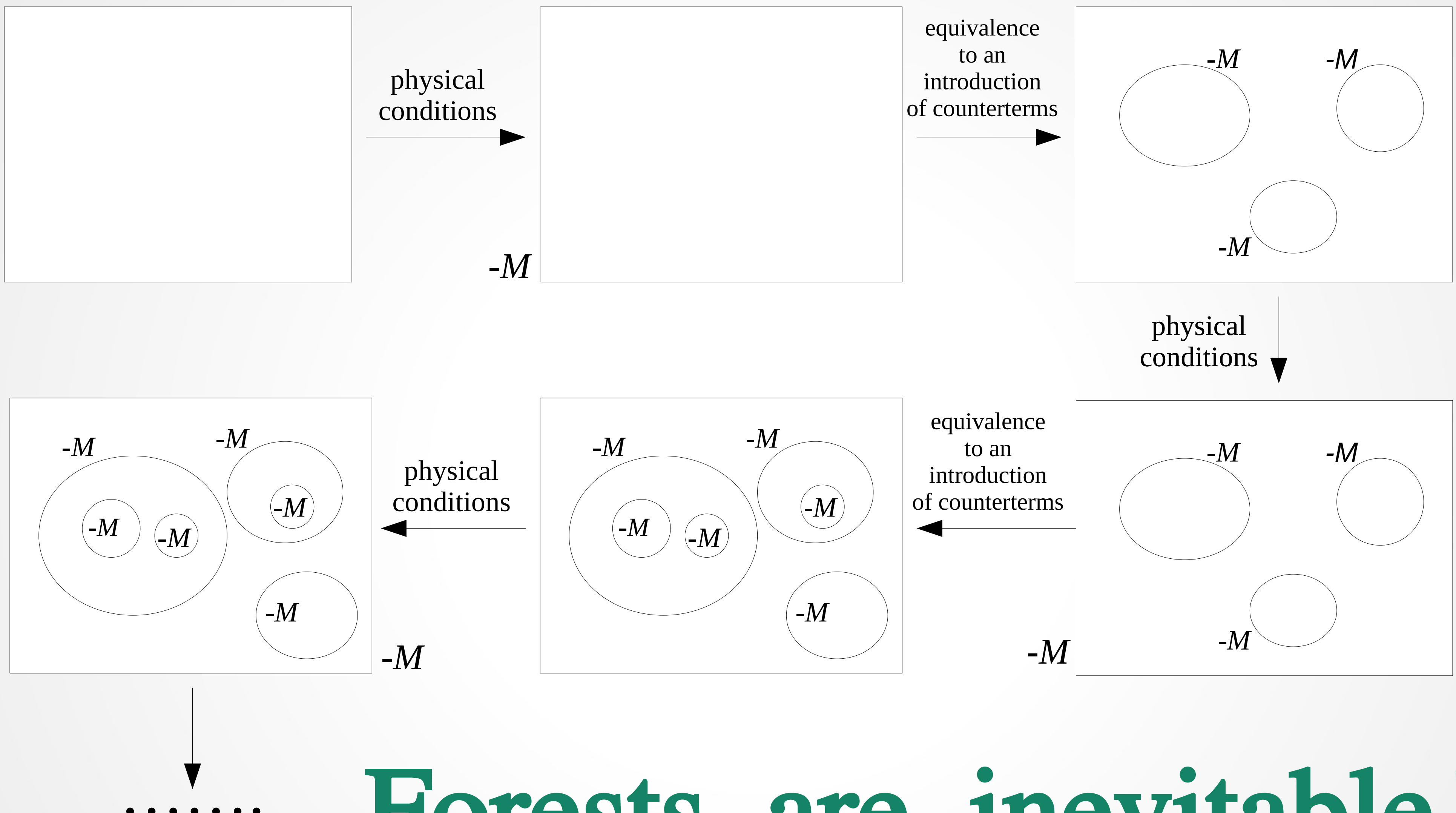
Physical conditions?

No problem. The linear conditions are satisfied, because $(1-M_G)$ is factorized.

Equivalence to the introduction of counterterms?

FAIL,
an expansion of several counterterm vertexes should be possible.

Renormalization of quantum electrodynamics: the in-place renormalization forest formula and **physics**



Forests are inevitable!

Renormalization of quantum electrodynamics: a numerical example of the in-place renormalization

The coefficient before $(\alpha/\pi)^2$ in the electron anomalous magnetic moment.

[A. Petermann, Helv. Phys. Acta 30, 407 (1957)]

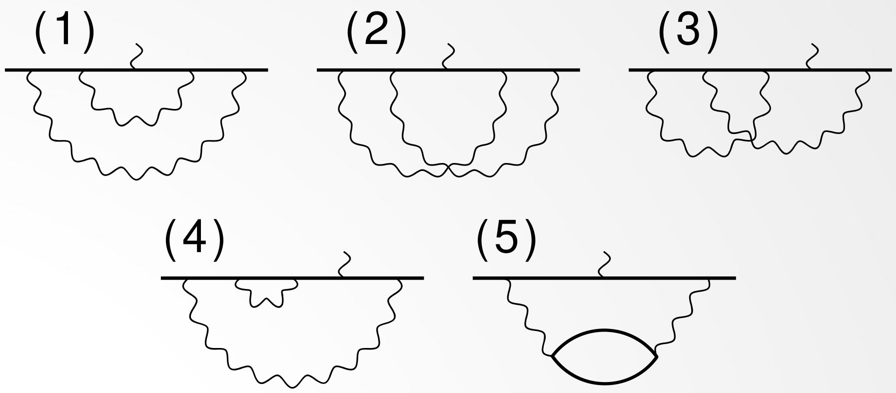
λ is the photon mass.
Unfortunately, some of the contributions are IR divergent.

It is interesting that both IR divergences come from **counterterms**!

IR divergences is the reason why the in-place subtraction is rarely used in calculations.

However, modified subtractions procedures that cover also IR divergences (in partial cases) are useful for high-order calculations:

[M. J. Levine, J. Wright, Phys. Rev. D 8, 3171 (1973)]
[R. Carroll, Y.-P. Yao, Phys. Lett. 48B, 125 (1974)]
[P. Cvitanović, T. Kinoshita, Phys. Rev. D 10, 3991 (1974)]
[T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Nucl. Phys. B 796, 184 (2008)]
[S. Volkov, Phys. Rev. D 100, 096004 (2019)]



№	Value
1	*0.77747802*
2	*-0.46764544*
3	$0.564021-(1/2)\log(\lambda^2/m^2)$
4	$-0.089978+(1/2)\log(\lambda^2/m^2)$
5	*0.0156874*
Σ	*-0.328478966*

Renormalization of quantum electrodynamics: a note about an interchange between L_0 and L_1

The Lagrangian: $L=L_0+L_1$

L_0 is the *free* part (the propagators come from it).

L_1 is the *interaction* part (the vertices come from it).

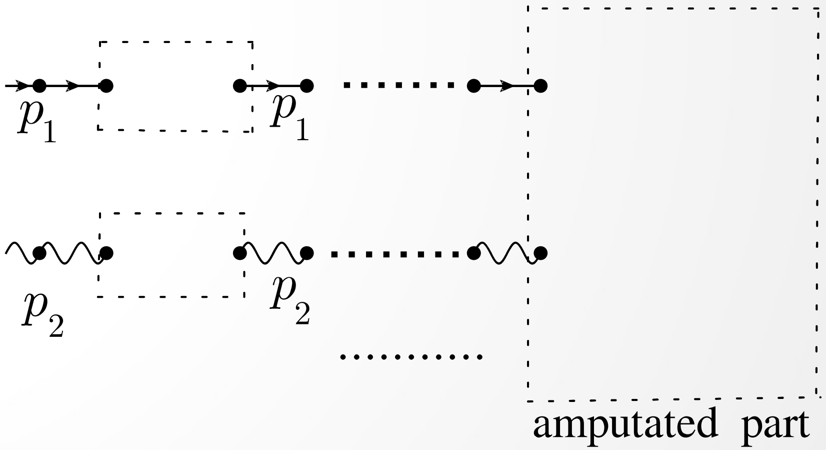
The renormalization by the forest formula is equivalent to the adding counterterms to L_1.

Moving quadratic terms between L_0 and L_1 changes nothing:

a geometric progression sum of the chains with the corresponding vertices in L_1 equals the propagator with this term in L_0.

However, there are two possible unpleasant situations:

- The new vertex can fall into an external line.
  This geometric progression idea does not work for external lines.
  Solution: only two types of Feynman diagrams have a physical meaning:
  1) amputated (in this case, a vertex can't be on an external line);
  2) surrounded by propagators (in this case, the external lines are incident to the vertexes that do not follow the Feynman rules for the ordinary vertices and therefore an insertion is impossible).
- The new vertex can become the only vertex in a diagram.
  Solution: amputated Feynman diagrams with ≤2 external lines do not have a physical meaning as describing physical processes. They have a meaning only as a part of the full propagator.



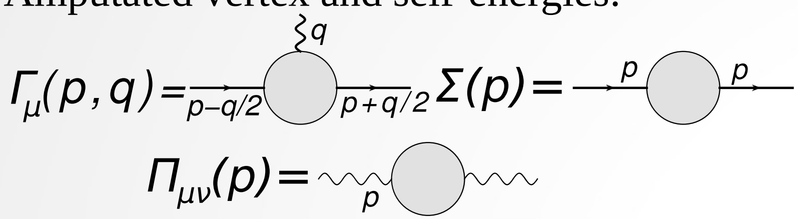
not amputated Feynman diagram

Renormalization of quantum electrodynamics: different ways of the in-place renormalization

A minimalistic approach:

to subtract only the fermion mass and the quadratic part of the photon self-energy

Amputated vertex and self-energies:



$$\Sigma(p) = r(p^2) + s(p^2)\not{p}$$

$$\Pi_{\mu\nu}(p) = \Pi(p^2) g_{\mu\nu} + h(p^2) p_\mu p_\nu$$

$$M\Sigma(p) = r(m^2) + s(m^2)m, \quad M\Pi_{\mu\nu}(p) = \frac{1}{2} \left. \frac{\partial \Pi_{\mu\nu}(p)}{\partial \xi \partial \eta} \right|_{p=0} p_\xi p_\eta$$

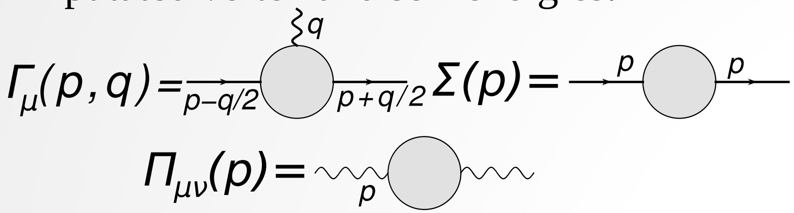
No other subtractions!

- UV divergences are removed by the external line renormalization: Z_1 and Z_2 are UV-divergent (and, strictly speaking, also IR-divergent).
- The charge renormalization is easy: $Z_1=Z_2$, $Z_3=1$, $e_{\text{phys}}=e$.
- A regularization for working with infinities is required.
- No nonphysical subtractions.
- No overlaps.
- Very convenient for studying properties (like the gauge invariance at the level of Feynman diagrams).

An <u>intermediate</u> approach:

simpler formulas for the vertex and fermion self-energy subtractions

Amputated vertex and self-energies:



$$\Gamma_\mu(p,0) = a(p^2)\gamma_\mu + b(p^2)p_\mu + c(p^2)\not{p}p_\mu + d(p^2)(\not{p}\gamma_\mu - \gamma_\mu\not{p})$$

$$\Sigma(p) = r(p^2) + s(p^2)\not{p} \qquad \Pi_{\mu\nu}(p) = \Pi(p^2)g_{\mu\nu} + h(p^2)p_\mu p_\nu$$

The subtractions are the same as in the <u>*Z=1*</u> approach, but

$$M\Sigma(p) = a(m^2) + b(m^2)\not{p}, \quad M\Gamma_\mu(p,q) = a(m^2)\gamma_\mu$$

- All UV divergences are removed in each individual Feynman diagram, as well as in the *Z=1* approach.
- $Z_1=Z_2\neq 1$, an external line renormalization is required (in contrast to the *Z=1* approach).
- No IR divergences in the counterterms (in contrast to the *Z=1* approach).
- However, Z_1 and Z_2 are IR divergent.
- The charge renormalization is easy: $Z_1=Z_2$, $Z_3=1$, $e_{\text{phys}}=e$ (as well as in the other approaches).

See an example of the application of a *similar* method:
[S.Volkov, arXiv:2308.11560 (2023)]

Outline

- Introduction
- General ideas of handling UV divergences
- Application to the renormalization of quantum electrodynamics
- **Formulations in terms of finite integrals**
  - approaches to the regularization of Minkowsky-space propagators
  - the Schwinger parameters
  - the formulation of the BPHZ theorem
  - issues and difficulties related to Schwinger-parametric integrals
  - power counting theorems
- The proof of the BPHZ theorem
- Conclusions

Formulations in terms of finite integrals: regularization of Minkowsky-space propagators

Integrals with Minkowsky-space propagators

$$\frac{P(q)}{q^2 - m^2 + i\varepsilon}$$

where $P(q)$ is a polynomial, do not exist. The divergence subtractions do not help.

The approaches:

- An additional Euclidean-based term in the denominator.
  For example, with an additional regulator $\varepsilon_{\text{mink}}$:
  $$\frac{P(q)}{q^2 - m^2 + i(\varepsilon_{\text{IR}} + \varepsilon_{\text{Mink}}|q|^2_{\text{Eucl}})}$$
  Or Zimmermann's simultaneous approach:
  $$\frac{P(q)}{q^2 - m^2 + i\varepsilon(\mathbf{q}^2 + m^2)}$$
  The Lorentz-covariance after taking $\varepsilon_{\text{mink}} \to 0$ (or $\varepsilon \to 0$) is not obvious.
  W. Zimmermann proved that the forest formula applied directly in momentum space leads to finite integrals and gives a Lorentz-covariant distribution as $\varepsilon \to 0$:
  [W. Zimmermann, Commun. Math. Phys. 15, 208 (1969)]

- The Schwinger parameters: $\frac{P(q)}{q^2 - m^2 + i\varepsilon} = \frac{P(q)}{i} \int_0^{+\infty} e^{i\alpha(p^2 - m^2) - \alpha\varepsilon} d\alpha$
  First integrate over loop momenta with a fixed α (analytically, ignoring the non-existence).
  The subtraction applied directly in the Schwinger-parametric integrals leads to finite integrals:
  [N. N. Bogoliubov and O. S. Parasiuk, Acta Math. 97, 227 (1957)]
  [K. Hepp, Commun. Math. Phys. 2, 301 (1966)]

Formulations in terms of finite integrals: the Schwinger parameters

Suppose we have a loop integral

$$I(p_1,p_2,\ldots,p_r,\varepsilon_{\mathrm{IR}}) = \lim_{\varepsilon_{\mathrm{Mink}}\to+0} \int \frac{P}{Q_1\ldots Q_M} d^4k_1\ldots d^4k_L$$

$p_1,\ldots,p_r$ are the external momenta.

$k_1,\ldots,k_L$ are the loop momenta.

$P(k,p)$ is polynomial of k and p.

$Q_j(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})=s_j(k,p)+i\varepsilon_{\mathrm{Mink}} r_j(k,p)+i\varepsilon_{\mathrm{IR}}$,

where s_j are Lorentz-covariant real-valued quadratic functions (not obligatory homogeneous), r_j are quadratic functions, $r_j(k,p)>0$, $r_j(k,p)\to+\infty$ as $k\to\infty$.

The integral in I and the limit do not exist, but we can redefine it by

$$I(p_1,\ldots,p_r,\varepsilon_{\mathrm{IR}}) = \int_0^{+\infty} F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})d\alpha_1\ldots d\alpha_M$$

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}}) = \lim_{\varepsilon_{\mathrm{Mink}}\to+0} F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})$$

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}}) = \int \frac{1}{i^M} P(k,p) e^{i[\alpha_1 Q_1(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})+\ldots+\alpha_M Q_M(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})]} d^4k_1\ldots d^4k_L$$

This integral exists for any $\alpha>0$ (because $r_j(k,p)\to+\infty$ make a multiplier tending to 0 at infinity; it suppresses all other multipliers).

$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})$ exists and saves the Lorentz-invariance (it can be proved using analytical formulas).

$\alpha_1,\ldots,\alpha_M$ are called **the Schwinger parameters**.

This swap of the limit and integration and the integration order is incorrect, but we use it as a definition!

$I(p_1,\ldots,p_r,\varepsilon_{\mathrm{IR}})$ exists for $\varepsilon_{\mathrm{IR}}>0$ provided that we don't have UV divergences.

Formulations in terms of finite integrals: the BPHZ theorem formulation

Suppose we have a Feynman diagram with r external, M internal lines, L independent loops.
$p_1,\ldots,p_r$ are the external momenta.
$k_1,\ldots,k_L$ are the loop momenta.
$P(k,p)$ is the product of all propagator numerators and vertex polynomials.
$Q_j(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})=q_j(k,p)^2-(m_j)^2+i\varepsilon_{\mathrm{Mink}}r_j(k,p)+i\varepsilon_{\mathrm{IR}}$ (a regularized propagator denominator).
q_j is the momentum passing the line j.
m_j is the particle mass of the line j.
r_j are quadratic functions, $r_j(k,p)>0$, $r_j(k,p)\to+\infty$ as $k\to\infty$.

