Inclusive semileptonic, radiative and rare B decays

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P3H
Particle Physics Phenomenology after the Higgs Discovery

On behalf of the PIs of project C1a (Huber, Mannel, Steinhauser)

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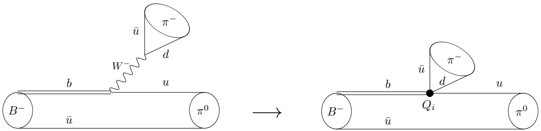
Outline

- Introduction
- Inclusive semileptonic decays
- Inclusive radiative decays
- Inclusive rare decays
- Miscellaneous
- Conclusion

Effective theory for B decays



- $M_W, M_Z, m_t, m_H \gg m_b$: integrate out heavy gauge bosons, t-quark, Higgs
- Effective Weak Hamiltonian: [Buras,Buchalla,Lautenbacher'96; Chetyrkin,Misiak,Münz'98]

$$\mathcal{H}_{eff} = \frac{4G_F}{\sqrt{2}} \sum_{p=u,c} \lambda_p \left[C_1 Q_1^p + C_2 Q_2^p + \sum_{k=3}^{10} C_k Q_k \right] + \text{h.c.}$$

$$Q_1^p = (\bar{d}_L \gamma^\mu T^a p_L)(\bar{p}_L \gamma_\mu T^a b_L) \qquad Q_4 = (\bar{d}_L \gamma^\mu T^a b_L) \textstyle\sum_q (\bar{q} \gamma_\mu T^a q)$$

$$Q_2^p = (\bar{d}_L \gamma^\mu p_L)(\bar{p}_L \gamma_\mu b_L) \qquad Q_5 = (\bar{d}_L \gamma^\mu \gamma^\nu \gamma^\rho b_L) \textstyle\sum_q (\bar{q} \gamma_\mu \gamma_\nu \gamma_\rho q)$$

$$Q_3 = (\bar{d}_L \gamma^\mu b_L) \textstyle\sum_q (\bar{q} \gamma_\mu q) \qquad Q_6 = (\bar{d}_L \gamma^\mu \gamma^\nu \gamma^\rho T^a b_L) \textstyle\sum_q (\bar{q} \gamma_\mu \gamma_\nu \gamma_\rho T^a q)$$

$$Q_7 = \frac{e}{16\pi^2} m_b \, \bar{s}_L \, \sigma_{\mu\nu} F^{\mu\nu} b_R \qquad Q_8 = \frac{g_s}{16\pi^2} m_b \, \bar{s}_L \, \sigma_{\mu\nu} G^{\mu\nu} b_R$$

$$Q_9 = (\bar{s}_L \gamma^\mu b_L)(\bar{\ell} \gamma_\mu \ell) \qquad Q_{10} = (\bar{s}_L \gamma^\mu b_L)(\bar{\ell} \gamma_\mu \gamma_5 \ell) \qquad \lambda_p = V_{pb} V_{pd}^*$$

- Size of Wilson coeffcients

$$C_1 = -0.25 \qquad C_7 = -0.30$$

$$C_2 = 1.01 \qquad C_8 = -0.15$$

$$|C_{3,5,6}| < 0.01 \qquad C_9 = 4.06$$

$$C_4 = -0.08 \qquad C_{10} = -4.29$$

Inclusive B decays, generalities

- Main tool for inclusive decays: Heavy Quark Expansion

[Khoze,Shifman,Voloshin,Bigi,Uraltsev,Vainshtein,Blok,Chay,Georgi,Grinstein,Luke,... '80s and '90s]

$$\Gamma(B_q \to X) = \frac{1}{2m_{B_q}} \sum_X \int_{\text{PS}} (2\pi)^4 \delta^{(4)}(p_{B_q} - p_X) |\langle X|\hat{\mathcal{H}}_{eff}|B_q\rangle|^2$$

- Use optical theorem $\Gamma(B_q \to X) = \frac{1}{2m_{B_q}} \langle B_q|\hat{\mathcal{T}}|B_q\rangle$ with $\hat{\mathcal{T}} = \text{Im}\, i \int d^4x \hat{T}\left[\hat{\mathcal{H}}_{eff}(x)\hat{\mathcal{H}}_{eff}(0)\right]$