If the Schwinger parameters $\alpha_1,\ldots,\alpha_M>0$ are fixed, we can replace the usual propagators $\frac{P_j(q)}{Q_j(q,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})}$ with $P_j(q)e^{i\alpha_j Q_j(q,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})}$

After that, we apply Zimmermann's forest formula to the diagram with these propagators and obtain

$$F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})$$

It is correctly defined, because all the needed integrals exist.
Put

$$F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})=\lim_{\varepsilon_{\mathrm{Mink}}\to+0}F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})$$

$$I_{\mathrm{Sub}}(p_1,\ldots,p_r,\varepsilon_{\mathrm{IR}})=\int_0^{+\infty}F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})d\alpha_1\ldots d\alpha_M$$

Theorem. $I_{\mathrm{Sub}}(p_1,\ldots,p_r,\varepsilon_{\mathrm{IR}})$ **exists.** We prove this later.

Formulations in terms of finite integrals: Schwinger parameters and Gaussian integrals

To obtain an explicit formula for

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})=\int\frac{1}{i^M}P(k,p)e^{i[\alpha_1 Q_1(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})+\ldots+\alpha_M Q_M(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})]}d^4k_1\ldots d^4k_L$$

we rewrite it as

$$\frac{1}{i^M}P\left(\frac{-i\partial}{\partial\xi_1},\ldots,\frac{-i\partial}{\partial\xi_L},p\right)\int e^{i[\alpha_1 Q_1(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})+\ldots+\alpha_M Q_M(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})+\xi_1k_1+\ldots+\xi_Lk_L]}d^4k_1\ldots d^4k_L\bigg|_{\xi=0}$$

What we need is to obtain the integrals like

$$\int e^{x^TAx+f^Tx}d^Nx,$$

where A is an arbitrary matrix $n\times n$, Re $A<0$; f is an n-dimensional vector. If all the matrices are real, by the change of variables $y=x+\frac{1}{2}A^{-1}f$ we rewrite the integral as

$$\int e^{y^TAy-\frac{1}{4}f^TA^{-1}f}d^Ny=e^{-\frac{1}{4}f^TA^{-1}f}\int e^{y^TAy}d^Ny$$

By diagonalization and using the formula for 1-dimensional Gaussian integral:

$$\int e^{x^TAx+f^Tx}d^Nx=\frac{\pi^{N/2}}{\sqrt{\det(-A)}}e^{-\frac{1}{4}f^TA^{-1}f}$$

If the matrices are complex, use the analytic continuation (both the left and right side are analytic, but the continuation from the real axis is unique).
Another approach to complex matrices: first diagonalize Re(A) by multiplying by Re$(A)^{-1/2}$ in both sides, after that diagonalize Im(A) by rotations.
Note. One should be careful with the det(-A) square root **sign**. It is **not** determined by det(-A) and should be taken to make the function continuous.

Formulations in terms of finite integrals: a general form of Schwinger-parametric integrands

We have an integrand

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})=\int \frac{1}{i^M}P(k,p)e^{i[\alpha_1 Q_1(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})+\ldots+\alpha_M Q_M(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})]}d^4k_1\ldots d^4k_L$$

$$=\frac{1}{i^M}\,P\left(\frac{-i\partial}{\partial\xi_1},\ldots,\frac{-i\partial}{\partial\xi_L},p\right)\int e^{i[\alpha_1 Q_1(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})+\ldots+\alpha_M Q_M(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})+\xi_1 k_1+\ldots+\xi_L k_L]}d^4k_1\ldots d^4k_L\bigg|_{\xi=0}$$

The explicit formula

$$\int e^{x^T A x+f^T x}d^N x=\frac{\pi^{N/2}}{\sqrt{\det(-A)}}e^{-\frac{1}{4}f^T A^{-1} f}$$

for the integrals like this allows us to take the limit $\varepsilon_{\mathrm{Mink}}\to +0$: the same formula remains valid for the limit (but some care with the sign is required). After taking the limit:

- The matrices *A* and *f* become imaginary.
- The matrix A consists of ``Minkowsky'' 4*4 blocks; thus, the square root of det*(-A)* is the square of the *reduced matrix* determinant (a rational function).

Since $Q_j(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})=q_j(k,p)^2-(m_j)^2+i\varepsilon_{\mathrm{Mink}}r_j(k,p)+i\varepsilon_{\mathrm{IR}}$, and A^{-1} is a polynomial on *A* divided by det(*A*), it is convenient to write

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})=C\frac{W(p,\alpha)}{U(\alpha)^2}e^{i\frac{V(p,\alpha)}{U(\alpha)}-i\sum_j m_j\alpha_j^2-\varepsilon_{\mathrm{IR}}\sum_j\alpha_j},$$

where *U* is a real polynomial; *V* is a real quadratic form on *p* and a polynomial on *α; W* is a polynomial on *p* and rational on *α; C* is a coefficient.

Formulations in terms of finite integrals: singularities of Schwinger-parametric integrals

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}}) = C\frac{W(p,\alpha)}{U(\alpha)^2}e^{i\frac{V(p,\alpha)}{U(\alpha)}-i\sum_j m_j\alpha_j^2-\varepsilon_{\mathrm{IR}}\sum_j\alpha_j},$$

where U is a real polynomial; V is a real quadratic form on p and a polynomial on α; W is a polynomial on p and rational on α; C is a coefficient.

The exponential factor is bounded and fastly tends to *0* as $\alpha \to +\infty$.
Moreover, all coefficients of U are positive (we will see this later).

Thus, singularities in the integral may occur when $\alpha \to 0$. And it is governed by the rational part.

Formulations in terms of finite integrals: discovering singularities of rational integrals

Suppose, for simplicity, we have an integral

$$\int_0^\Lambda F(\alpha_1, \ldots, \alpha_M) d\alpha_1 \ldots d\alpha_M,$$

where F is a rational function with positive coefficients, Λ serves as an IR cut-off.

How to determine if it is convergent?

One approach:
take **all** possible nonempty sets S of indexes $\{1,2,\ldots,M\}$;
for each S put $\alpha_j = t \to 0$ for j in S, $a_j > 0$ are fixed for j outside S;
if $F(t) = \Omega(t^{-|S|})$, then the integral is divergent.

Yes, it recognizes divergences, but...

 sergey.volkov@partner.kit.edu

Formulations in terms of finite integrals: discovering singularities of rational integrals

Suppose, for simplicity, we have an integral

$$\int_0^\Lambda F(\alpha_1,\ldots,\alpha_M)d\alpha_1\ldots d\alpha_M,$$

where F is a rational function with positive coefficients, Λ serves as an IR cut-off.

How to determine if it is convergent?

<u>One approach</u>:
take **all** possible nonempty sets S of indexes $\{1,2,\ldots,M\}$;
for each S put $\alpha_j=t\rightarrow 0$ for j in S, $a_j>0$ are fixed for j outside S;
if $F(t)=\Omega(t^{-|S|})$, then the integral is divergent.

Yes, it recognizes divergences, but... **It is not enough!**

Formulations in terms of finite integrals: discovering singularities of rational integrals

Suppose, for simplicity, we have an integral

$$\int_0^\Lambda F(\alpha_1,\ldots,\alpha_M)d\alpha_1\ldots d\alpha_M,$$

where F is a rational function with positive coefficients, Λ serves as an IR cut-off.

How to determine if it is convergent?

One approach:
take **all** possible nonempty sets S of indexes $\{1,2,\ldots,M\}$;
for each S put $\alpha_j=t\to 0$ for j in S, $a_j>0$ are fixed for j outside S;
if $F(t)=\Omega(t^{-|S|})$, then the integral is divergent.

Yes, it recognizes divergences, but... **It is not enough!**

An example: $F(x,y) = x/(x^4+y^2)$.
$x\to 0$, $F\approx x$ (no divergence). $y\to 0$, $F\approx 1$ (no divergence). $t=x=y\to 0$, $F\approx t^{-1}$ (no divergence).
But it is divergent!
In the area $x^2<y<2x^2$ it behaves like $1/x^3$.
Thus, the whole integral in this area behaves like $\int x^{-1}dx$.

One must be **careful** even in the **simplest** cases!

Formulations in terms of finite integrals: power counting theorems

To handle difficulties with a divergence recognition in rational integrals, **power counting theorems** were developed:

- Weinberg's theorem

  [S. Weinberg, Physical Review 118, N 3, 838-849 (1960)]

  Applicable for Feynman integrals in Euclidean momentum space.

- Zimmermann's theorem

  [W. Zimmermann, Commun. Math. Phys. 11, 1-8 (1968)]

  A modification of Weinberg's theorem working in Minkowsky space with Zimmermann's regularization.

- Speer's lemma, its consequences and generalizations ← the most useful

  [E. Speer, Journal of Mathematical Physics 9, N 9, 1404-1410 (1968)]

$$|F(p,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})| \leq e^{-\varepsilon_{\mathrm{IR}}(\alpha_1+\ldots+\alpha_M)}\frac{P(p,\sum_j \alpha_j)}{\alpha_1\ldots\alpha_M}\left(\frac{\alpha_1}{\alpha_2}\right)^{\lceil -\omega(\{1\})/2\rceil}\ldots\left(\frac{\alpha_{M-1}}{\alpha_M}\right)^{\lceil -\omega(\{1,\ldots,M-1\})/2\rceil}(\alpha_M)^{\lceil -\omega(\{1,\ldots,M\})/2\rceil}$$

where $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_M$ are the Schwinger parameters; the ultraviolet divergence index *ω(S)* is *defined on sets of lines, including not connected*; *P* is a polynomial.
It guarantees that the Schwinger-parametric integral is convergent, if there are no UV-divergent subdiagrams.
The asymptotic behavior of the constituting terms can also be obtained exactly (it was used earlier as a basis for the dimensional and other regularizations).

Outline

- Introduction
- General ideas of handling UV divergences
- Application to the renormalization of quantum electrodynamics
- Formulations in terms of finite integrals
- **The proof of the BPHZ theorem**
- Conclusions

The proof of the BPHZ theorem: remember the formulation

Suppose we have a Feynman diagram with r external, M internal lines, L independent loops.
$p_1,\ldots,p_r$ are the external momenta.
$k_1,\ldots,k_L$ are the loop momenta.
$P_l(k,p)$ is the propagator numerator (a polynomial).
$Q_j(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})=q_j(k,p)^2-(m_j)^2+i\varepsilon_{\mathrm{Mink}}r_j(k,p)+i\varepsilon_{\mathrm{IR}}$ (a regularized propagator denominator).
q_j is the momentum passing the line j.
m_j is the particle mass of the line j.
r_j are quadratic functions, $r_j(k,p)>0$, $r_j(k,p)\to+\infty$ as $k\to\infty$.

Introduce the Schwinger parameters $\alpha_1,\ldots,\alpha_M$. If the values $\alpha>0$ are fixed, use the propagator

$$P_l(k,p)e^{i\alpha_l Q_l(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})}$$

for the line l.

If $\varepsilon_{\mathrm{IR}}>0$, $\varepsilon_{\mathrm{Mink}}>0$ are also fixed, all the integrals are well-defined. Apply the forest formula, obtain

$$F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})$$

Take the limit

$$F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})=\lim_{\varepsilon_{\mathrm{Mink}}\to+0}F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})$$

We can use explicit analytical formulas. However, when we apply the forest formula, we have to work with diagrams containing special polynomial vertices (that correspond to the Taylor expansion terms).

After that, the integral $I_{\mathrm{Sub}}(p_1,\ldots,p_r,\varepsilon_{\mathrm{IR}})=\int_0^{+\infty}F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})d\alpha_1\ldots d\alpha_M$ **exists**.

The proof of the BPHZ theorem: remember the formulation

More precisely,…

Earlier we assumed that each vertex v had its polynomial P_v, as well as each line l has its polynomial P_l.
It also implies tensors, matrices and other algebraic objects in a diagram.
It is not convenient for the analysis...

Suppose that we use **monomials** instead of polynomials.
Each line monomial P_l has a form $(q_0)^{a_0}(q_1)^{a_1}(q_2)^{a_2}(q_3)^{a_3}$, where q is the line momentum. Each vertex monomial P_v is the product of monomials like this for the incident to v lines.
The forest formula is linear; thus, more complicated constructions can be obtained as linear combinations.

The monomials can have degrees smaller that of the original polynomials, but we always use the ultraviolet degrees of divergence $\omega(G)$ calculated for the original ones.

The proof of the BPHZ theorem: the ideas

- Use explicit combinatorial formulas for the construction blocks of the Schwinger-parametric integrals.

  Symanzik polynomials and so on.
  Independent on the loop basis.
  This makes the power counting possible…

- Split the Schwinger-parametric space into areas with asymptotically different behavior of the functions: Hepp sectors and “Hanoi towers”.

- Reduce the forest formula to the form with sets of lines.

  because the “Hanoi towers’’ technique works with sets of lines…

- Factorize the forest formula in each Hepp’s sector differently, split it into parts.

  It eliminates the problem with overlapping divergences…
  The combinatorial ideas are easy to check, but hard to invent...

- Replace the remaining subtractions with differentiations.

- Count the powers and estimate the integrand absolute value.

 Sergey Volkov sergey.volkov@partner.kit.edu

Outline

- Introduction
- General ideas of handling UV divergences
- Application to the renormalization of quantum electrodynamics
- Formulations in terms of finite integrals
- **The proof of the BPHZ theorem**
  - the formulation, ideas
  - Schwinger-parametric integrals, combinatorial formulas
  - power counting, Hepp sectors, "Hanoi" towers
  - reduction to the forest formula with sets of lines
  - elimination of overlaps
  - the case when all the subtractions fit
- Conclusions

The proof of the BPHZ theorem: schwinger-parametric integrals, combinatorial formulas

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})=\int\frac{1}{i^M}P_1(q_1(k,p))\ldots P_m(q_m(k,p))e^{i[\alpha_1 Q_1(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})+\ldots+\alpha_M Q_M(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})]}d^4k_1\ldots d^4k_L$$

$k_1,\ldots,k_L$ are the loop momenta

$p_1,\ldots,p_r$ are the external momenta

$q_j(p,k)$ is the momentum passing through the line j

$Q_j(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})=q_j(k,p)^2-(m_j)^2+i\varepsilon_{\mathrm{Mink}}r_j(k,p)+i\varepsilon_{\mathrm{IR}}$ (a regularized propagator denominator)

P_j is the monomial corresponding to the propagator numerator (and parts of the vertex monomials)

We will use a more convenient formula with ξ:

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})=\frac{1}{i^M}P_1\left(\frac{-i\partial}{\partial\xi_1}\right)\ldots P_M\left(\frac{-i\partial}{\partial\xi_M}\right)$$
$$\times\left(\lim_{\varepsilon_{\mathrm{Mink}}\to 0}\int e^{i[\alpha_1 Q_1(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})+\ldots+\alpha_M Q_M(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})+\xi_1 q_1(k,p)+\ldots+\xi_M q_M(k,p)]}d^4k_1\ldots d^4k_L\right)\Big|_{\xi=0}$$

It equals

$$\frac{1}{i^M}P_1\left(\frac{-i\partial}{\partial\xi_1}\right)\ldots P_M\left(\frac{-i\partial}{\partial\xi_M}\right)\left(\lim_{\varepsilon_{\mathrm{Mink}}\to 0}\int e^{k^TAk+f^Tk+g}d^4k_1\ldots d^4k_L\right)\Big|_{\xi=0}$$
$$=\frac{\pi^{2L}}{i^M\sqrt{\det(-A)}}P_1\left(\frac{-i\partial}{\partial\xi_1}\right)\ldots P_M\left(\frac{-i\partial}{\partial\xi_M}\right)e^{g-\frac{1}{4}f^TA^{-1}f}\Big|_{\xi=0}$$

Here A is independent on ξ; f and g depend on ξ linearly.

Thus, we have a quadratic function on ξ in the exponent.

The proof of the BPHZ theorem: schwinger-parametric integrals, combinatorial formulas

Multiple differentiation of the exponent of the quadratic function

$$\frac{\partial}{\partial \xi_{i_1}} \cdots \frac{\partial}{\partial \xi_{i_n}} e^{\xi^T \Lambda \xi + \xi^T \beta + \gamma} \Big|_{\xi=0}$$

$$= \left[\sum_{\{\{r[1],s[1]\},\ldots,\{r[m],s[m]\}\}} (2\Lambda_{i_{r[1]} i_{s[1]}}) \ldots (2\Lambda_{i_{r[m]} i_{s[m]}}) \prod_{l \neq r[j],s[j]} \beta_{i_l} \right] e^{\gamma},$$

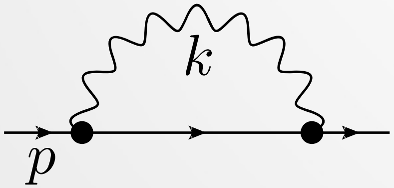
where Λ, β, γ are a symmetric matrix, a vector and a number; the summation goes over all sets of non-intersecting pairs in $\{1,2,\ldots,n\}$ (including the empty one).

Example:

$$\frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} \frac{\partial^2}{\partial (\xi_3)^2} e^{\xi^T \Lambda \xi + \xi^T \beta + \gamma} \Big|_{\xi=0} = (\beta_1 \beta_2 (\beta_3)^2 + 2\Lambda_{12} (\beta_3)^2 + 4\Lambda_{13} \beta_2 \beta_3 + 4\Lambda_{23} \beta_1 \beta_3 + 4\Lambda_{12} \Lambda_{33} + 8\Lambda_{13} \Lambda_{23}) e^{\gamma}$$

Application to the Schwinger-parametric expressions.

$i_1,\ldots,i_n$ correspond to the propagator numerator multipliers or vertex monomial multipliers.
In one term, some of the multipliers are paired; the remaining ones are unpaired.
Each pair increases the degree of divergence.



$\omega=1$, but we have only a logarithmic UV divergence,
because there is only one numerator multiplier, no possibility to make a pair

The proof of the BPHZ theorem: schwinger-parametric integrals, combinatorial formulas

Thus, we have an integral

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}}) = \frac{1}{i^M} P_1\left(\frac{-i\partial}{\partial\xi_1}\right)\ldots P_M\left(\frac{-i\partial}{\partial\xi_M}\right)$$

$$\times\left(\lim_{\varepsilon_{\mathrm{Mink}}\to 0}\int e^{i[\alpha_1 Q_1(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})+\ldots+\alpha_M Q_M(k,p,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})+\xi_1 q_1(k,p)+\ldots+\xi_M q_M(k,p)]} d^4k_1\ldots d^4k_L\right)\Big|_{\xi=0}$$

$$= C\frac{1}{\sqrt{\det(-A)}} P_1\left(\frac{\partial}{\partial\xi_1}\right)\ldots P_M\left(\frac{\partial}{\partial\xi_M}\right) e^{g-\frac{1}{4}f^T A^{-1} f}\Big|_{\xi=0}$$

We can perform the multiple differentiation with respect to ξ, but we need exact formulas for A, f, g.

They are:

$$A = iS_{\mathrm{Loop}}^T \mathrm{Diag}[\alpha] S_{\mathrm{Loop}} \quad \text{(of Minkowsky } 4\times 4\text{-blocks)},$$

$$\mathrm{Diag}[\alpha] = \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_M \end{bmatrix},$$

$$f = iS_{\mathrm{Loop}}^T[2\mathrm{Diag}[\alpha]S_{\mathrm{Flow}}p+\xi] \quad \text{(a vector of 4-vectors, upper tensor indices are used, whereas lower indices are used for } k),$$

$$g = g_0+g_1, \quad g_0 = -\varepsilon_{\mathrm{IR}}\sum_j \alpha_j - i\sum_j \alpha_j m_j,$$

$$g_1 = ip^T S_{\mathrm{Flow}}^T[\mathrm{Diag}[\alpha]S_{\mathrm{Flow}}p+\xi] \quad \text{(Minkowsky scalar products are implied)},$$

where $p=[p_1,\ldots,p_r]^T$;

S_{Loop} is a loop basis (a matrix of $0,1,-1$, one column is one independent loop);

S_{Flow} is a flow basis (a number matrix; $S_{\mathrm{Flow}}p$ = the vector of the line momenta that corresponds to $k=0$ and the external momenta p).

The proof of the BPHZ theorem: schwinger-parametric integrals, combinatorial formulas

We should combine the formulas

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}}) = \frac{C}{\sqrt{\det(-A)}} P_1\left(\frac{\partial}{\partial\xi_1}\right)\ldots P_M\left(\frac{\partial}{\partial\xi_M}\right) e^{g-\frac{1}{4}f^T A^{-1} f}\Big|_{\xi=0} = C\frac{W(p,\alpha)}{U(\alpha)^2} e^{i\frac{V(p,\alpha)}{U(\alpha)} - i\sum_j m_j\alpha_j^2 - \varepsilon_{\mathrm{IR}}\sum_j \alpha_j},$$

$$A = iS^T_{\mathrm{Loop}}\mathrm{Diag}[\alpha]S_{\mathrm{Loop}} \quad \text{(of Minkowsky } 4\times 4\text{-blocks)},$$

$$\mathrm{Diag}[\alpha] = \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_M \end{bmatrix},$$

$$f = iS^T_{\mathrm{Loop}}[2\mathrm{Diag}[\alpha]S_{\mathrm{Flow}}p + \xi] \quad \text{(a vector of 4-vectors, upper tensor indices are used, whereas lower indices are used for } k\text{)},$$

$$g = g_0 + g_1, \quad g_0 = -\varepsilon_{\mathrm{IR}}\sum_j \alpha_j - i\sum_j \alpha_j m_j,$$

$$g_1 = ip^T S^T_{\mathrm{Flow}}[\mathrm{Diag}[\alpha]S_{\mathrm{Flow}}p + \xi] \quad \text{(Minkowsky scalar products are implied)},$$

$$\frac{\partial}{\partial\xi_{i_1}}\ldots\frac{\partial}{\partial\xi_{i_n}} e^{\xi^T\Lambda\xi + \xi^T\beta + \gamma}\Big|_{\xi=0} = \left[\sum_{\{\{r[1],s[1]\},\ldots,\{r[m],s[m]\}\}} (2\Lambda_{i_{r[1]}i_{s[1]}})\ldots(2\Lambda_{i_{r[m]}i_{s[m]}}) \prod_{l\neq r[j],s[j]} \beta_{i_l}\right] e^{\gamma},$$

to obtain *U(α), W(p,α), V(p,α)*.

We arrive at

$$U(\alpha) = S^T_{\mathrm{Loop}}\mathrm{Diag}[\alpha]S_{\mathrm{Loop}}, \quad \frac{V(p,\alpha)}{U(\alpha)} = p^T S^T_{\mathrm{Flow}}[\mathrm{Diag}[\alpha] - \mathrm{Diag}[\alpha]S_{\mathrm{Loop}}(S^T_{\mathrm{Loop}}\mathrm{Diag}[\alpha]S_{\mathrm{Loop}})^{-1}S^T_{\mathrm{Loop}}\mathrm{Diag}[\alpha]]S_{\mathrm{Flow}}p$$

W(p,α) is obtained as a sum (with coefficients) over all sets of nonintersecting pairs of the numerator and vertex multipliers. The multiplier corresponds to *(l,μ)*, where *l* is a line number, *μ* is a coordinate index. The pair *[(l,μ),(j,ν)]* gives $(B_{lj}g_{\mu\nu})/U(\alpha)$, the unpaired multiplier *(l,μ)* gives $((Y_l)_\mu)/U(\alpha)$, where

$$B(\alpha)/U(\alpha) = S_{\mathrm{Loop}}(S^T_{\mathrm{Loop}}\mathrm{Diag}[\alpha]S_{\mathrm{Loop}})^{-1}S^T_{\mathrm{Loop}},$$

$$Y(p,\alpha)/U(\alpha) = [1 - (B(\alpha)/U(\alpha))\mathrm{Diag}[\alpha]]\, S_{\mathrm{Flow}}p.$$

The proof of the BPHZ theorem: schwinger-parametric integrals, combinatorial formulas

$U(\alpha)=\det(S^T_{\text{Loop}}\text{Diag}[\alpha]S_{\text{Loop}})$, $\quad \dfrac{V(p,\alpha)}{U(\alpha)}=p^T S^T_{\text{Flow}}[\text{Diag}[\alpha]-\text{Diag}[\alpha]S_{\text{Loop}}(S^T_{\text{Loop}}\text{Diag}[\alpha]S_{\text{Loop}})^{-1}S^T_{\text{Loop}}\text{Diag}[\alpha]]S_{\text{Flow}}p$

$$B(\alpha)/U(\alpha)=S_{\text{Loop}}(S^T_{\text{Loop}}\text{Diag}[\alpha]S_{\text{Loop}})^{-1}S^T_{\text{Loop}},$$
$$Y(p,\alpha)/U(\alpha)=[1-(B(\alpha)/U(\alpha))\text{Diag}[\alpha]]\,S_{\text{Flow}}p.$$

U, *V*, *B*, *Y* are polynomials in *α*, *p*. There exist basis-independent combinatorial formulas for them that allow us to easily analyze properties and to count the powers.

Ideas of obtaining these formulas:

- An independence on the loop basis.
- Two formulas from linear algebra:

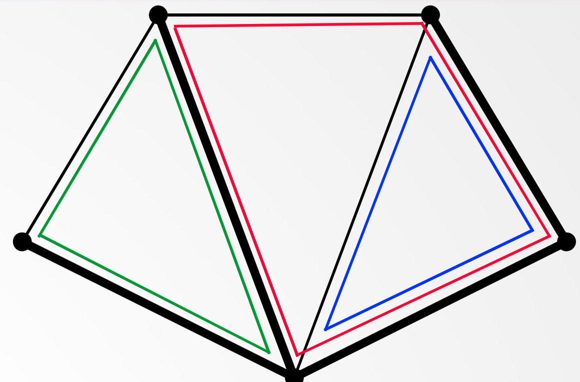
$$\det(AB)=\sum_{i_1<i_2<\ldots<i_n}\det(a_{i_1}\ldots a_{i_n})\det(b_{i_1}\ldots b_{i_n}),\quad A=[a_1\ldots a_m],\ B=[b_1,\ldots,b_m]^T;$$

$$A^{-1}=\frac{C^T}{\det A},\quad C=\begin{bmatrix}C_{11}&\ldots&C_{1n}\\ \ldots&\ldots&\ldots\\ C_{n1}&\ldots&C_{nn}\end{bmatrix},\quad C_{ij}=(-1)^{i+j}\det(A\text{ without }i\text{-th row and }j\text{-th column}).$$

- Graph connectivity and linear independence.
- An accurate consideration of the *graph flows*.

Loop basis independence:

if the bases S_{Loop} and S'_{Loop} are constructed from the 1-trees (spanning trees) R and R', then there exists a matrix Q such that $S'_{Loop}=S_{Loop}Q$, $\det Q=\pm 1$.
(note: *the basis elements are the columns*)



An 1-tree (**bold lines**) and a corresponding loop basis (coloured);
Each loop is one line outside the tree and the path in the tree connecting its ends.

Proof.

Construct the sequence of 1-trees $R_0,\ldots,R_n$, where $R_0=R$, $R_n=R'$, each tree is obtained from the previous one by adding one line and removing one.
It is always possible: add a line that is in R', but not in the current 1-tree, and remove one on the *emerged loop* that is not in R'; repeat the operation until it equals R'.

Let S_j be the loop basis that corresponds to R_j. By $[S_j]_l$ we denote the basis element (column) that corresponds to the line l (that is not in R_j).
Suppose R_{j+1} is obtained from R_j by adding b and removing a.
- $[S_{j+1}]_l = \pm[S_j]_l$, if R_j has a path that connects the ends of l and not contains a.
- $[S_{j+1}]_l = \pm[S_j]_l\pm[S_j]_b$, if $l\neq a$, and the path in R_j , that connects the ends of l, contains a (we go around).
- $[S_{j+1}]_a = \pm[S_j]_b$.

The corresponding matrix is triangle with 0, 1, -1 elements. Therefore $det=\pm 1$.

 Sergey Volkov sergey.volkov@partner.kit.edu

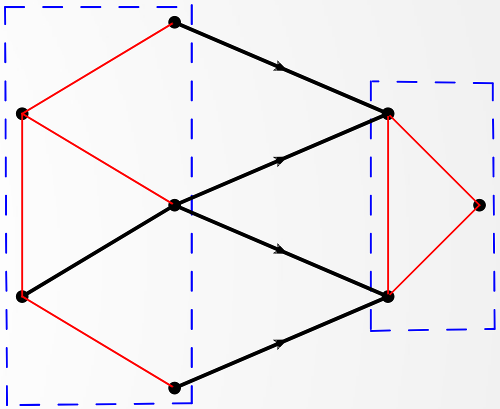
The proof of the BPHZ theorem: schwinger-parametric integrals, combinatorial formulas

$$U(\alpha) = \det(S^T_{\text{Loop}} \text{Diag}[\alpha] S_{\text{Loop}})$$

It equals $\sum_{1\le i_1 < i_2 < \ldots, i_L \le M} \alpha_{i_1} \ldots \alpha_{i_L} (\det[s_{i_1}, \ldots, s_{i_L}])^2$, where $S_{\text{Loop}} = [s_1, \ldots, s_M]^T$

Let us take one term. There are two cases:

1. The remaining *M-L* lines (that are not in $i_1,\ldots,i_L$) form an 1-tree (spanning tree) *R*.
   Take a basis S'_{Loop} based on *R*. Due to the basis independence, we have
   $S_{\text{Loop}} = S'_{\text{Loop}} Q \Longrightarrow s_j = Q^T s'_j$, where $S'_{\text{Loop}} = [s'_1, \ldots, s'_M]^T$,
   $\det[s_{i_1}, \ldots, s_{i_L}] = \det[Q^T s'_{i_1}, \ldots, Q^T s'_{i_L}] = \pm \det Q = \pm 1$
   (since S'_{Loop} is diagonal with ± 1 elements on the lines $i_1,\ldots,i_L$).
2. The remaining *M-L* lines do not connect all vertices.
   In this case, $\{i_1,\ldots,i_L\}$ contains a ***cut*** that separates the vertices into 2 parts.
   The linear combination of $s_{i_1}, \ldots, s_{i_L}$ that corresponds to this cut (*directed from one part to another*) gives 0 (because each loop passes the cut in one direction the same number of times than in the opposite direction). Thus, $\det[s_{i_1}, \ldots, s_{i_L}] = 0$.



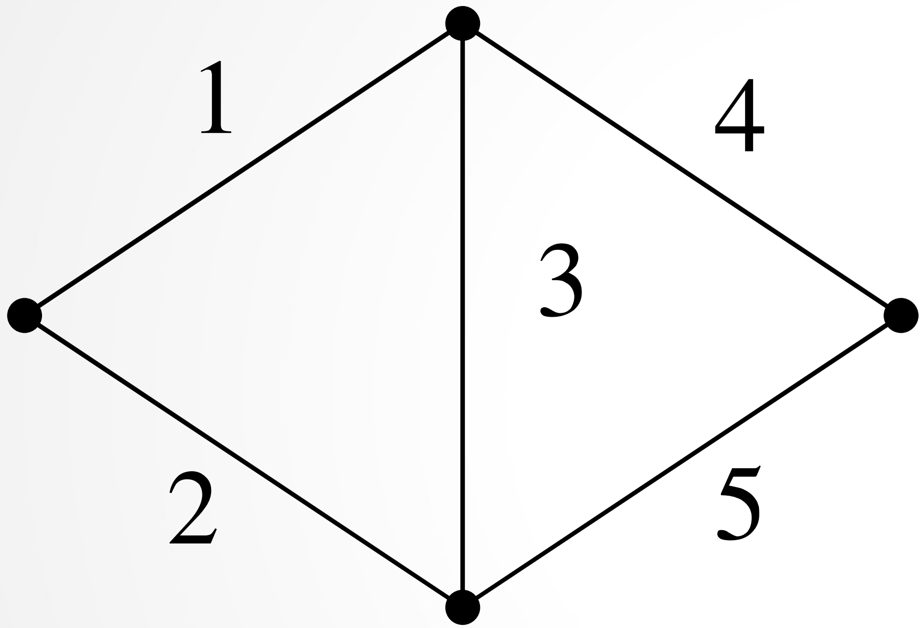
If the remaining *M-L* lines (red) do not span all vertices, the set $i_1,\ldots,i_L$ (**black bold**) has a **cut** that separates into 2 parts

---

The first Symanzik polynomial: $U(\alpha) = \sum_{R \text{ is 1-tree}} \prod_{j \notin R} \alpha_j$

the term signs are constant!

The first Symanzik polynomial: an example



$$U(\alpha) = \sum_{R \text{ is 1-tree}} \prod_{j \notin R} \alpha_j$$

The 1-trees:
{3,1,4}, {3,1,5}, {3,2,4}, {3,2,5},
{1,4,5}, {4,5,2}, {5,2,1}, {2,1,4}

$$U(\alpha) = \alpha_2\alpha_5 + \alpha_2\alpha_4 + \alpha_1\alpha_5 + \alpha_1\alpha_4$$
$$+\alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_3\alpha_4 + \alpha_3\alpha_5$$

The proof of the BPHZ theorem: schwinger-parametric integrals, combinatorial formulas

$\frac{B(\alpha)}{U(\alpha)} = S_{\text{Loop}}(S_{\text{Loop}}^T \text{Diag}[\alpha] S_{\text{Loop}})^{-1} S_{\text{Loop}}^T,$

$B(\alpha) = S_{\text{Loop}} C^T (S_{\text{Loop}}^T \text{Diag}[\alpha] S_{\text{Loop}}) S_{\text{Loop}}^T,$

where $C(X) = \begin{bmatrix} C_{11} & \dots & C_{1L} \\ \dots & \dots & \dots \\ C_{L1} & \dots & C_{LL} \end{bmatrix}, \quad C_{ij} = (-1)^{i+j} \det(X \text{ without } i\text{-th row and } j\text{-th column}).$

The formula is:

$$B(\alpha) = \sum_{R \text{ is a tree with cycle}} B_R \prod_{l \notin R} \alpha_l,$$

where $(B_R)_{ab}=1$, if a and b go in the same direction in the loop of R; -1 if in the opposite direction, 0 in the other cases. A *tree with cycle* is a set that is obtained from an 1-tree by adding a line.