- Expand non-local double insertion of effective Hamiltonian in local operators

$$\begin{aligned}\Gamma \quad &= \quad \Gamma_0 \langle O_{D=3}\rangle + \Gamma_2 \frac{\langle O_{D=5}\rangle}{m_b^2} + \Gamma_3 \frac{\langle O_{D=6}\rangle}{m_b^3} + \ldots \\ &\quad + 16\pi^2 \left[\tilde{\Gamma}_3 \frac{\langle \tilde{O}_{D=6}\rangle}{m_b^3} + \tilde{\Gamma}_4 \frac{\langle \tilde{O}_{D=7}\rangle}{m_b^4} + \tilde{\Gamma}_5 \frac{\langle \tilde{O}_{D=8}\rangle}{m_b^5} + \ldots\right]\end{aligned}$$

- Each term can be expanded in a perturbative series: $\Gamma_i = \Gamma_i^{(0)} + \frac{\alpha_s}{4\pi}\Gamma_i^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \Gamma_i^{(2)} + \ldots$.

HQE expansion parameters

- Γ_0: Decay of a free quark, known to $\mathcal{O}(\alpha_s^3)$
- Γ_1: Vanishes due to Heavy Quark Symmetry

- Two terms in Γ_2
  - Kinetic energy μ_π: $2M_B\mu_\pi^2 = -\langle B(v)|\bar{b}_v(iD)^2 b_v|B(v)\rangle$
  - Chromomagnetic moment μ_G: $2M_B\mu_G^2 = -i\langle B(v)|\bar{b}_v\sigma_{\mu\nu}(iD^\mu)(iD^\nu)b_v|B(v)\rangle$

- Two more terms in Γ_3
  - Darwin term ρ_D: $2M_H\rho_D^3 = -\langle B(v)|\bar{b}_v(iD_\mu)(ivD)(iD^\mu)b_v|B(v)\rangle$
  - Spin-orbit term ρ_{LS}: $2M_H\rho_{LS}^3 = -i\langle B(v)|\bar{b}_v\sigma_{\mu\nu}(iD^\mu)(ivD)(iD^\nu)b_v|B(v)\rangle$

- At higher orders: proliferation of number of matrix elements
  - Reparametrization invariance (RPI) allows to reduce number of independent terms

Inclusive semileptonic decays

[Mannel,Milutin,Vos'23]

- Investigate HQE parameters in $b \to c\ell\bar{\nu}$ at $\mathcal{O}(1/m_b^5)$
- Identify 10 RPI parameters at $\mathcal{O}(1/m_b^5)$

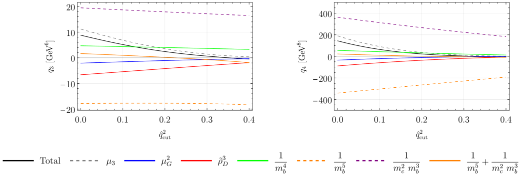
$$2m_B X_1^5 = \langle \bar{b}_v \Big[(ivD), [(ivD), (iD_\mu)] \Big] [(ivD), (iD^\mu)] b_v \rangle$$

- Concentrate on q^2-moments (also RPI)

$$q_1 = \frac{m_b^2}{\mu_3}\Big(0.22\mu_3 - 0.57\frac{\mu_G^2}{m_b^2} - 1.4\frac{(\mu_G^2)^2}{m_b^4\mu_3} - 5.5\frac{\tilde{\rho}_D^3}{m_b^3} + 16\frac{\tilde{r}_E^4}{m_b^4} - 5.7\frac{r_G^4}{m_b^4} - 1.7\frac{\tilde{s}_E^4}{m_b^4}$$
$$+ 0.097\frac{s_B^4}{m_b^4} - 0.064\frac{s_{qB}^4}{m_b^4} - 24\frac{\mu_G^2\tilde{\rho}_D^3}{m_b^5\mu_3} - 19\frac{X_1^5}{m_b^5} + 18\frac{X_2^5}{m_b^5} - 15\frac{X_3^5}{m_b^5} + 2.3\frac{X_4^5}{m_b^5}$$
$$+ 6.5\frac{X_5^5}{m_b^5} + 0.91\frac{X_6^5}{m_b^5} - 7.0\frac{X_7^5}{m_b^5} + 8.0\frac{X_8^5}{m_b^5} + 5.2\frac{X_9^5}{m_b^5} - 4.4\frac{X_{10}^5}{m_b^5} + 0.047\frac{X_{\rm IC}^5}{m_b^3 m_c^2}\Big)$$

- Also include "intrinsic charm" terms $\mathcal{O}(\log(m_c/m_b)/m_b^3)$ and $\mathcal{O}(1/(m_b^3 m_c^2))$ (also RPI)