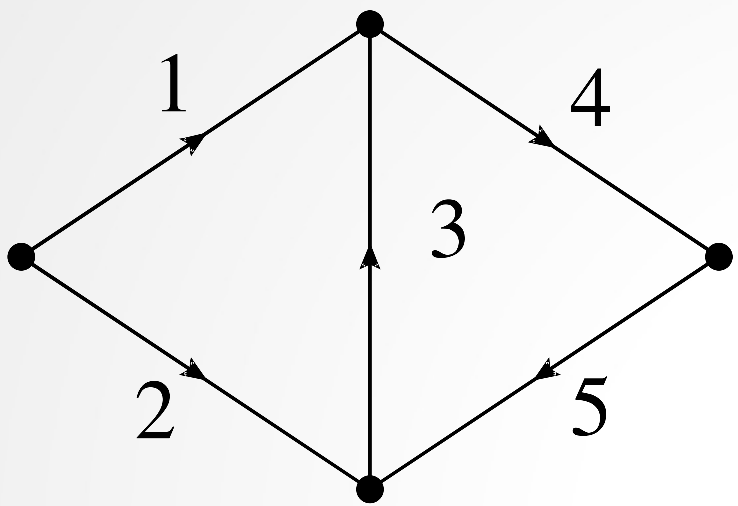
The idea of the proof is the same as for $U(\alpha)$, but with cofactor matrices:

$$B(\alpha) = \sum_{1 \leq i_1 < \ldots < i_{L-1} \leq M} S_{\text{Loop}} C^T (\alpha_{i_1} s_{i_1} s_{i_1}^T + \ldots + \alpha_{i_{L-1}} s_{i_{L-1}} s_{i_{L-1}}^T) S_{\text{Loop}}^T,$$

where $S_{\text{Loop}}=[s_1,\ldots,s_M]^T$; for each term we have two cases:

- The remaining lines form a tree with cycle R. Since $C^T(Q^TAQ)=(Q^{-1}C^T(A)Q)(\det Q)^2$, each term does not depend on the loop basis. Take the basis based on a tree that is obtained from R by removing one line. The argument of C^T becomes diagonal with one zero, everything becomes simple; thus, C^T is a matrix with only one nonzero element.
- The remaining lines do not connect all vertices. Then we have a **cut**, and the term is zero (the rank of the C^T argument $\leq L-2$).

The $B(\alpha)$ matrix: an example



$$B(\alpha) = \sum_{R \text{ is a tree with cycle}} B_R \prod_{l \notin R} \alpha_l,$$

where $(B_R)_{ab}=1$, if *a* and *b* go in the same direction in the loop of *R*; *-1* if in the opposite direction, *0* in the other cases. A *tree with cycle* is a set that is obtained from an 1-tree by adding a line.

The trees with cycle are:
{1,2,3,4}, {1,2,3,5}, {1,2,4,5}, {1,3,4,5}, {2,3,4,5}

$$B(\alpha) = (\alpha_5+\alpha_4)\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 & 1 \end{bmatrix} + (\alpha_2+\alpha_1)\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The proof of the BPHZ theorem: schwinger-parametric integrals, combinatorial formulas

$$\frac{Y(p,\alpha)}{U(\alpha)} = \left[1 - \frac{B(\alpha)}{U(\alpha)}\mathrm{Diag}[\alpha]\right] S_{\mathrm{Flow}} p,$$

or

$$Y(p,\alpha) = [U(\alpha) - B(\alpha)\mathrm{Diag}[\alpha]]\, S_{\mathrm{Flow}} p$$

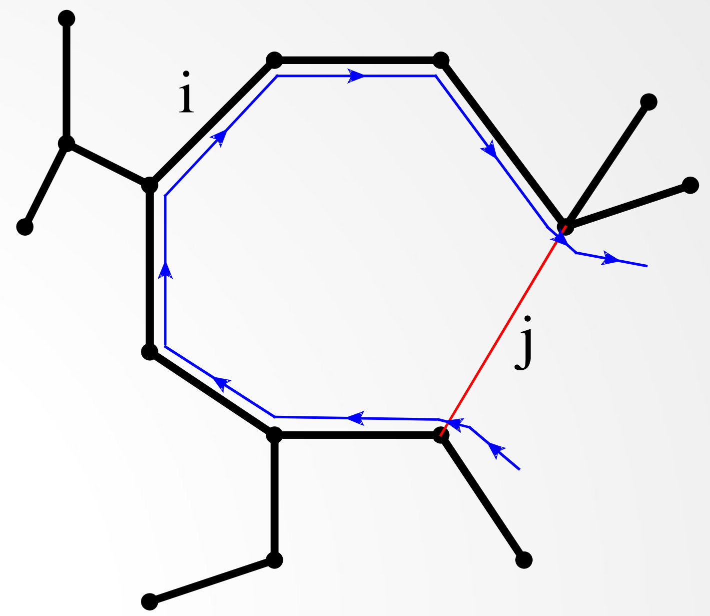
If $i \neq j$,

$$(B(\alpha)\mathrm{Diag}[\alpha])_{ij} = \sum_{R \text{ is 1-tree}} \mathrm{Flow}_{R,i}[\text{end of } j \to \text{begin of } j] \prod_{l \notin R} \alpha_l,$$

where $\mathrm{Flow}_{R,i}[a \to b]$ means the flow through i in R that comes from a and goes to b with the intensity 1.

For $i=j$ it works a little bit differently, but it is compensated by the $U(\alpha)$ term:

$$U(\alpha)\delta_{ij} - (B(\alpha)\mathrm{Diag}[\alpha])_{ij} = \sum_{R \text{ is 1-tree}} \mathrm{Flow}_{R,i}[\text{begin of } j \to \text{end of } j] \prod_{l \notin R} \alpha_l,$$



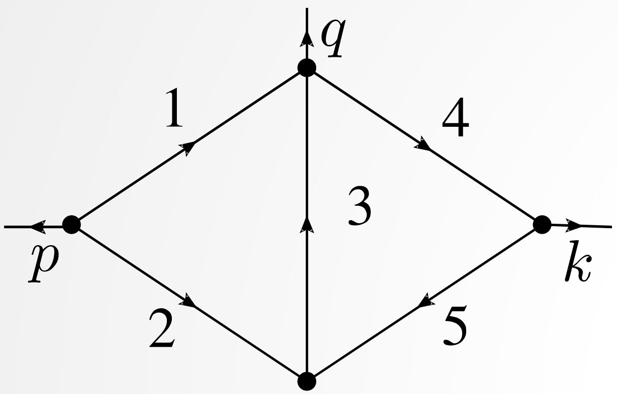
If a tree with cycle has i and j on a loop, then in the tree obtained by removing j, the flow going from the end of j to its begin passes through i.

An accurate analysis of the flow incomes and outcomes leads to

$$Y_i(p,\alpha) = \sum_{R \text{ is 1-tree}} (\text{the flow of } p \text{ through } i \text{ in } R) \prod_{l \notin R} \alpha_l$$

The Y(p,α) vector: an example

$$Y_i(p,\alpha) = \sum_{R \text{ is 1-tree}} (\text{the flow of } p \text{ through } i \text{ in } R) \prod_{l \notin R} \alpha_l$$



The 1-trees:
{3,1,4}, {3,1,5}, {3,2,4}, {3,2,5},
{1,4,5}, {4,5,2}, {5,2,1}, {2,1,4}

$$Y(p,q,k,\alpha) = \alpha_2\alpha_5 \begin{bmatrix} -p \\ 0 \\ p+q+k \\ k \\ 0 \end{bmatrix} + \alpha_2\alpha_4 \begin{bmatrix} -p \\ 0 \\ p+q \\ 0 \\ -k \end{bmatrix} + \alpha_1\alpha_5 \begin{bmatrix} 0 \\ -p \\ q+k \\ k \\ 0 \end{bmatrix} + \alpha_1\alpha_4 \begin{bmatrix} 0 \\ -p \\ q \\ 0 \\ -k \end{bmatrix}$$

$$+\alpha_2\alpha_3 \begin{bmatrix} -p \\ 0 \\ 0 \\ -p-q \\ -p-q-k \end{bmatrix} + \alpha_1\alpha_3 \begin{bmatrix} 0 \\ -p \\ 0 \\ -q \\ -k-q \end{bmatrix} + \alpha_3\alpha_4 \begin{bmatrix} q \\ -p-q \\ 0 \\ 0 \\ -k \end{bmatrix} + \alpha_3\alpha_5 \begin{bmatrix} q+k \\ -p-q-k \\ 0 \\ k \\ 0 \end{bmatrix}$$

The proof of the BPHZ theorem: schwinger-parametric integrals, combinatorial formulas

$$\frac{V(p,\alpha)}{U(\alpha)} = p^T S_{\mathrm{Flow}}^T [\mathrm{Diag}[\alpha] - \mathrm{Diag}[\alpha] S_{\mathrm{Loop}} (S_{\mathrm{Loop}}^T \mathrm{Diag}[\alpha] S_{\mathrm{Loop}})^{-1} S_{\mathrm{Loop}}^T \mathrm{Diag}[\alpha]] S_{\mathrm{Flow}} p,$$

or

$$V(p,\alpha) = p^T S_{\mathrm{Flow}}^T \mathrm{Diag}[\alpha] Y(p,\alpha).$$

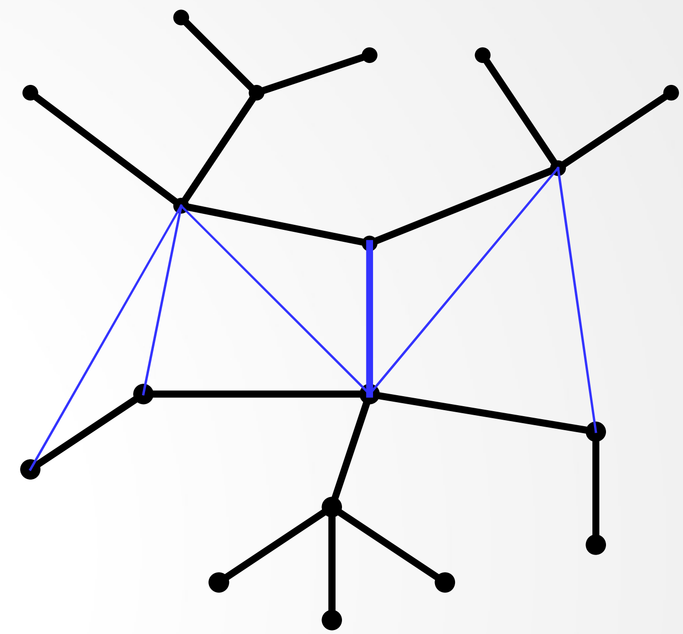
We have

$$\mathrm{Diag}[\alpha] Y(p,\alpha) = \sum_{R \text{ is a 2-tree}} (\text{the flow of } p \text{ between the components of } R) \mathrm{Cut}[R] \prod_{l \notin R} \alpha_l,$$

where Cut*[R]* is a vector of the *cut* that correponds to the connectivity components of *R* (oriented in the same way as in the calculation of the flow).

A **2-tree** is an acyclic subgraph that connects the whole set of vertexes into 2 components.

Calculating $p^T S_{\mathrm{Flow}}^T \mathrm{Cut}[R]$ accurately, we arrive at



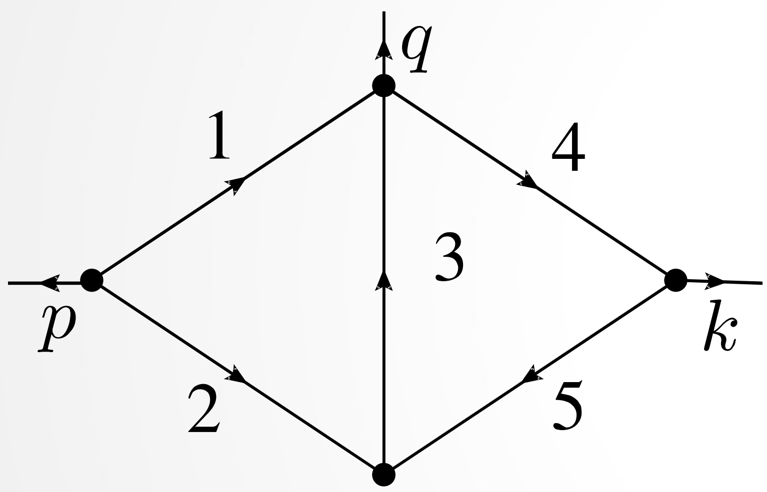
A 1-tree (**bold lines**) without a line (**blue bold**) is a 2-tree (**black bold**). All possible ways to insert a line back (blue) form a *cut*.

$$V(p,\alpha) = \sum_{R \text{ is a 2-tree}} (\text{the flow of } p \text{ between the components of } R)^2 \prod_{l \notin R} \alpha_l$$

It is called **the second Symanzik polynomial**

The second Symanzik polynomial V(p,α): an example

$$V(p,\alpha) = \sum_{R \text{ is a 2-tree}} (\text{the flow of } p \text{ between the components of } R)^2 \prod_{l \notin R} \alpha_l$$



The 2-trees are:

{2,5}	(the cut flow = q),
{1,2}, {1,3}, {2,3}	(the cut flow = k),
{1,4}	(the cut flow = p+q+k),
{3,4}, {3,5}, {4,5}	(the cut flow = p),
{2,4}	(the cut flow = k+q),
{1,5}	(the cut flow = p+q).

$$V(p,q,k,\alpha) = \alpha_1\alpha_3\alpha_4 q^2 + \alpha_4\alpha_5(\alpha_1+\alpha_2+\alpha_3)k^2 + \alpha_2\alpha_3\alpha_5(p+q+k)^2$$
$$+\alpha_1\alpha_2(\alpha_3+\alpha_4+\alpha_5)p^2 + \alpha_1\alpha_3\alpha_5(k+q)^2 + \alpha_2\alpha_3\alpha_4(p+q)^2$$

Combinatorial formulas: a summary

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}}) = C\frac{W(p,\alpha)}{U(\alpha)^2}e^{i\frac{V(p,\alpha)}{U(\alpha)}-i\sum_j m_j\alpha_j^2-\varepsilon_{\mathrm{IR}}\sum_j\alpha_j},$$

W(p,α) is obtained as a sum (with coefficients) over all sets of nonintersecting pairs of the numerator or vertex multipliers. The multiplier corresponds to *(l,μ)*, where *l* is a line number, *μ* is a coordinate index. The pair *[(l,μ),(j,ν)]* gives $(B_{lj}g_{\mu\nu})/U(\alpha)$, the unpaired multiplier *(l,μ)* gives $((Y_l)_\mu)/U(\alpha)$.

$$U(\alpha) = \sum_{R\text{ is 1-tree}}\prod_{j\notin R}\alpha_j$$ (the first Symanzik polynomial)

$$B(\alpha) = \sum_{R\text{ is a tree with cycle}} B_R\prod_{l\notin R}\alpha_l,$$

where $(B_R)_{ab}=1$, if *a* and *b* go in the same direction in the loop of *R*; *-1* if in the opposite direction, *0* in the other cases.

$$Y_i(p,\alpha) = \sum_{R\text{ is 1-tree}}(\text{the flow of } p \text{ through } i \text{ in } R)\prod_{l\notin R}\alpha_l$$

$$V(p,\alpha) = \sum_{R\text{ is a 2-tree}}(\text{the flow of } p \text{ between the components of } R)^2\prod_{l\notin R}\alpha_l$$

(the second Symanzik polynomial)

Combinatorial formulas: the literature

[V. A. Smirnov, *Renormalization and Asymptotic Expansions*, PPH'14, Progress in Mathematical Physics, Birkhäuser, 2000]

[O. I. Zavyalov, *Renormalized Quantum Field Theory*, Mathematics and Its Applications. Soviet Series, vol. 21, Kluwer, Dordrecht, Netherlands, 1990, 524 pp.]

[P. Cvitanovic, T. Kinoshita, *Feynman-Dyson rules in parametric space*, Phys. Rev. D 10 (1974) 3978]

There is also an **electric circuit analogy**:

$Y_j(p,\alpha)/U(\alpha)$ = the current passing through j in the circuit with the element resistances α and the external currents p;

$V(p,\alpha)/U(\alpha)$ = the dissipated power of this circuit.

[J. D. Bjorken, S. D. Drell, *Relativistic Quantum Fields*, McGraw-Hill College, New York, 1965, Chapter 18 "Dispersion Relations", Section 18.4 "Generalization to Arbitrary Graphs and the Electrical Circuit Analogy"]

Applications of the **electric circuit analogy** exist:

[S. Volkov, J. Exp. Theor. Phys. 122 (6) (2016) 1008–1031]

[S. Volkov, Nuclear Physics B 961, 115232 (2020)]

Outline

- Introduction
- General ideas of handling UV divergences
- Application to the renormalization of quantum electrodynamics
- Formulations in terms of finite integrals
- **The proof of the BPHZ theorem**
  - the formulation, ideas
  - Schwinger-parametric integrals, combinatorial formulas
  - power counting, Hepp sectors, "Hanoi" towers
  - reduction to the forest formula with sets of lines
  - elimination of overlaps
  - the case when all the subtractions fit
- Conclusions

The proof of the BPHZ theorem: power counting, Hepp sectors, ``Hanoi towers"

Definitions.

A ***Hepp sector*** is an order in the Schwinger parameters.
There are *M!* Hepp sectors (the number of all permutations of *M* elements).
If we work inside one Hepp sector, we suppose for convenience $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_M$.

It is convenient for the power counting to consider one Hepp's sector as "**Hanoi towers**''.

Each initial segment of lines $\{1,2,\ldots,j\}$ $(1 \leq j \leq M)$ gives "tower'' disks of thickness α_j/α_{j+1} (or α_M for $j=M$).
A ***disk*** is an 1-particle irreducible component (as a nonempty *set of lines*) of the initial segment (or a bridge).

More precisely, two lines a,b of $\{1,2,\ldots,j\}$ are called equivalent, if $a=b$ or there is a cycle in $\{1,2,\ldots,j\}$ passing through a and b and not passing lines twice; A disk is a class of equivalence.

There may be several disks with the same sets of lines (they correspond to different segments).

By $h(D)$ we denote the thickness of the disk D.
$h(D) = \alpha_j/\alpha_{j+1} \leq 1$. The thicknesses are multiplied!