- To estimate size of $\mathcal{O}(1/m_b^5)$ and IC HQE parameters, use LLSA
- For q^2 moments, genuine $1/m_b^5$ terms and IC terms similar in size but of opposite sign



Alternative Treatment of the Quark Mass in the HQE

[Boushmelev,Mannel,Vos'23]

- Treatment of heavy quark mass crucial for precision in heavy-hadron inclusive decays
- Various short-distance mass schemes on the market
  - Need to be extracted from other, independent observables
- Idea: Replace heavy quark mass and matrix elements by observables
  - E.g. inverse moments M_n of the cross section for $e^+e^- \to$ hadrons

$$M_n = \int \frac{ds}{s} \frac{1}{s^n} R(s)$$

- Hope: Later onset of asymptotic behaviour of perturbative expansion

$$M(m_Q) = \sum_n \sum_i C_n^{(i)}(m_Q) \langle O_n^{(i)} \rangle \, , \qquad C_n^{(i)} \sim \frac{1}{m_Q^n}$$

$$m_Q = \frac{1}{2} \left(\frac{9}{4} Q_Q^2 \right)^{1/(2n)} \left(\frac{C_n}{M_n} \right)^{1/(2n)}$$

- Investigate for inclusive $\bar{B} \to X_u \ell \nu$

$$\Gamma(B \to X_u \ell \bar{\nu}) = \frac{G_F |V_{ub}|^2 m_{\rm pole}^5}{192\pi^3} \times \left(1 + \frac{\alpha_s}{\pi} b_1 + \left(\frac{\alpha_s}{\pi} \right)^2 \left[b_2 + \beta_0 b_1 \ln \left(\frac{\mu^2}{m_Q^2} \right) \right] + \cdots \right)$$

- Have $b_1 = -2.4, b_2 = -21.3$ and $b_2/b_1 = 8.8$

Alternative Treatment of the Quark Mass in the HQE

[Boushmelev,Mannel,Vos'23]

- Trade pole mass for inverse moments of R-ratio

$$\Gamma(B \to X_u \ell \bar{\nu}) \sim \left(\frac{C_n^{(0)}}{M_n}\right)^{5/(2n)} \times \left(1 + \frac{\alpha_s}{\pi} d_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \left[d_n^{(2)} + d_n^{(1)} \beta_0 \ln\left(\frac{\mu^2}{m_Q^2}\right)\right] + \cdots\right)$$

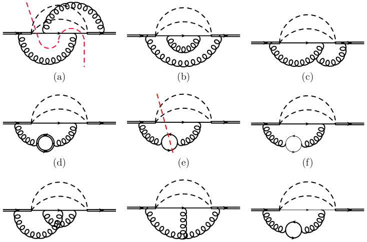
	n						
	1	2	3	4	5	6	7
$d_n^{(1)}$	10.24	7.29	5.85	4.94	4.29	3.80	3.41
$d_n^{(2)}$	70.41	49.45	39.69	33.70	29.52	26.40	23.93
$d_n^{(2)}/d_n^{(1)}$	6.87	6.79	6.78	6.81	6.89	6.95	7.03

- In this case, convergence of the pertrurbative series not strongly improved

Semileptonic B decays at NNLO in QCD

[Egner,Fael,Schönwald,Steinhauser'23]

- NNLO corrections to semileptonic decay rate of B mesons for arbitrary values of the final-state quark mass
- Flow of the calculation
  - Generate diagrams with `qgraf`
  - Process further with `tapir`, `exp` and `FORM` to obtain scalar integrals
  - Use `Kira`, `FireFly` and `ImproveMasters.m` for reduction to 129 master integrals
  - Check cancellation of gauge parameter ξ



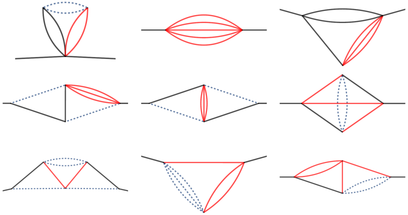
Semileptonic B decays at NNLO in QCD

[Egner,Fael,Schönwald,Steinhauser'23]

- Solve master integrals with method of DE in $\rho = m_c/m_b$
- For contributions with one massive quark use DE in canonical form. Alphabet reads

$$\rho = \frac{1-t^2}{1+t^2}, \qquad \left\{ \frac{1}{1+t}, \frac{1}{t}, \frac{1}{1-t}, \frac{t}{1+t^2}, \frac{t^3}{1+t^4} \right\}$$

  - Boundary condition from asymptotic expansions and method of regions
  - Analytic expressions in terms of iterated integrals
- For three massive quarks in the final state apply semi-analytic method
  - Expand DE about several points in $\rho \in [0,1]$ [see Steinhauser's talk]
- Results agree with available expansions for $b \to c\ell\nu$ and $b \to u\ell\nu$

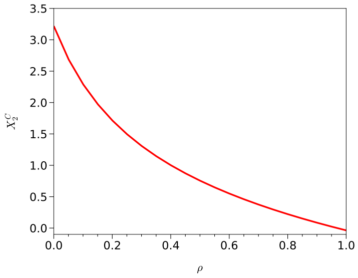


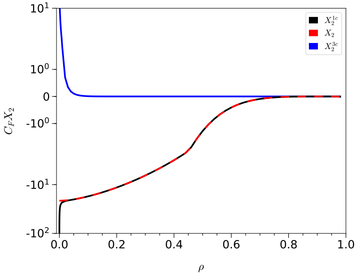
Semileptonic B decays at NNLO in QCD

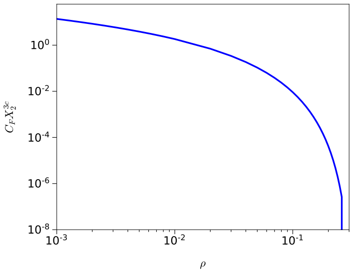
[Egner,Fael,Schönwald,Steinhauser'23]

- Results, e.g. charm-quark contribution in $b \to u\ell\nu$

$$\Gamma(B \to X_u \ell \bar{\nu}) \quad = \quad \Gamma_0 \left[1 + \left(\frac{\alpha_s}{\pi} \right)^2 C_F T_F X_2^C + \ldots \right] + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{m_b^2} \right)$$



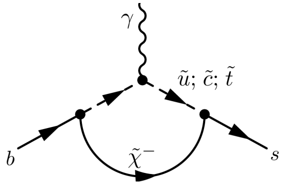




Inclusive radiative decays

Inclusive $\bar{B} \to X_s\gamma$

- One of the standard candles in the search for NP in the quark flavour sector
- Flavour-changing neutral current process
- Dominant contribution is loop-induced
  - Indirectly sensitive to new particles
- Plays a prominent role in global fits



- Current CP- and isospin-averaged SM prediction vs. measurement (for $E_\gamma > 1.6$ GeV)

$$\mathcal{B}_{s\gamma}^{\rm SM} = (3.40 \pm 0.17) \times 10^{-4}$$ [Misiak,Rehman,Steinhauser'20]

$$\mathcal{B}_{s\gamma}^{\rm exp.} = (3.49 \pm 0.19) \times 10^{-4}$$ [HFLAV,PDG'23]

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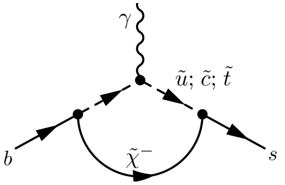

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- Error budget of SM prediction
  - Interpolation in m_c: $\pm 3\%$
  - Unknown higher-order effects: $\pm 3\%$
  - Input params. + non-pert. unc.: $\pm 2.5\%$

Inclusive $\bar{B} \to X_s\gamma$

[Czaja,Czakon,Huber,Misiak,Niggetiedt,Rehman,Schönwald,Steinhauser'23]

- Unrenormalized $Q_{1,2}$ – Q_7 interference contributions at $\mathcal{O}(\alpha_s^2)$ for physical value of m_c

$$\Gamma(b \to X_s^p \gamma) = \frac{G_F^2 \alpha_{\rm em} m_{b,\rm pole}^5}{32\pi^4} |V_{ts}^* V_{tb}|^2 \sum_{i,j=1}^{8} C_i(\mu_b)\, C_j(\mu_b)\, \hat{G}_{ij}$$

$$\hat{G}_{ij} = \hat{G}_{ij}^{(0)} + \frac{\alpha_s}{4\pi} \hat{G}_{ij}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \hat{G}_{ij}^{(2)} + \mathcal{O}(\alpha_s^3)$$

$$\hat{G}_{27}^{(2)\rm bare} = \hat{G}_{27}^{(2)2P} + \hat{G}_{27}^{(2)3P} + \hat{G}_{27}^{(2)4P}$$

$$\hat{G}_{27}^{(2)2P} = \Delta_{30}\hat{G}_{27}^{(2)2P} + \Delta_{21}\hat{G}_{27}^{(2)2P}$$