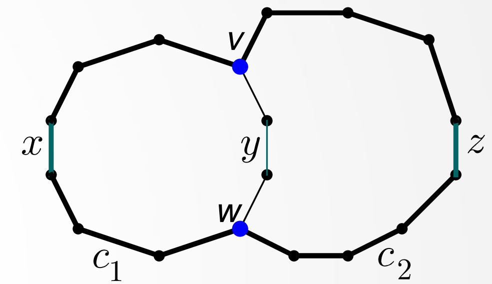
The proof of the BPHZ theorem: disks as classes of equivalence

We suppose $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_M$

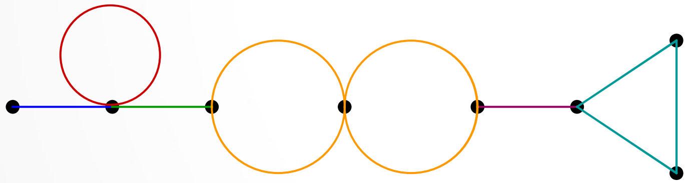
The lines a,b of $\{1,2,\ldots,j\}$ are called equivalent, if $a=b$ or there is a cycle in $\{1,2,\ldots,j\}$ passing through a and b and not passing lines twice; A disk is a class of equivalence.

The defined relation $x \sim y$ is really an equivalence relation, because it is transitive: if $x \sim y$ and $y \sim z$, then $x \sim z$.

Proof of the transitivity. Take a cycle c_1 that passes through x and y. If z is not on this cycle, take a cycle c_2 that passes y and z. Take the part of c_2 from v to w, where v and w are the closest (to z) intersections with c_1. Continue this path with the part of c_1 from w to v that contains x.



An example of equivalence classes:



The orange set is one class, although for the left and right lines only a self-intersecting cycle exists.

Properties of disks.

1. If a disk D is not 1-particle irreducible, then D is one line with different ends.
2. If the disks D_1 and D_2 intersect, then one of them is contained in the other one.

   The idea of the proof: consider the largest initial segment and its classes of equivalence.
3. If the 1-particle irreducible disks D_1 and D_2 have a common vertex, then one of them contains the other one.

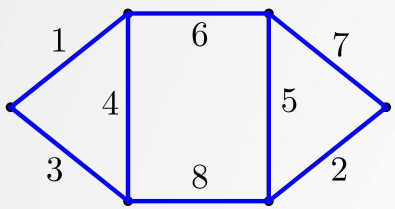
   The idea of the proof: consider the largest initial segment and cycles passing the common vertex.

 sergey.volkov@partner.kit.edu

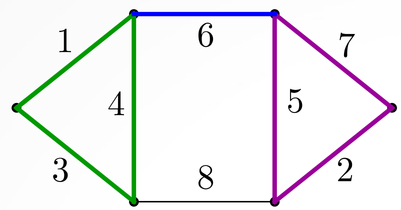
The proof of the BPHZ theorem: power counting, Hepp sectors, ``Hanoi towers”

“Hanoi towers” and disks: an example

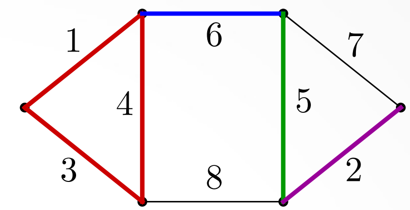
We suppose $\alpha_1 \le \alpha_2 \le \ldots \le \alpha_8$.



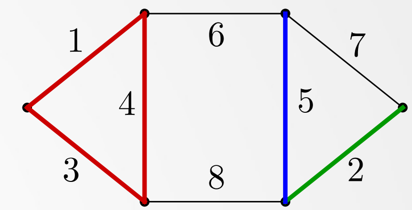
Initial segment:{1,2,3,4,5,6,7,8}
Disks:
{1,2,3,4,5,6,7,8}
Thickness: $h=\alpha_8$



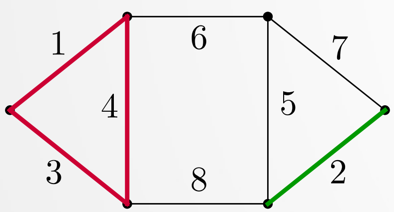
Initial segment:{1,2,3,4,5,6,7}
Disks:
{1,3,4}, {6}, {2,5,7}
Thickness: $h=\alpha_7 / \alpha_8$



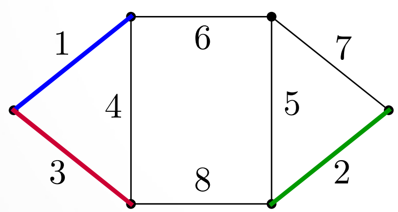
Initial segment:{1,2,3,4,5,6}
Disks:
{1,3,4}, {6}, {5}, {2}
Thickness: $h=\alpha_6 / \alpha_7$



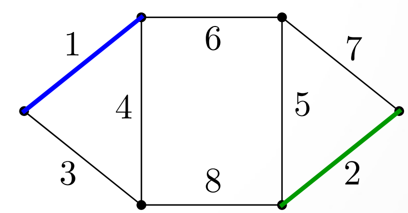
Initial segment:{1,2,3,4,5}
Disks:
{1,3,4}, {5}, {2}
Thickness: $h=\alpha_5 / \alpha_6$



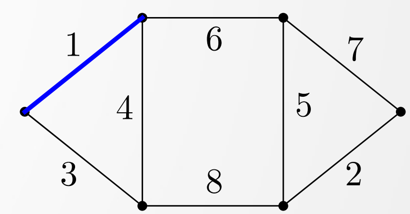
Initial segment:{1,2,3,4}
Disks:
{1,3,4}, {2}
Thickness: $h=\alpha_4 / \alpha_5$



Initial segment:{1,2,3}
Disks:
{1}, {3}, {2}
Thickness: $h=\alpha_3 / \alpha_4$



Initial segment:{1,2}
Disks:
{1}, {2}
Thickness: $h=\alpha_2 / \alpha_3$



Initial segment:{1}
Disks:
{1}
Thickness: $h=\alpha_1 / \alpha_2$

The proof of the BPHZ theorem: power counting, Hepp sectors, ``Hanoi towers”

Basics of power counting

$$U(\alpha)=\sum_{R\text{ is 1-tree}}\prod_{j\notin R}\alpha_j$$ is in the denominator! One must be accurate!

If T is a 1-tree, s is a set of diagram lines, put

$$\mathrm{Defect}_s(T)=\max_{T'\text{ is 1-tree}}|s\cap T'|-|s\cap T|.$$

We call it the **defect** of the 1-tree T in s.
If s is connected,

$$\mathrm{Defect}_s(T)=|\mathrm{Vertex}(s)|-1-|s\cap T|.$$

The main statement needed for the power counting.
if $D_1,\ldots,D_n$ are all disks of the Hepp sector, there exists a 1-tree T such that

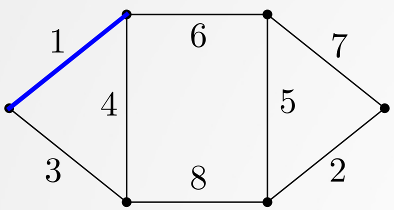
$$\mathrm{Defect}_{D_i}(T)=0\text{ for all }D_i.$$

Proof. Let us describe the algorithm of constructing T line by line. Start from the empty set.
Take firsts all disks of the initial segment $\{1\}$, then of $\{1,2\}$ and so on.
For each disk D extend T to a spanning tree inside D.
Cycles do not emerge, because there is no cycle inside one initial segment $\{1,\ldots,j\}$ passing several disks of this segment.

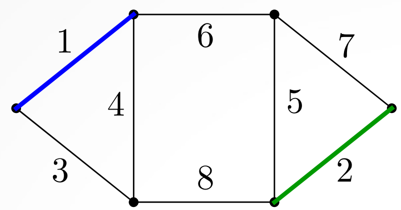
The proof of the BPHZ theorem: power counting, Hepp sectors, ``Hanoi towers”

The optimal tree for all disks simultaneously: an example

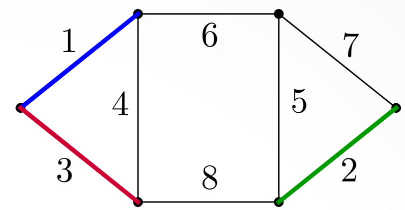
We suppose $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_8$.



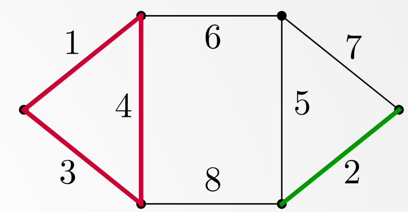
Initial segment:{1}
Disks: {1}
Tree (state): {1}



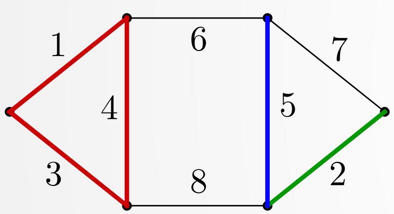
Initial segment:{1,2}
Disks: {1}, {2}
Tree (state): {1,2}



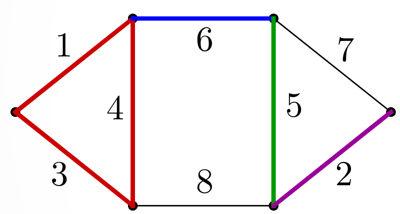
Initial segment:{1,2,3}
Disks: {1}, {3}, {2}
Tree (state): {1,2,3}



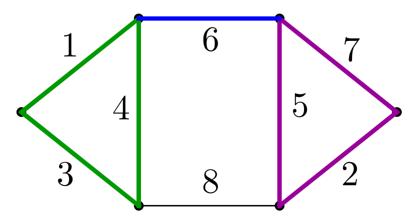
Initial segment:{1,2,3,4}
Disks: {1,3,4}, {2}
Tree (state): {1,2,3}



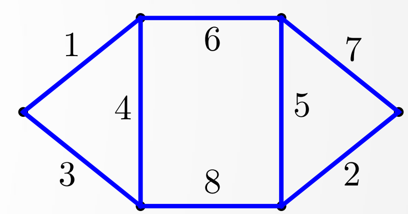
Initial segment:{1,2,3,4,5}
Disks: {1,3,4}, {5}, {2}
Tree (state): {1,2,3,5}



Initial segment:{1,2,3,4,5,6}
Disks: {1,3,4}, {6}, {5}, {2}
Tree (state): {1,2,3,5,6}

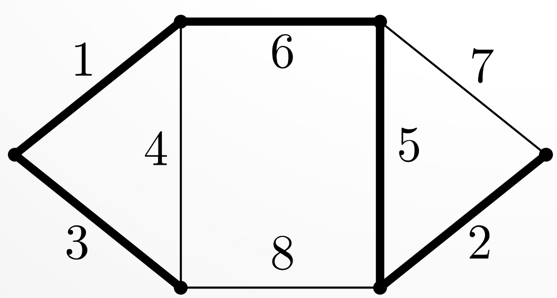


Initial segment:{1,2,3,4,5,6,7}
Disks: {1,3,4}, {6}, {2,5,7}
Tree (state): {1,2,3,5,6}



Initial segment:{1,2,3,4,5,6,7,8}
Disks: {1,2,3,4,5,6,7,8}
Tree (state): {1,2,3,5,6}

We arrive at
T = {1,2,3,5,6}



The proof of the BPHZ theorem: power counting, Hepp sectors, ``Hanoi towers”

Power counting in Schwinger parametric space

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}}) = C\frac{W(p,\alpha)}{U(\alpha)^2} e^{i\frac{V(p,\alpha)}{U(\alpha)} - i\sum_j m_j\alpha_j^2 - \varepsilon_{\mathrm{IR}}\sum_j \alpha_j},$$

W(p,α) is obtained as a sum (with coefficients) over all sets of nonintersecting pairs of the numerator and vertex multipliers. Each multiplier corresponds to a line. The pair of lines *(l,j)* gives $B_{lj}(\alpha)/U(\alpha)$, the unpaired line *l* gives $Y_l(\alpha)/U(\alpha)$.

Since a zero-defect 1-tree for all disks exists, we have for $\max(\alpha_1,\ldots,\alpha_M)\leq 1$

$$U(\alpha) = \sum_{R\text{ is 1-tree}} \prod_{j\notin R} \alpha_j \geq C \prod_{D\text{ is a disk}} h(D)^{|D|-|\mathrm{Vertex}(D)|+1}.$$

Also,

$$|Y(p,\alpha)/U(\alpha)| = \left| \sum_{R\text{ is 1-tree}} (\text{the flow of } p \text{ through } i \text{ in } R) \prod_{l\notin R} \alpha_l \right| \leq C(p),\ \text{ where } C(p) \text{ is a polynomial},$$

$$|B_{ij}(\alpha)/U(\alpha)| = \left| \sum_{R\text{ is a tree with cycle}} (B_R)_{ij} \prod_{l\notin R} \alpha_l \right| \leq C \prod_{D\text{ is a disk}} h(D)^{-I(i,j\in D\ \&\ D\text{ is 1PI})},$$

where B_R is a *{0,1,-1}*-matrix, *I* is an indicator (=1 if the statement is true, 0 otherwise).

Since $\omega(D) \geq 4 + 2|D| - 4|\mathrm{Vertex}(D)| + \sum_{l\in D} \deg P_l + \sum_{v\in\mathrm{Vertex}(D)} \deg P_{v[\text{in } D]},$

where $P_{v[\text{in } D]}$ means the part of P_v corresponding to the lines inside *D*. we arrive at

$$|F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})| \leq \frac{C(p,\sum_j \alpha_j)}{\alpha_1\ldots\alpha_M} e^{-\varepsilon_{\mathrm{IR}}\sum_j\alpha_j} \prod_{D\text{ is a disk}} h(D)^{\lceil -\omega(D)/2\rceil},$$

where *C* is a polynomial ($\Sigma_j\alpha_j$ is needed if max*(α)>1*).

Outline

- Introduction
- General ideas of handling UV divergences
- Application to the renormalization of quantum electrodynamics
- Formulations in terms of finite integrals
- **The proof of the BPHZ theorem**
  - the formulation, ideas
  - Schwinger-parametric integrals, combinatorial formulas
  - power counting, Hepp sectors, "Hanoi" towers
  - reduction to the forest formula with sets of lines
  - elimination of overlaps
  - the case when all the subtractions fit
- Conclusions

The proof of the BPHZ theorem: a forest formula with sets of lines

A necessity to work with arbitrary sets of lines as subdiagrams

The forest formula we deal with works with subdiagrams as sets of vertexes.
Since a ***disk*** is a set of lines, it is convenient to have a forest formula working with sets of lines as subdiagrams.

Usually we define

$$F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})$$

using the forest formula

$$\sum_{\{G_1,\ldots,G_n\}\in F} (-1)^n M_{G_1} M_{G_2}\ldots M_{G_n}$$

applied to the diagram with Schwinger-like exponential regularized propagators; here F is the set of all forests of UV-divergent 1-particle irreducible subdiagrams of the diagram; a subdiagram includes all lines connecting its vertices (having both ends in its sets of vertices).
After that we take the limit

$$F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}}) = \lim_{\varepsilon_{\mathrm{Mink}}\to+0} F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})$$

and calculate the integral

$$I_{\mathrm{Sub}}(p_1,\ldots,p_r,\varepsilon_{\mathrm{IR}}) = \int_0^{+\infty} F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})d\alpha_1\ldots d\alpha_M$$

The proof of the BPHZ theorem: a forest formula with sets of lines

A definition of the forest formula working with sets of lines

We say that the sets of lines s_1 and s_2 are nested, if $s_1 \subseteq s_2$ or $s_2 \subseteq s_1$.

We say that the sets of lines s_1 and s_2 are independent, if $\mathrm{Vertex}(s_1) \cap \mathrm{Vertex}(s_2) = \emptyset$.

Yes, the nestedness is defined with sets of lines, but the independence with sets of vertices!

The sets of lines s_1 and s_2 are said to overlap, if they are not nested and not independent.

A forest of sets of lines is a set of sets of lines, each of them do not overlap.

By F_{lines} we denote the set of all forests of UV-divergent 1-particle irreducible sets of the diagram lines;
the UV degree of divergence is defined in the same way as for usual subdiagrams.

The formula is almost the same:

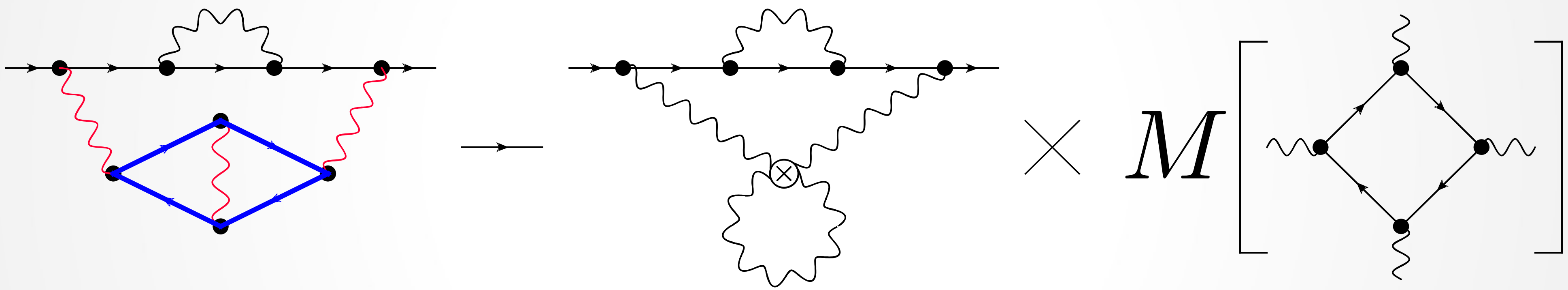
$$\sum_{\{s_1,\ldots,s_n\}\in F_{\text{lines}}} (-1)^n M_{s_1} M_{s_2} \ldots M_{s_n}$$

 Sergey Volkov sergey.volkov@partner.kit.edu

A note about applying subtractions based on sets of lines

A misunderstanding is possible about how to apply the Taylor expansion projectors M to arbitrary 1-particle irreducible sets of lines…

It is demonstrated by this example:



The operator M is applied to the **blue bold** set of lines s.
The red lines are external lines relative to s.
The cross in a circle is a special vertex expressing to the corresponding Taylor expansion monomial.