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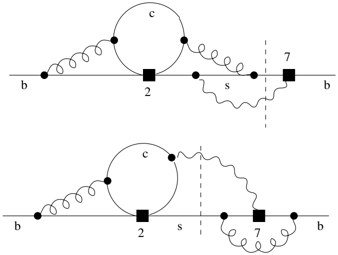

[Czaja,Czakon,Huber,Misiak,Niggetiedt,Rehman,Schönwald,Steinhauser'23]

- Calculation of $\Delta_{30}\hat{G}_{27}^{(2)2P}$ and $\Delta_{21}\hat{G}_{27}^{(2)2P}$
  - Use cut-propagator approach and reverse unitarity

$$-2\pi i\delta(p^2 - m^2) = \frac{1}{p^2 - m^2 + i\varepsilon} - \frac{1}{p^2 - m^2 - i\varepsilon}$$

  - Generate ~ 200 four-loop propagator diagrams with `QGRAF`, `FeynArts` and in-house codes
  - Perform Dirac algebra with `FORM` to obtain scalar integrals
  - Reduce scalar integrals with `Kira` to 447 master integrals
  - Solve master integrals using `AMFlow`

Inclusive $\bar{B} \to X_s\gamma$

[Czaja,Czakon,Huber,Misiak,Niggetiedt,Rehman,Schönwald,Steinhauser'23]

- Results at $z = m_c^2/m_b^2 = 0.04$

$$\begin{aligned}
\Delta_{30}\hat{G}_{27}^{(2)2P}(z=0.04) &\simeq \frac{0.181070}{\epsilon^3} - \frac{6.063805}{\epsilon^2} - \frac{34.087329}{\epsilon} - 127.624515 \\
&+ \left(\frac{0.482853}{\epsilon^2} + \frac{4.093615}{\epsilon} + 10.984004\right) n_b \\
&+ \left(\frac{0.482853}{\epsilon^2} + \frac{4.185427}{\epsilon} + 19.194053\right) n_c \\
&+ \left(\frac{0.482853}{\epsilon^2} + \frac{4.135795}{\epsilon} + 19.647238\right) n_l
\end{aligned}$$

$$\begin{aligned}
\Delta_{21}\hat{G}_{27}^{(2)2P}(z) &= \frac{368}{243\epsilon^3} + \frac{736 - 324 f_0(z)}{243\epsilon^2} \\
&+ \frac{1}{\epsilon}\left(\frac{1472}{243} + \frac{92}{729}\pi^2 - \frac{8f_0(z) + 4f_1(z)}{3}\right) + p(z)
\end{aligned}$$

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- NLO functions $f_0(z)$ and $f_1(z)$ from

$$\hat{G}_{27}^{(1)2P} = -\frac{92}{81\epsilon} + f_0(z) + \epsilon f_1(z) + \mathcal{O}(\epsilon^2)$$

  - Known analytically

[Fael,Lange,Schönwald,Steinhauser'23]

- Numerical values at $z = 0.04$

$$\begin{aligned}
f_0(z=0.04) &\simeq -6.371045 \\
f_1(z=0.04) &\simeq -18.545805 \\
p(z=0.04) &\simeq 144.959811
\end{aligned}$$

- Agreement with parallel calculations where applicable

[Greub,Asatrian,Saturnino,Wiegand'23; Fael,Lange,Schönwald,Steinhauser'23]

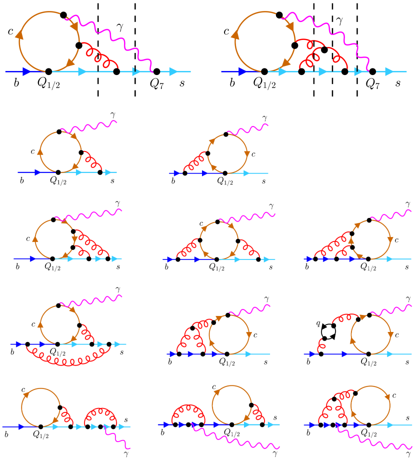
[Fael,Lange,Schönwald,Steinhauser'23]

- Three-loop $b \to s\gamma$ vertex with current-current operators
- Focus on two-particle cuts with no loop on Q_7 side

$$A = \frac{4G_F m_b^2}{\sqrt{2}} V_{ts}^\star V_{tb}\, M^\mu \varepsilon_\mu$$

$$M^\mu = \bar{u}_s(p_s) P_R \left(t_1 \frac{q_\gamma^\mu}{m_b} + t_2 \frac{p_b^\mu}{m_b} + t_3 \gamma^\mu \right) u_b(p_b)$$

$$\hat{G}_{i7}^{2P,Q_7^{\rm tree}} = -\,{\rm Re}\left[\frac{t_2^{Q_i}}{2} + (3-2\epsilon)\, t_3^{Q_i} \right] \frac{e^{\gamma_E \epsilon}}{8} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}$$



Inclusive $\bar{B} \to X_s\gamma$

[Fael,Lange,Schönwald,Steinhauser'23]

- Computation is performed in a well-etablished setup
  - Generate diagrams with `qgraf`
  - Process further with `tapir`, `exp` and `FORM` to obtain scalar integrals (10 (181) families and 2 (3) loops)
  - Use `Kira`, `Fermat` and `ImproveMasters.m` for reduction to 14 (479) master integrals
  - Check cancellation of gauge parameter ξ in final result
- Analytic computation of two-loop master integrals using DE
  - Use variables

    $x = m_c/m_b$, $y = 1/x$ and $w = \frac{1-\sqrt{1-4x^2}}{1+\sqrt{1-4x^2}}$

    to rationalize all roots in the alphabet $\implies$ HPLs

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- Computation of three-loop master integrals
  - Use DE to construct deep series expansions about several values of x_0
  - Plug ansatz of Taylor or power-log expansion in $x - x_0$ in DE
  - Obtain/solve linear system of equations for expansion coefficients
  - Here, using $x_0 = 0$ and $x_0 = 1/5$ is sufficient
  - Use `AMFlow` for boundary conditions and numerical check at $x = 1/10$

[see Steinhauser's talk]

[Fael,Lange,Schönwald,Steinhauser'23]

- Results
  - Two-loop results are completely analytic
  - Three-loop results semi-analytic as expansion in x
  - All checks work, e.g.
    - Analytic expansion for $x \to 0$
    - Comparison w/ parallel calculations
    - Ward identity

$$
\begin{aligned}
\mathrm{Re}(t_2^{Q_1}) = n_l \Bigg\{ & -\frac{0.643804}{\epsilon^2} - \frac{6.31123}{\epsilon} - 27.9137 + x^2 \Bigg[\frac{1}{\epsilon}\Big(2.107 l_x^3 + 3.16049 l_x^2 - 27.8263 l_x \\
& - 11.7523 \Big) - 7.37449 l_x^4 + 3.51166 l_x^3 + 25.8566 l_x^2 - 201.543 l_x - 247.57 \Bigg] \Bigg\} \\
& + n_c \Bigg\{ -\frac{0.643804}{\epsilon^2} - \frac{6.31123}{\epsilon} - 27.9137 + x^2 \Bigg[\frac{1}{\epsilon}\Big(2.107 l_x^3 + 3.16049 l_x^2 - 24.6658 l_x \\
& - 9.61098 \Big) - 7.37449 l_x^4 + 12.9931 l_x^3 + 54.3011 l_x^2 - 224.155 l_x - 335.398 \Bigg] \Bigg\} \\
& + n_b \Bigg\{ -\frac{0.643804}{\epsilon^2} - \frac{6.25499}{\epsilon} - 14.2846 + x^2 \Bigg[\frac{1}{\epsilon}\Big(2.107 l_x^3 + 3.16049 l_x^2 - 27.8263 l_x \\
& - 11.7523 \Big) - 5.26749 l_x^4 + 23.7497 l_x^2 - 104.437 l_x - 132.539 \Bigg] \Bigg\} \\
& - \frac{2.0192}{\epsilon^3} + \frac{87.3997}{\epsilon} + 256.363 + \frac{8.17904}{\epsilon^2} + x\left(\frac{374.314}{\epsilon} - 1497.26 l_x + 669.332 \right) \\
& + x^2 \Bigg[\frac{1}{\epsilon^2}\Big(4.21399 l_x^3 + 6.32099 l_x^2 - 55.6525 l_x - 23.5046 \Big) + \frac{1}{\epsilon}\Big(- 13.6955 l_x^4 \\
& - 36.8724 l_x^3 - 209.669 l_x^2 + 1407.45 l_x + 233.132 \Big) + 27.8123 l_x^5 + 142.222 l_x^4 \\
& + 402.206 l_x^3 - 2492.03 l_x^2 + 7662.75 l_x + 8375.85 \Bigg]
\end{aligned}
$$

Inclusive rare decays

[Huber,Hurth,Lunghi,Jenkins'23]