The proof of the BPHZ theorem: a forest formula with sets of lines

The equivalence of the usual forest formula and the formula with sets of lines

The forest formula with sets of lines is

$$\sum_{\{s_1,\ldots,s_n\}\in F_{\text{lines}}} (-1)^n M_{s_1} M_{s_2} \ldots M_{s_n}$$

A subset s of the diagram lines is said to be **closed**, if *s* contains all lines that have both ends in Vertex*(s)*. By F_{closed} we denote the set of all forests from F_{lines} containing only closed sets. The usual forest formula is

$$\sum_{\{s_1,\ldots,s_n\}\in F_{\text{closed}}} (-1)^n M_{s_1} M_{s_2} \ldots M_{s_n}$$

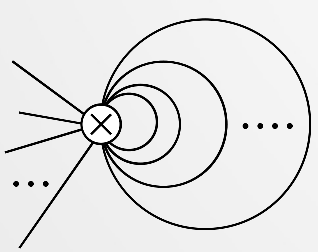
Let $\{s_1,\ldots,s_n\}$ be one element of F_{lines} containing at least one not closed element. Suppose s_1, …,s_k are all **maximal** (with respect to inclusion) not closed elements in it.

By s_1' ,…, s_k' we denote the corresponding closed sets: Vertex(s_j')=Vertex(s_j).

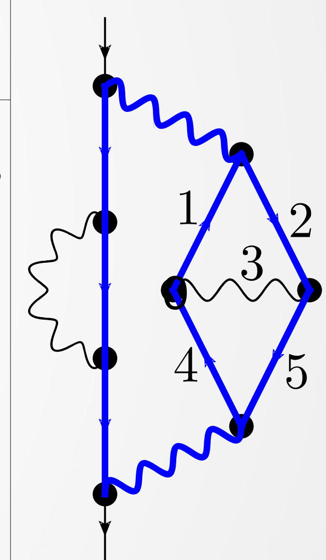
The corresponding forest formula element is a part of an expression, where the multiplier

$$(1-M_{s_1'})M_{s_1}\ldots(1-M_{s_k'})M_{s_k}$$

is factorized. The maximality (with respect to inclusion) is needed to avoid a pathological situation when some set *s* overlaps with s_j' , but not with s_j (that destroys the factorization).



Before applying $(1-M_{s_j'})$ the corresponding diagram looks like in the picture to the left. The special vertex is a monomial of degree ≤*ω(s)*, where $s_j \subseteq s \subset s_j'$. Since $\omega(s)=4+2|s|-4|\text{Vertex}(s)|+\sum_{l\in s}\deg P_l$, we have $\omega(s_{j'})\geq\omega(s)$ and $(1-M_{s_j'})$ nullifies the monomial.



A pathological example: the **blue bold** set overlaps with {1,2,3,4,5}, but does not with {1,2,4,5}

 Sergey Volkov sergey.volkov@partner.kit.edu

The proof of the BPHZ theorem: a forest formula with sets of lines

The equivalence of the usual forest formula and the formula with sets of lines: some details

The usual forest formula gives

$$F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})$$

The forest formula with sets of lines gives:

$$F_{\mathrm{SubL}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})$$

Since all the integrals are well defined, the proved equivalence works for all $\alpha_1,\ldots,\alpha_M>0$, $\varepsilon_{\mathrm{IR}}>0$, $\varepsilon_{\mathrm{Mink}}>0$:

$$F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})=F_{\mathrm{SubL}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{Mink}})$$

Due to the explicit analytical formulas, both the limits $\varepsilon_{\mathrm{Mink}}\to 0$ exist. Therefore,

$$F_{\mathrm{Sub}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})=F_{\mathrm{SubL}}(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}})$$

Outline

- Introduction
- General ideas of handling UV divergences
- Application to the renormalization of quantum electrodynamics
- Formulations in terms of finite integrals
- **The proof of the BPHZ theorem**
  - the formulation, ideas
  - Schwinger-parametric integrals, combinatorial formulas
  - power counting, Hepp sectors, "Hanoi" towers
  - reduction to the forest formula with sets of lines
  - elimination of overlaps
  - the case when all the subtractions fit
- Conclusions

The proof of the BPHZ theorem: the elimination of overlaps

We have a formula:

$$\sum_{\{s_1,\ldots,s_n\}\in F_{\mathrm{lines}}} (-1)^n M_{s_1} M_{s_2} \ldots M_{s_n}$$

here F_{lines} is the set of all forests of UV-divergent 1-particle irreducible sets of the diagram lines;
a forest of sets of lines is a set of sets of lines, for each s_1 and s_2 of it one of the following properties is satisfied: $s_1 \subseteq s_2,\ s_2 \subseteq s_1,\ \mathrm{Vertex}(s_1)\cap\mathrm{Vertex}(s_2)=\emptyset$.

Really, the subtractions are needed only on disks!

If all forests consist of only disks, the formula is factorized like

$$(1-M_{s_1})\ldots(1-M_{s_n})$$

and everything becomes simpler (we will prove it later).

However, it is not possible to remove not needed elements for each Hepp sector separately.
Subtractions on not disks make serious problems!

The proof of the BPHZ theorem: the elimination of overlaps

A general idea of removing not-on-disk subtractions

Split the whole sum

$$\sum_{\{s_1,\ldots,s_n\}\in F_{\text{lines}}} (-1)^n M_{s_1} M_{s_2} \ldots M_{s_n}$$

into parts, each of them satisfies the following properties:

- Some sets $s_1,\ldots,s_r$ present in all terms.
- The sets $s_1,\ldots,s_r$ split the diagram into parts, and the sum can be factorized into the product of forest formulas in these parts.

Separations like this allow us to reduce the problem to smaller diagrams or numbers of terms. The separations may be different in different Hepp sectors.

However, it is very nontrivial to make it working in every case.

For example, if we just fix all not-disk subdiagrams to which M is applied, we obtain a factorization into smaller forest formulas, but the subtractions in the parts would cover not all UV divergent subdiagrams.

The proof of the BPHZ theorem: the elimination of overlaps

The idea № 1: to formulate what we will prove by induction

Suppose the Hepp sector in the diagram is fixed: $\alpha_1 \leq \ldots \leq \alpha_M$.
Suppose the integer numbers $\Delta_v \geq 0$ are defined for each vertex v (to allow us to do consistent oversubtractions: an oversubtraction at a subdiagram increases the UV degree of the larger diagrams).
We say that a set of the diagram lines s is enough divergent, if s is 1-particle irreducible and

$$\omega'(s) = \omega(s) + \sum_{v \in \mathrm{Vertex}(s)} \Delta_v \geq 0.$$

Let S be a set of 1-particle irreducible sets of the diagram lines satisfying the conditions:

- The closure under union with an initial segment:
  If $s_1, \ldots, s_n \in S$ $(n \geq 0)$, l is a number, s is a maximal (with respect to inclusion) 1-particle irreducible subset of $s_1 \cup \ldots \cup s_n \cup \{1, 2, \ldots, l\}$, then $s \in S \cup \{\Lambda\}$, where Λ is the set of all diagram lines.
  From this it automatically follows that all disks except Λ are in S (the case $n=0$).
- The closure under intersection: If $s_1, s_2 \in S$, s is a maximal (with respect to inclusion) 1-particle irreducible subset of $s_1 \cap s_2$, then $s \in S$.
- The whole set Λ is not in S.

By $F_{\mathrm{lines}}[S]$ we denote the set of all forests of enough divergent elements of S.
Applying the forest formula based on F_{lines} and Taylor expansions up to the degree $\omega'(s)$, we obtain

$$F_{\mathrm{SubL}[S,\Delta]}(p_1, \ldots, p_r, \alpha_1, \ldots, \alpha_M, \varepsilon_{\mathrm{IR}})$$

 Sergey Volkov sergey.volkov@partner.kit.edu

The idea № 1: to formulate what we will prove by induction

The whole set of lines Λ is not included to S.

There are two cases what we do with the whole diagram:

- Take the Taylor expansion (at zero momenta) coefficients of degree d.
  In this case, we also require that each s from S does not contain the line M with the largest α.
- Apply the usual subtraction $(1-M_G)$, where M_G is the Taylor expansion at zero momenta up to the degree $\omega'(G)$.

We will prove that

$$|F_{\mathrm{SubL}[S,\Delta]}(p,\alpha,\varepsilon_{\mathrm{IR}})| \leq \mathrm{e}^{-\varepsilon_{\mathrm{IR}}\sum_j \alpha_j} \times \frac{P(p,\sum_j \alpha_j)}{\alpha_1\ldots\alpha_M} \times \prod_{j=1}^{M-1} \left(\frac{\alpha_j}{\alpha_{j+1}}\right)^{(1+\sum_{v\in\mathrm{cycl}(\{1,2,\ldots,j\})}\Delta_v)/2} \times (\alpha_M)^z,$$

where P is some polynomial giving positive values and non-decreasing with respect to $\Sigma\alpha$;
$\mathrm{cycl}(x)$ is the set of all vertices v such that for any $s\in S\cup\{\Lambda\}$ satisfying $v\in\mathrm{Vertex}(s)$ there exist a **cycle** (without self-intersections) in $s\cap x$ that passes v;

$$z=(d-\omega(G))/2,$$

if we take the Taylor expansion coefficients of degree d;

$$z=(1+\sum_v \Delta_v)/2,$$

if we subtract up to the degree $\omega'(G)$.

We will prove this by induction on $|S|$.
The induction base case: all the elements of S are disks (we will prove the basis later).

 sergey.volkov@partner.kit.edu

The proof of the BPHZ theorem: the elimination of overlaps

The idea № 2: a "**minimax**" separation of the forest formula

The Hepp sector: $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_M$.

1. Take the **maximal** line number τ contained in at least one not-a-disk element of S.
2. In each term of the forest formula

$$\sum_{\{s_1,\ldots,s_n\}\in F_{\text{lines}}[S]} (-1)^n M^{\Delta}_{s_1} M^{\Delta}_{s_2} \ldots M^{\Delta}_{s_n}$$

   take the **minimal** (with respect to inclusion) not-a-disk set s that contains the line τ and the operator is applied to s.

The formula is split into parts with different s.
One part corresponds to the case when the minimal element does not exist (because of the empty set).
We analyze all the parts separately.

If the element s is fixed, all the diagram is split into two parts: the external and internal part. Separate forest formulas can be written for these parts. We will prove that the whole sum with a fixed s is the product of the expressions for the external and internal parts.

The proof of the BPHZ theorem: the elimination of overlaps

The proof by induction: **the case when not-a-disk elements containing τ are not used**

τ is the maximal line number contained in not-a-disk elements of S.

This case corresponds to the same diagram and the same numbers Δ_v, but with another S'.

S' is the set of all elements of S that are disks or not containing τ.

We have to prove the following properties of S':

- The closure under union with an initial segment: if $s_1,\ldots,s_n$ are in S', l is a number, $s_0 = s_1 \cup \ldots \cup s_n \cup \{1, 2, \ldots, l\}$, s' is a maximal (with respect to inclusion) 1-particle irreducible subset of s_0, then s' in S'.

  To prove this, we swap the elements and express s_0 as the union $u_1 \cup \ldots \cup u_{n+1}$, where for $1 \leq i \leq r$ the set u_i is the initial segment $\{1,2,\ldots,l_i\}$, $l_i \geq \tau$ or a disk generated by this segment containing τ; each of the sets $u_{r+1},\ldots,u_{n+1}$ is a disk not containing τ or not-a-disk contained in $\{1,2,\ldots,\tau-1\}$ [we use that τ is the **maximal** line number contained in not-disks from S].

  If τ is in s', then s' is exactly the disk generated by $\{1,2,\ldots,\max(l_1,\ldots,l_r)\}$ and containing τ (because the 1-particle irreducible disks not containing τ do not have common vertices with this disk; the part of $\{1,2,\ldots,\tau-1\}$ not contained in this disk is covered by another disks of $\{1,2,\ldots,\max(l_1,\ldots,l_r)\}$, i.e., can't be in the same 1PI component).

  If τ is not in s', the set s' is obviously in S'.

- The closure under intersection: if s_1, s_2 are in S', then each maximal 1-particle irreducible subset of the intersection is in S'.

  Proof.

  If s_1 and s_2 are disks and intersect, one of them is inside the other.

  If one of them does not contain τ, all the obtained sets don't contain τ.

The proof of the BPHZ theorem: the elimination of overlaps

The proof by induction: **the internal part relative to *s***

The Hepp sector: $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_M$.

τ is the maximal line number contained in at least one not-a-disk element of S.
In each term, s is the minimal not-a-disk element containing τ.

Let us consider the **internal** part of the diagram relative to s.
The corresponding part of the forest formula is factorized. It corresponds to the subdiagram s. The numbers Δ_v are the same as for the original diagram. The set S' is defined as

$$S' = \{s_1 \subsetneq s, s_1 \in S : \ \tau \notin s_1 \text{ or } s_1 \text{ is a disk}\}$$

(we use here that s is the **minimal** not-a-disk element containing τ, this is true for each term).

All elements of S' don't contain τ: we need this for induction.
Indeed, if $s' \in S'$ and $\tau \in s'$, then s' is a disk generated by an initial segment $\{1,2,\ldots,l\}$, $l \geq \tau$.
Since $s \subseteq \{1,\ldots,\tau\}$ [we use that τ is the **maximal** line number contained in not-disks from S]
and s is 1PI, s is contained in one of the disks of $\{1,\ldots,l\}$. Therefore, $s \subseteq s'$.

$$S' = \{s_1 \subseteq s, s_1 \in S : \ \tau \notin s_1\}$$

We also have to prove the following properties of S':

- The closure under union with an initial segment. An internal order initial segment is $s \cap \{1,2,\ldots,l\}$.
  Suppose $s_1,\ldots,s_n \in S'$, $s_0 = s_1 \cup \ldots \cup s_n \cup (s \cap \{1,\ldots,l\})$, and s' is a is a maximal 1PI set contained in s_0 .
  We have to prove that $s' \in S' \cup \{s\}$.
  First we prove that s' in S. The set $s_1 \cup \ldots \cup s_n \cup \{1,\ldots,l\}$ has maximal 1PI sets contained in it. Since s' is 1PI, we have $s' \subseteq u$, where u is one of these sets. Thus, s' is a maximal 1PI set contained in $u \cap s$. From the closure of S under union with an initial segment it follows that $u \in S$. From the closure of S under intersection it follows that $s' \in S$.
  The case $\tau \in s'$ is possible only if $l \geq \tau$. Since $s \subseteq \{1,\ldots,\tau\}$, $s'=s_0=s$ in this case. [the **maximality** of τ is used]
- The closure under intersection.
  It is obvious (because S satisfies this property).

The proof of the BPHZ theorem: the elimination of overlaps

The proof by induction: **the external part relative to *s***

The Hepp sector: $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_M$.

τ is the maximal line number contained in at least one not-a-disk element of S.

In each term, s is the minimal not-a-disk element containing τ.

We also suppose that we take only the Taylor expansion terms of degree d' on s, *where* $d' \leq \omega'(s)$.

Let us consider the **external** part of the diagram relative to s.

The corresponding part of the forest formula is factorized. It corresponds to the diagram obtained from the whole diagram G by shrinking s to a point. The numbers Δ'_v are defined as

$$\Delta'_v = \begin{cases} \omega(s) - d' + \sum_{w \in \mathrm{Vertex}(s)} \Delta_w, & \text{if } v \text{ corresponds to the shrinked } s, \\ \Delta_v & \text{otherwise.} \end{cases}$$

We define the corresponding set S' as

$$S' = \{u \subseteq \Lambda \backslash s : \; f(u) \in S\},$$

(we use that any changes in the external part don't affect the **minimality** of s)

where Λ is the set of all diagram lines,

$$f(u) = \begin{cases} u \cup s, & \text{if } \mathrm{Vertex}(u) \cap \mathrm{Vertex}(s) \neq \emptyset, \\ u & \text{otherwise.} \end{cases}$$

The definition of enough divergent sets for the external part with Δ' is consistent with the one for the whole diagram with Δ. It is clear taking into account

$$\omega_{\mathrm{ext}}(u) = \begin{cases} \omega(f(u)), & \text{if } \mathrm{Vertex}(u) \cap \mathrm{Vertex}(s) = \emptyset, \\ \omega(f(u)) + d' - \omega(s), & \text{if } \mathrm{Vertex}(u) \cap \mathrm{Vertex}(s) \neq \emptyset, \end{cases}$$

where $\omega_{\mathrm{ext}}(u)$ is the UV degree of divergence of u in the external part relative to s, where s is shrunk to a point with a polynomial of degree d' (cumbersome, but easy).