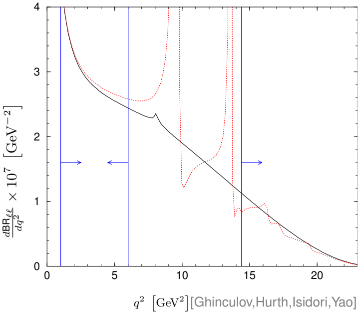
$$\frac{d^3\mathcal{B}}{ds\,du\,dz} = \frac{3}{8}\left[(1+z^2)\frac{d^2\mathcal{H}_T}{ds\,du} + 2z\frac{d^2\mathcal{H}_A}{ds\,du} + 2(1-z^2)\frac{d^2\mathcal{H}_L}{ds\,du}\right] + O(\alpha_e)$$

$$s = \frac{q^2}{m_b^2}\,, \qquad u = \frac{(m_b v - q)^2}{m_b^2}$$

$$z = \cos\theta = \frac{v\cdot(p_{\ell^-} - p_{\ell^+})}{\sqrt{(v\cdot q)^2 - q^2}}$$

$$\frac{d^2\mathcal{B}}{ds\,du} = \frac{d^2\mathcal{H}_L}{ds\,du} + \frac{d^2\mathcal{H}_T}{ds\,du}$$

- q^2 spectrum of inclusive $\bar{B} \to X_s \ell^+ \ell^-$



[Ghinculov,Hurth,Isidori,Yao]

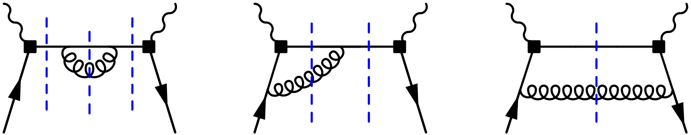
- Low-q^2 region: $1\,\text{GeV}^2 < q^2 < 6\,\text{GeV}^2$
- High-q^2 region: $q^2 > 14.4\,\text{GeV}^2$

$\bar{B} \to X_s \ell^+ \ell^-$ with a hadronic mass cut

[Huber,Hurth,Lunghi,Jenkins'23]

- The suppression of background from $b \to c\ (\to s\ell\nu)\ \ell\nu$ requires a cut on M_{X_s}
- Have $M_{X_s} < 1.8\ (2.0)$ GeV at BaBar (Belle)
- Also analyses at Belle II will require a cut
- High-q^2 region hardly affected by the cut

- Investigate hadronic mass spectrum at NLO in the heavy quark expansion

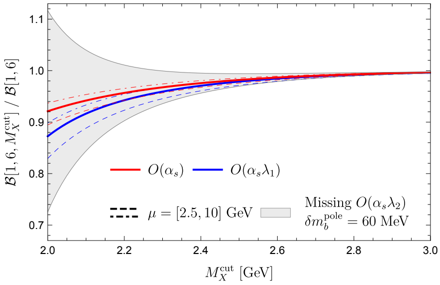


$\bar{B} \to X_s \ell^+ \ell^-$ with a hadronic mass cut

$$\mathcal{B}[q_1^2, q_2^2, M_X^{\rm cut}] = \int_{s_1}^{s_2} ds \int_0^{(1-\sqrt{s})^2} du\, \theta(M_X^{\rm cut} - M_X)\, \frac{d^2\mathcal{B}}{ds\, du}$$

$$\mathcal{B}[q_1^2, q_2^2] = \int_{s_1}^{s_2} ds \int_0^{(1-\sqrt{s})^2} du\, \frac{d^2\mathcal{B}}{ds\, du}$$

[Huber,Hurth,Lunghi,Jenkins'23]



$\bar{B} \to X_s \ell^+ \ell^-$ with a hadronic mass cut

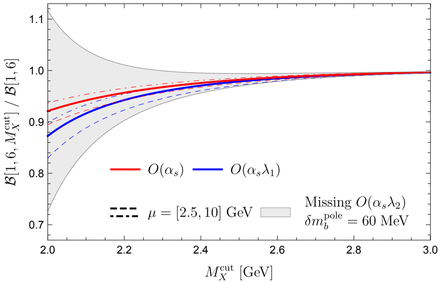
$$\mathcal{B}[q_1^2, q_2^2, M_X^{\rm cut}] = \int_{s_1}^{s_2} ds \int_0^{(1-\sqrt{s})^2} du\, \theta(M_X^{\rm cut} - M_X)\, \frac{d^2\mathcal{B}}{ds\, du}$$

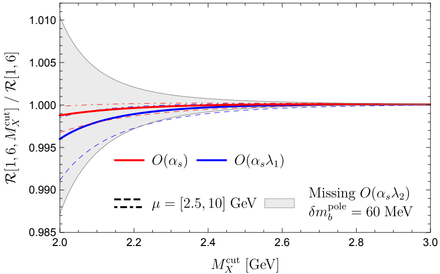
$$\mathcal{B}[q_1^2, q_2^2] = \int_{s_1}^{s_2} ds \int_0^{(1-\sqrt{s})^2} du\, \frac{d^2\mathcal{B}}{ds\, du}$$

$$\mathcal{R}[q_1^2, q_2^2, M_X^{\rm cut}] = \frac{\displaystyle\int_{s_1}^{s_2} ds \int_0^{(1-\sqrt{s})^2} du\, \theta(M_X^{\rm cut} - M_X)\, \frac{d^2\mathcal{B}(\bar{B} \to X_s \ell^+ \ell^-)}{ds\, du}}{\displaystyle\int_{s_1}^{s_2} ds \int_0^{(1-\sqrt{s})^2} du\, \theta(M_X^{\rm cut} - M_X)\, \frac{d^2\mathcal{B}(\bar{B} \to X_u \ell^- \nu)}{ds\, du}}$$

$$\mathcal{R}[q_1^2, q_2^2] = \frac{\displaystyle\int_{s_1}^{s_2} ds \int_0^{(1-\sqrt{s})^2} du\, \frac{d^2\mathcal{B}(\bar{B} \to X_s \ell^+ \ell^-)}{ds\, du}}{\displaystyle\int_{s_1}^{s_2} ds \int_0^{(1-\sqrt{s})^2} du\, \frac{d^2\mathcal{B}(\bar{B} \to X_u \ell^- \nu)}{ds\, du}}$$

[Huber,Hurth,Lunghi,Jenkins'23]





Miscellaneous

Master integrals for massless four-loop form factors

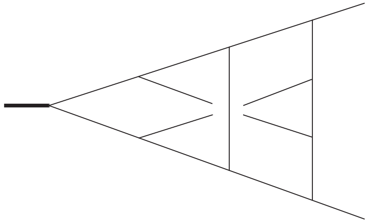
- The four-loop form factors for $\gamma^* \to q\bar{q}$, $gg \to H$ and $H \to b\bar{b}$ have recently become available
  [Lee,Manteuffel,Schabinger,Smirnov,Smirnov,Steinhauser'22; Chakraborty,Huber,Lee,Manteuffel,Schabinger,Smirnov,Smirnov,Steinhauser'22]

- All results are analytic in terms of transcendental constants up to weight eight (e.g. π^8, $\pi^2\zeta_3^2$, $\zeta_3\zeta_5$, $\zeta_{5,3}$)

- Recently, master integrals have been presented separately [Lee,Manteuffel,Schabinger,Smirnov,Smirnov,Steinhauser'23]

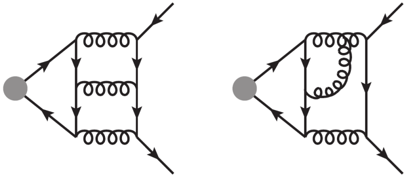
- Kinematics: $q = p_1 + p_2$ with $p_1^2 = p_2^2 = 0, \quad q^2 \neq 0$

- Two main integration strategies
  - Take one more leg off-shell,
    apply/solve DE in $x = p_2^2/q^2$,
    take limit $x \to 0$
- Direct integration over Feynman parameters
  - Transform masters to ϵ-finite basis
    Use `HyperInt` for integration, `FIESTA` for checks

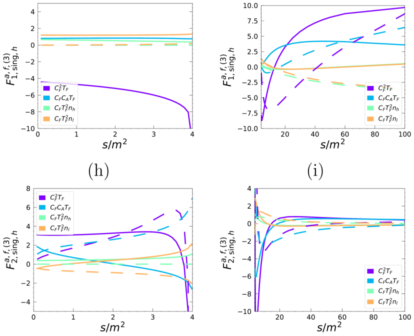


Anomaly contribution to massive three-loop form factors

[Fael,Lange,Schönwald,Steinhauser'23]



- Contribution of massive and massless singlet contributions to three loops
- Use Larin scheme for γ_5
- Computational techniques similar to other projects
- Perform Chiral-Ward-identity and other checks



Conclusion

- Many new results on inclusive B-decays have recently become available in the CRC, addressing
  - higher orders in the HQE
  - higher orders in perturbation theory
  - phenomenology
- Expect many more interesting results in FP2