The proof of the BPHZ theorem: the elimination of overlaps

The proof by induction: **the external part relative to *s***

We have $S' = \{u \subsetneq \Lambda \backslash s : \ f(u) \in S\}$,
where *Λ* is the set of all diagram lines,

$$f(u) = \begin{cases} u \cup s, \ \text{if Vertex}(u) \cap \text{Vertex}(s) \neq \emptyset, \\ u \text{ otherwise.} \end{cases}$$

We have to prove that S' satisfies the following properties:

- The closure under union with an initial segment. The external order initial segment is *{1,2,…,l}\s*. Suppose $s_1, \ldots, s_n \in S'$, $s_0 = s_1 \cup \ldots \cup s_n \cup (\{1, \ldots, l\} \backslash s)$, the set s' is a maximal 1-particle irreducible set contained in s_0, and **the connectivity where *s* is shrunk to a point is implied**. We have to prove that $s' \in S' \cup \{\Lambda \backslash s\}$.

  There are two cases:

  1) $\text{Vertex}(s') \cap \text{Vertex}(s) = \emptyset$. In this case, *s'* is a maximal 1PI set contained in $f(s_1) \cup \ldots \cup f(s_n) \cup \{1, \ldots, l\}$, where the connectivity of the original graph is implied. Thus, *s'* in *S*. Since *f(s')=s'*, it is also in *S'*.
  2) $\text{Vertex}(s') \cap \text{Vertex}(s) \neq \emptyset$. In this case, *f(s')* is a maximal 1PI set contained in $f(s_1) \cup \ldots \cup f(s_n) \cup s \cup \{1, \ldots, l\}$, where the connectivity of the original graph is implied. Thus, $f(s') \in S$. This means that $s' \in S' \cup \{\Lambda \backslash s\}$.

- The closure under intersection. Suppose $s_1, s_2 \in S'$, the set s' is a maximal 1PI set contained in $s_1 \cap s_2$, and **the connectivity where *s* is shrunk to a point is implied**. We have to prove that $s' \in S'$.

  For proving this note that $f(s_1) \cap f(s_2)$ equals $f(s_1 \cap s_2)$ or $f(s_1 \cap s_2) \cup s$, the last case is possible only if $\text{Vertex}(s_1 \cap s_2) \cap \text{Vertex}(s) = \emptyset$. Thus, *f(s')* is a maximal 1PI set contained in $f(s_1) \cap f(s_2)$, where the connectivity of the original graph is implied. This means that *f(s')* in *S*.

The proof of the BPHZ theorem: the elimination of overlaps

The proof by induction: **power counting**

We have to prove that

$$|F_{\mathrm{SubL}[S,\Delta]}(p,\alpha,\varepsilon_{\mathrm{IR}})| \leq \mathrm{e}^{-\varepsilon_{\mathrm{IR}}\sum_j \alpha_j} \times \frac{P(p,\sum_j \alpha_j)}{\alpha_1\ldots\alpha_M} \times \prod_{j=1}^{M-1} \left(\frac{\alpha_j}{\alpha_{j+1}}\right)^{(1+\sum_{v\in\mathrm{cycl}(\{1,2,\ldots,j\})}\Delta_v)/2} \times (\alpha_M)^z,$$

where *P* is some polynomial giving positive values and non-decreasing with respect to $\Sigma\alpha$;
$\mathrm{cycl}(x)$ is the set of all vertices *v* such that for any $y\in S\cup\{\Lambda\}$ satisfying $v\in\mathrm{Vertex}(y)$ there exist a **cycle** (without self-intersections) in $y\cap x$ that passes *v*;

$z=(d-\omega(G))/2$, if we take the Taylor expansion coefficients of degree d for the whole diagram *G*;
$z=(1+\sum_v \Delta_v)/2$, if we subtract up to the degree *ω'(G)*.

Suppose we take the Taylor coefficient of degree *d* (the other case is analogous).
The set of lines *s* splits the diagram into the external and internal part.
The Taylor expansion coefficients of degree *d'* are taken for the internal part, $d'\leq\omega(s)+\sum_{v\in\mathrm{Vertex}(s)}\Delta_v$.
The estimation is valid for both parts.

The following values contribute to the power of (α_j/α_{j+1}), if the conditions are satisfied:

- $\{1,\ldots,j\}\cap s\neq\emptyset \rightarrow$ *+1/2*;
- $\{1,\ldots,j\}\backslash s\neq\emptyset \rightarrow$ *+1/2*;
- for each $v\in\mathrm{cycl}_{\mathrm{int}}(\{1,\ldots,j\}\cap s)\rightarrow +\Delta_v/2$;
- for each $v\notin\mathrm{Vertex}(s)$ such that $v\in\mathrm{cycl}_{\mathrm{ext}}(\{1,\ldots,j\}\backslash s)\rightarrow +\Delta_v/2$;
- the shrunk *s* is in $\mathrm{cycl}_{\mathrm{ext}}(\{1,\ldots,j\}\backslash s)\rightarrow +\frac{1}{2}\left(w(s)-d'+\sum_{v\in\mathrm{Vertex}(s)}\Delta_v\right)$;
- $s\subseteq\{1,\ldots,j\}\rightarrow -\frac{1}{2}\left(1+w(s)-d'+\sum_{v\in\mathrm{Vertex}(s)}\Delta_v\right)$;
- $s\cup\{1,\ldots,j\}=\Lambda\rightarrow -\frac{1}{2}\left(1+\sum_v\Delta_v+\omega(G)-d\right)$ never happens, because we have a requirement $M\notin s$ or it gives 0 .

Here $\mathrm{cycl}_{\mathrm{int}}$, $\mathrm{cycl}_{\mathrm{ext}}$ are defined as cycl, but in the internal and external parts; the connectivity when *s* is shrunk to a point is implied in the external part.

The proof of the BPHZ theorem: the elimination of overlaps

The proof by induction: **power counting, $\Delta_v/2$**

Suppose $v\in\mathrm{cycl}(\{1,2,\ldots,l\})$, where
$\mathrm{cycl}(x)$ is the set of all vertices *v* such that for any $y\in S\cup\{\Lambda\}$ satisfying $v\in\mathrm{Vertex}(y)$
there exist a **cycle** (without self-intersections) in $y\cap x$ that passes *v;*

There are two cases:

- $v\in\mathrm{Vertex}(s)$, where *s* is the set that separates the diagram into the external and internal parts.
  In this case, for all $y\in S$ such that $y\subseteq s,\ v\in\mathrm{Vertex}(y)$ there exist a cycle in $\{1,\ldots,l\}\cap y$ that passes *v*.
  From this it follows that $v\in\mathrm{cycl}_{\mathrm{int}}(\{1,\ldots,l\}\cap s)$.
  Thus, *v* is counted in the internal part.
- $v\notin\mathrm{Vertex}(s)$.
  In this case, for all $y\in S\cup\{\Lambda\},\ v\in\mathrm{Vertex}(y)$ such that $s\subseteq y$ or $\mathrm{Vertex}(y)\cap\mathrm{Vertex}(s)=\emptyset$
  there is a cycle in $\{1,\ldots,l\}\cap y$ that passes *v*.
  We can use this cycle for proving that $v\in\mathrm{cycl}_{\mathrm{ext}}(\{1,\ldots,l\}\backslash s)$,
  the contained in *s* part of the cycle should be shrunk to a point.
  Thus, *v* is counted in the external part.

The definitions of $\mathrm{cycl}_{\mathrm{int}}$ and $\mathrm{cycl}_{\mathrm{ext}}$ repeat the definition of cycl, but in the internal and external parts, respectively.

The proof of the BPHZ theorem: the elimination of overlaps

The proof by induction: **power counting, the case when *s* is contained in *{1,2,…,j}***

We also have two rules:

$$\text{the shrunk } s \text{ is in } \mathrm{cycl}_{\mathrm{ext}}(\{1,\ldots,l\}\backslash s) \to +\frac{1}{2}\left(w(s)-d'+\sum_{v\in\mathrm{Vertex}(s)}\Delta_v\right);$$

$$s\subseteq\{1,\ldots,l\} \to -\frac{1}{2}\left(1+w(s)-d'+\sum_{v\in\mathrm{Vertex}(s)}\Delta_v\right);$$

We have to prove that the "negative" rule is always compensated by the "positive" one (-*1/2* can't be compensated, but we have an extra *+1/2* for this case) .

Suppose $s\subseteq\{1,\ldots,l\}$. We have to prove that the shrunk *s* is in $\mathrm{cycl}_{\mathrm{ext}}(\{1,\ldots,l\}\backslash s)$. It is enough to prove that for each $y\in S$ such that $s\subsetneq y$ there exist a cycle in $y\cap\{1,\ldots,l\}$ that has vertices in Vertex*(s)* as well as lines not in *s* (the needed cycle in the external part is obtained by shrinking the part of it inside *s*).

There are two cases:

- *y* is a disk generated by the initial segment *{1,2,…,*l_1*}*.
  In this case, there exists a disk *D* generated by *{1,2,…,*min*(l,*l_1*)}* that contains *s* and is contained in *y*.
  Since *s* is not a disk, *s* does not coincide with *D*; thus, there is a cycle in *D* that has vertexes in Vertex(*s)* as well as lines not in *s*. This cycle lies in *{1,2,…,l}*.
- y is not a disk.
  In this case, $y\subseteq\{1,\ldots,l\}$ [we use here that *s* contains the **maximal** line number contained in not-disks from *S*].
  Since $s\subsetneq y$ and both of them are 1PI, there is a cycle in *y* that has vertices in Vertex(*s)* and lines not in *s*.
  This cycle also lies in *{1,2,…,l}*.

The proof of the BPHZ theorem: the elimination of overlaps

The proof by induction: **the remaining multipliers**

We have to prove that

$$|F_{\mathrm{SubL}[S,\Delta]}(p,\alpha,\varepsilon_{\mathrm{IR}})| \le \mathrm{e}^{-\varepsilon_{\mathrm{IR}}\sum_j \alpha_j} \times \frac{P(p,\sum_j \alpha_j)}{\alpha_1\dots\alpha_M} \times \prod_{j=1}^{M-1} \left(\frac{\alpha_j}{\alpha_{j+1}}\right)^{(1+\sum_{v\in \mathrm{cycl}(\{1,2,\dots,j\})}\Delta_v)/2} \times (\alpha_M)^z,$$

P is a polynomial giving positive values and non-decreasing with respect to Σ*α;*

$z=(d-\omega(G))/2$, if we take the Taylor expansion coefficients of degree d for the whole diagram *G*;

$z=(1+\sum_v \Delta_v)/2$, if we subtract up to the degree *ω'(G)*.

<u>The remaining multipliers</u> are:

- $(\alpha_M)^z$ is obtained exactly.
- $e^{-\varepsilon_{\mathrm{IR}}\sum_j \alpha_j}$ is obtained exactly.
- $1/(\alpha_1\alpha_2\dots\alpha_M)$ is obtained exactly.
- $P(p,\sum_j \alpha_j)$. After multiplying the internal and external part expressions we have $P_{\mathrm{ext}}(p,\sum_{j\notin s}\alpha_j)P_{\mathrm{int}}(\sum_{j\in s}\alpha_j)$. Since the polynomials are non-decreasing with respect to Σ*α* and positive-valued, we can put $P(p,a)=P_{\mathrm{ext}}(p,a)P_{\mathrm{int}}(a)$.

Approaches to handling overlapping divergences

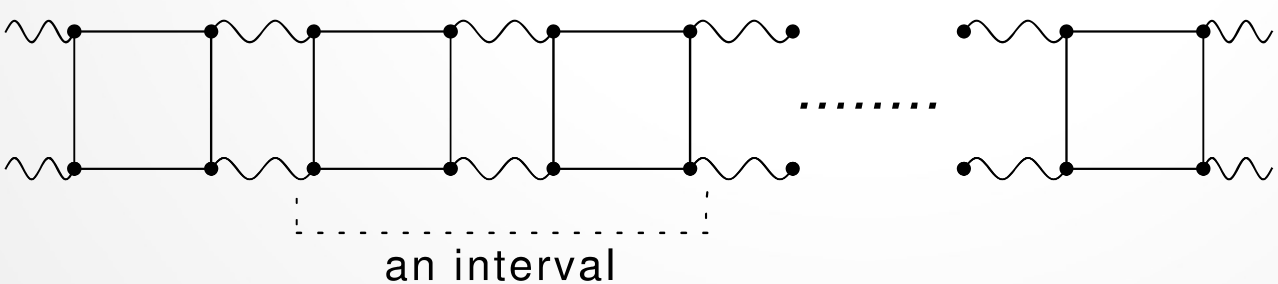
The ideas similar to ''minimax'' or the induction hypothesis described here come from Klaus Hepp [K. Hepp, Commun. Math. Phys. 2, 301 (1966)]

There exist **completely different** approaches.

For example, based on the replacement the subtractions with differentiations for all the needed subdiagrams simultaneously.
[S. A. Anikin, O. I. Zav'yalov, M. K. Polivanov, Theor. Math. Phys. 17, 1082–1088 (1973)]
[M. C. Bergère, J. B. Zuber, Commun. math. Phys. 35, 113-140 (1974)]

The subtraction-free approach **has applications for numerical calculations**:
[L. T. Adzhemyan, M. V. Kompaniets, Journal of Physics: Conference Series, 15th International Workshop on Advanced Computing and Analysis Techniques in Physics Research (ACAT2013), Vol. 523, 012049 (2014)]
(the absence of subtractions prevents from round-off errors, but φ^4-theory only)

A **good test** from QED for alternative approaches:



Each interval is divergent and needs a subtraction.

Outline

- Introduction
- General ideas of handling UV divergences
- Application to the renormalization of quantum electrodynamics
- Formulations in terms of finite integrals
- **The proof of the BPHZ theorem**
  - the formulation, ideas
  - Schwinger-parametric integrals, combinatorial formulas
  - power counting, Hepp sectors, "Hanoi" towers
  - reduction to the forest formula with sets of lines
  - elimination of overlaps
  - the case when all the subtractions fit
- Conclusions

The remaining part is the induction base case

$$|F_{\mathrm{SubL}[S,\Delta]}(p,\alpha,\varepsilon_{\mathrm{IR}})| \leq \mathrm{e}^{-\varepsilon_{\mathrm{IR}}\sum_j \alpha_j} \times \frac{P(p,\sum_j \alpha_j)}{\alpha_1\ldots\alpha_M} \times \prod_{j=1}^{M-1} \left(\frac{\alpha_j}{\alpha_{j+1}}\right)^{(1+\sum_{v\in\mathrm{cycl}(\{1,2,\ldots,j\})}\Delta_v)/2} \times (\alpha_M)^z,$$

P is a polynomial giving positive values and non-decreasing with respect to Σ*α*;
$z=(d-\omega(G))/2$, if we take the Taylor expansion coefficients of degree d for the whole diagram *G*;
$z=(1+\sum_v \Delta_v)/2$, if we subtract up to the degree *ω'(G)*.

The **base case**: *S* is the set of all 1PI disks except the whole set *Λ*.

We will prove a stronger statement:

$$|F_{\mathrm{SubL}[S,\Delta]}(p,\alpha,\varepsilon_{\mathrm{IR}})| \leq \mathrm{e}^{-\varepsilon_{\mathrm{IR}}\sum_j \alpha_j} \times \frac{P(p,\sum_j \alpha_j)}{\alpha_1\ldots\alpha_M} \times \prod_{s\in S} h(s)^{(1+\sum_{v\in\mathrm{Vertex}(s)}\Delta_v)/2} \times \prod_{s\text{ is not 1PI disk}} h(s) \times (\alpha_M)^z,$$

where *h(s)* is the thickness of the disk *s*; it equals α_j/α_{j+1}, if the disk is generated by *{1,2,…,j}*.

The <u>forest formula</u> for obtaining $F_{\mathrm{SubL}[S,\Delta]}$ can be expressed as

$$O_\Lambda \prod_{s\in S,\ \omega'(s)\geq 0} (1-M_s^\Delta),$$

where $\omega'(s)=\omega(s)+\sum_{v\in\mathrm{Vertex}(s)}\Delta_v$, *ω(s)* is the UV degree of divergence of *s*, the operator M_s^Δ takes the Taylor expansion at the subdiagram *s* up to the degree *ω'(s)*, the operator O_Λ takes the Taylor expansion coefficient of degree *d* for the whole diagram at zero momenta or equals $1-M_\Lambda^\Delta$.

 Sergey Volkov sergey.volkov@partner.kit.edu

The proof of the BPHZ theorem: the case when all subtractions fit into the Hepp sector

The idea: replace the subtractions with differentiations

The Taylor theorem with integral form of the remainder:

$$f(y) - \sum_{j=0}^{n} (\partial_y)^j \Big|_{y=0} \frac{y^j}{j!} = \frac{1}{n!} \int_0^1 (1-\chi)^n (\partial_\chi)^{n+1} f(\chi y) d\chi.$$

It works also for functions of many variables, only one χ is needed:

$$f(y_1, y_2, \ldots, y_m) - (\text{its Taylor expansion at 0 up to the degree } n)$$
$$= \frac{1}{n!} \int_0^1 (1-\chi)^n (\partial_\chi)^{n+1} f(\chi y_1, \ldots, \chi y_m) d\chi.$$

 sergey.volkov@partner.kit.edu

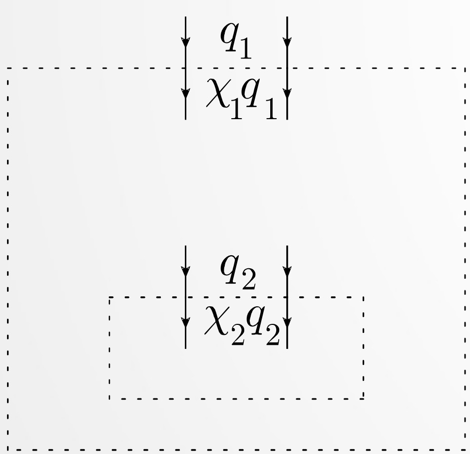
Feynman diagrams with momentum converters

We have the factorized forest formula

$$O_\Lambda \prod_{s\in S'} (1-M_s^\Delta), \text{ where } S' = \{s\in S:\ \omega'(s)\geq 0\}.$$

To replace each subtraction with a differentiation by Taylor's theorem with an integral remainder we have to introduce a parameter χ_s for each set of lines s for which we do the subtraction.

We need Feynman diagrams with **converters** on subdiagrams; each converter multiplies its external momenta by χ_s and transfers the multiplied momenta to the internal part.



Since the elements of S don't overlap, it is easy to obtain the Feynman amplitude for fixed $\alpha_1,\ldots,\alpha_M>0,\ \varepsilon_{\mathrm{IR}}>0$.

To do it, we introduce the change of variables

$$q_l' = \frac{q_l}{\prod_{s\in S':\ l\in s}\chi_s},$$

where q_l is the momentum of the line l. The momenta q' satisfy the 4-momentum conservation law. Thus, we can use the formulas for usual diagrams.

Important! Each vertex polynomial P_v belongs to the minimal (with respect to inclusion) set $s\in S'\cup\{\Lambda\}$ such that $v\in\mathrm{Vertex}(s)$ and uses its converted external momenta.

The Schwinger-parametric amplitude with χ-converters

$$F(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_M,\varepsilon_{\mathrm{IR}},\chi)=C\frac{W(p,\alpha,\chi)}{U(\alpha,\chi)^2}e^{i\frac{V(p,\alpha,\chi)}{U(\alpha,\chi)}-i\sum_j m_j\alpha_j^2-\varepsilon_{\mathrm{IR}}\sum_j\alpha_j},$$

W(p,α,χ) is obtained as a sum (with coefficients) over all sets of nonintersecting pairs of the numerator and vertex multipliers. The multiplier corresponds to *(l,μ)*, where *l* is a line number, *μ* is a coordinate index. The pair *[(l,μ),(j,ν)]* gives $(B_{lj}(\alpha,\chi)g_{\mu\nu})/U(\alpha,\chi)$, the unpaired multiplier *(l,μ)* gives $((Y_l(p,\alpha,\chi))_\mu)/U(\alpha,\chi)$.

Each set of pairs gives an **additional multiplier** $\prod_{s\in S'}(\chi_s)^{(\text{the number of the line multipliers in } P_l,\, l\in s \text{ or } P_v,\, v\in\mathrm{Vertex}(s) \text{ that are not in a pair } \textbf{inside } s)}$.

$$U(\alpha,\chi)=\sum_{R\text{ is 1-tree}}\prod_{j\notin R}\alpha_j\prod_{s\in S'}(\chi_s)^{2\times\mathrm{Defect}_s(R)},$$

$$B_{ab}(\alpha,\chi)=\sum_{R\text{ is a tree with cycle}}(B_R)_{ab}\prod_{l\notin R}\alpha_l\prod_{s\in S'}(\chi_s)^{2\times\mathrm{Defect}'_s(R,a,b)},$$

where $(B_R)_{ab}=1$, if *a* and *b* go in the same direction in the loop of *R*; *-1* if in the opposite direction, *0* in the other cases.

$$Y(p,\alpha,\chi)=\sum_{R\text{ is 1-tree}}(\text{the flow of } p \text{ through } i \text{ in } R)\prod_{l\notin R}\alpha_l\prod_{s\in S'}(\chi_s)^{2\times\mathrm{Defect}_s(R)},$$

$$V(p,\alpha,\chi)=\sum_{R\text{ is a 2-tree}}(\text{the flow of } p \text{ between the components of } R)^2\prod_{l\notin R}\alpha_l\prod_{s\in S'}(\chi_s)^{2\times\mathrm{Defect}_s(R)}.$$

$$\mathrm{Defect}_s(R)=\max_{R'\text{ is 1-tree}}|R'\cap s|-|R\cap s|$$

$$\mathrm{Defect}'_s(R,a,b)=\begin{cases}\mathrm{Defect}_s(R)+1, & \text{if } a,b\in s,\\ \mathrm{Defect}_s(R) & \text{otherwise.}\end{cases}$$

The proof of the BPHZ theorem: the case when all subtractions fit into the Hepp sector

The Schwinger-parametric integrand as an integral over χ

We have

$$F_{\mathrm{SubL}[S,\Delta]}(p,\alpha,\varepsilon_{\mathrm{IR}}) = C \times O_\Lambda \times e^{-i\sum_j m_j \alpha_j^2 - \varepsilon_{\mathrm{IR}}\sum_j \alpha_j}$$
$$\times \int_0^1 \prod_{s\in S'} [(1-\chi_s)^{\omega'(s)}(\partial_{\chi_s})^{\omega'(s)+1}] \frac{W(p,\alpha,\chi)}{U(\alpha,\chi)^2} e^{i\frac{V(p,\alpha,\chi)}{U(\alpha,\chi)}} \prod_{s\in S'} d\chi_s.$$

We obtained this correct answer without thinking about the correctness of the reasoning.

It can be derived correctly immediately at the level of analytical formulas, but it is better to work with correctly defined integrals: first to prove an analogous equality for $\varepsilon_{\mathrm{Mink}}>0$ and then take the limit $\varepsilon_{\mathrm{Mink}}\to 0$. Problems can occur at both steps: the diagram momentum space depends on χ and can become degenerate as $\chi\to 0$; the possibility to swap the operations with taking the limit $\varepsilon_{\mathrm{Mink}}\to 0$ also requires a justification.

To overcome this, one can introduce a χ-dependent loop integration basis $S_{\mathrm{Loop}}[\chi]$ in the following way:

- Take a 1-tree R such that $\mathrm{Defect}_s(R)=0$ for each s from S' and the loop basis S_{Loop} based on R (loops are columns).
- Put $(S_{\mathrm{Loop}}[\chi])_{lj} = (S_{\mathrm{Loop}})_{lj} \times \prod_{s\in S':\ l\in s \text{ and } j\text{-th loop is not contained in } s} \chi_s.$

In this case,

$$\det(S_{\mathrm{Loop}}[\chi]^T \mathrm{Diag}[\alpha] S_{\mathrm{Loop}}[\chi]) = U(\alpha,\chi) = \sum_{R \text{ is 1-tree}} \prod_{j\notin R} \alpha_j \prod_{s\in S'} (\chi_s)^{2\times \mathrm{Defect}_s(R)}$$

is separated from 0, a χ-dependent coefficient is not needed; everything becomes smooth and uniform.

The proof of the BPHZ theorem: the case when all subtractions fit into the Hepp sector

The powers of the disk thicknesses

We have $F_{\mathrm{SubL}[S,\Delta]}(p,\alpha,\varepsilon_{\mathrm{IR}}) = C \times O_\Lambda \times e^{-i\sum_j m_j \alpha_j^2 - \varepsilon_{\mathrm{IR}}\sum_j \alpha_j}$
$\times \int_0^1 \prod_{s\in S'} [(1-\chi_s)^{\omega'(s)} (\partial_{\chi_s})^{\omega'(s)+1}] \frac{W(p,\alpha,\chi)}{U(\alpha,\chi)^2} e^{i\frac{V(p,\alpha,\chi)}{U(\alpha,\chi)}} \prod_{s\in S'} d\chi_s,$

where *W(p,α,χ)* is constructed from the products of the blocks *B(α,χ)/U(α)* and *Y(p,a,χ)/U(α)* and powered *χ*.

The idea of power counting.

Suppose we took one term of *W*. Ignoring the part to the left from differentiations, we have before differentiation an expression of the form

$$\frac{1}{U^2} \times \chi \ldots \chi \times \frac{Y}{U} \ldots \frac{Y}{U} \times \frac{B}{U} \ldots \frac{B}{U} e^{iV/U}.$$

After applying ∂_χ several times we obtain the sum of terms of the form

$$\frac{1}{U^2} \times (\partial_\chi \ldots \partial_\chi \chi) \ldots (\partial_\chi \ldots \partial_\chi \chi) \times \frac{\partial_\chi \ldots \partial_\chi Y}{U} \ldots \frac{\partial_\chi \ldots \partial_\chi Y}{U} \times \frac{\partial_\chi \ldots \partial_\chi B}{U} \ldots \frac{\partial_\chi \ldots \partial_\chi B}{U}$$
$$\times \frac{\partial_\chi \ldots \partial_\chi U}{U} \ldots \frac{\partial_\chi \ldots \partial_\chi U}{U} \times \frac{\partial_\chi \ldots \partial_\chi V}{U} \ldots \frac{\partial_\chi \ldots \partial_\chi V}{U} \times e^{iV/U},$$

where the following conditions are satisfied:

- Each multiplier of *χ* type, *Y/U* type and *B/U* type corresponds (mutually exclusive) to the one of the original expression.
- For each *s* the total number of differentiations with respect to χ_s equals *ω'(s)+1*.

Note that:

- Each *U*, *Y*, *V* is the sum over graphs *R*, each term contains χ_s in the form $(\chi_s)^{2\times \mathrm{Defect}_s(R)}$. Thus, if we have *n* differentiations with respect to χ_s in the multiplier, only terms with $\mathrm{Defect}_s(R) \geq n/2$ survive.
- Analogously, in each B_{ab} only terms with $\mathrm{Defect}'_s(R,a,b) \geq n/2$ survive.
- In the multipliers $\partial_\chi \ldots \partial_\chi \chi_s$ the differentiations only with respect to χ_s are allowed and no more than one.
- The multipliers $(\partial_\chi \ldots \partial_\chi V/U)$ are bounded by polynomials in *p*; the multipliers $(\partial_\chi \ldots \partial_\chi U/U)$ are bounded by *1*.

Taking into account that each 1PI disk *s* has *ω'(s)+1* differentiations with respect to χ_s, we obtain the needed power of *h(s)*. Not 1PI disks have the power *1* (before and after differentiation).

 Sergey Volkov sergey.volkov@partner.kit.edu

The proof of the BPHZ theorem: the case when all subtractions fit into the Hepp sector

The power of α_M and the additional polynomial

We have $$F_{\mathrm{SubL}[S,\Delta]}(p,\alpha,\varepsilon_{\mathrm{IR}}) = C \times O_\Lambda \times e^{-i\sum_j m_j \alpha_j^2 - \varepsilon_{\mathrm{IR}}\sum_j \alpha_j}$$
$$\times \int_0^1 \prod_{s\in S'} [(1-\chi_s)^{\omega'(s)}(\partial_{\chi_s})^{\omega'(s)+1}] \frac{W(p,\alpha,\chi)}{U(\alpha,\chi)^2} e^{i\frac{V(p,\alpha,\chi)}{U(\alpha,\chi)}} \prod_{s\in S'} d\chi_s,$$
where $W(p,\alpha,\chi)$ is constructed from the products of the blocks $B(\alpha,\chi)/U(\alpha)$ and $Y(p,a,\chi)/U(\alpha)$ and powered χ.

Suppose O_Λ takes a Taylor expansion coefficient of degree d for the whole diagram.

Let us calculate the power of α_M, where M is the maximal line number (with maximal α).

The operator O_Λ consists of d differentiations with respect to p. Since before differentiations we had
$$\frac{1}{U^2} \times \chi \ldots \chi \times \frac{Y}{U} \cdots \frac{Y}{U} \times \frac{B}{U} \cdots \frac{B}{U} e^{iV/U}.$$
after differentiations we have a sum of expressions
$$\frac{1}{U^2} \times \chi \ldots \chi \times \frac{\partial_p \ldots \partial_p Y}{U} \cdots \frac{\partial_p \ldots \partial_p Y}{U} \times \frac{B}{U} \cdots \frac{B}{U} \times \frac{\partial_p \ldots \partial_p V}{U} \cdots \frac{\partial_p \ldots \partial_p V}{U} \times e^{iV/U},$$
where the following conditions are satisfied for one term:

- The total number of ∂_p is d.
- Each Y allows no more than *1* differentiation.
- Each V allows no more than *2* differentiations.

Taking into account that each V/U increases the power of α_M by *1*, we obtain that the total power is $z'\geq z$, where z is the needed power. In contrast to $h(s)$, α_M can be >1. Thus, we need a polynomial:
$(\alpha_M)^{z'} \leq (\alpha_M)^z (1+\sum_j \alpha_j)^n$, where n is an upper bound for $z'-z$.

 sergey.volkov@partner.kit.edu

The proof of the BPHZ theorem: the case when all subtractions fit into the Hepp sector

The integration with respect to the Schwinger parameters α

All the powers have been successfully calculated.
The estimation has been successfully proved:

$$|F_{\mathrm{SubL}[S,\Delta]}(p,\alpha,\varepsilon_{\mathrm{IR}})| \leq \mathrm{e}^{-\varepsilon_{\mathrm{IR}}\sum_j \alpha_j} \times \frac{P(p,\sum_j \alpha_j)}{\alpha_1 \ldots \alpha_M} \times \prod_{j=1}^{M-1} \left(\frac{\alpha_j}{\alpha_{j+1}}\right)^{\left(1+\sum_{v\in \mathrm{cycl}(\{1,2,\ldots,j\})} \Delta_v\right)/2} \times (\alpha_M)^z,$$

where $z \geq 1/2$.

From this it follows that

$$|F_{\mathrm{SubL}[S,\Delta]}(p,\alpha,\varepsilon_{\mathrm{IR}})| \leq \mathrm{e}^{-\varepsilon_{\mathrm{IR}}\sum_j \alpha_j} \times \frac{P(p,\sum_j \alpha_j)}{\alpha_1 \ldots \alpha_M} \times (\alpha_1)^{1/2} \times (\alpha_M)^z,$$

We have to integrate the function over α. To do this, we make the change of variables

$$\beta_1 = \frac{\alpha_1}{\alpha_2}, \ \ldots, \ \beta_{M-1} = \frac{\alpha_{M-1}}{\alpha_M}, \ \beta_M = \alpha_M.$$

The absolute value of the integrand (over β) does not exceed

$$\frac{e^{-\varepsilon_{\mathrm{IR}}\beta_M} P(p, M\beta_M)(\beta_M)^z}{(\beta_1)^{1/2} \ldots (\beta_M)^{1/2}}.$$

The corresponding integral, obviously, finite. The BPHZ theorem has been successfully proved.

Outline

- Introduction
- General ideas of handling UV divergences
- Application to the renormalization of quantum electrodynamics
- Formulations in terms of finite integrals
- The proof of the BPHZ theorem
- Conclusions

The BPHZ theorem: a literature

[N. N. **B**ogoliubov and O. S. **P**arasiuk, Acta Math. 97, 227 (1957)]

[K. **H**epp, Commun. Math. Phys. 2, 301 (1966)]

[W. **Z**immermann, Commun. Math. Phys. 15, 208 (1969)]

[V. A. Smirnov, *Renormalization and Asymptotic Expansions*, PPH'14, Progress in Mathematical Physics, Birkhäuser, 2000]

[O.I. Zavyalov, *Renormalized Quantum Field Theory*, Mathematics and Its Applications. Soviet Series, vol. 21, Kluwer, Dordrecht, Netherlands, 1990, 524 pp.]

[C. Itzykson, J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill Inc., 1980]

[N. N. Bogolyubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Nauka, Moscow, 1984; Wiley, New York, 1980)] – the equivalence to an introduction of counterterms

---

[M. C. Bergère, J. B. Zuber, Commun. math. Phys. 35, 113-140 (1974)]

[S. A. Anikin, O. I. Zav'yalov, M. K. Polivanov, Theor. Math. Phys. 17, 1082–1088 (1973)]

The BPHZ theorem: a discussion

All UV divergences can be removed in each Feynman diagram by a procedure that is equivalent to the renormalization.

- The theorem works for non-renormalizable theories as well.
- Subtractions at zero momenta are not obligatory: the handling overlaps does not use the structure of the operators at all; the reduction to differentiations is easily modifiable.
- The finiteness for each diagram leads to the finiteness of each coefficient in the perturbation series. However, the question about the whole series convergence remains open.
- These results do not provide the possibility to manipulate with finite objects needed for proving the gauge invariance at the level of Feynman diagrams and other symmetry properties. Additional tricks are required.
- All known proofs of the BPHZ theorem contain cumbersome combinatorics.
- Rigorously proved finiteness theorems that include also physical IR divergences don't exist.
- Several papers prove also that the limit $\varepsilon_{\mathrm{IR}} \to 0$ exists as a distribution. However, it is a distribution on the whole space of external 4-momenta, without taking into account that the external momenta are on the mass shell. Thus, this is useless.
- There exist theorems concerning non-physical IR divergences (that emerge, for example, when one uses a massless approximation for massive particles).
  [S.A. Anikin, O.I. Zavyalov, N.I. Karchev, Theor. Math. Phys. 44 (1980); Teor. Mat. Fiz. 44 (1980) 291]
  [J.H. Lowenstein, Commun. Math. Phys. 47 (1976) 53]
  [K.G. Chetyrkin, F.V. Tkachov, Phys. Lett. B 114 (1982) 340]
  [M.C. Bergere, Y.M.P. Lam, Commun. Math. Phys. 48 (1976) 267]
- All finiteness theorems concern the Feynman diagrams for scattering processes (or something close to it). There are no results in the form of equations of motion, processes in space-time and so on.