2. DEQs & Canonical Bases

Bingen '24
Lorenzo Tomasi

We have seen that we can define a family of integrals

$$I(a_1,\dots,a_t;-b_1,\dots,-b_\sigma)=\int\prod_{\ell=1}^{L}\frac{d^Dk_\ell}{(2\pi)^D}\,\frac{S_1^{b_1}\cdots S_\sigma^{b_\sigma}}{D_1^{a_1}\cdots D_t^{a_t}}$$

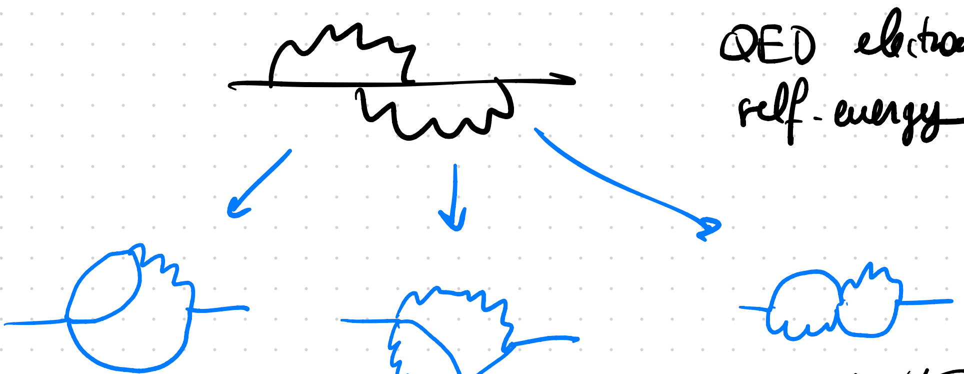
And derive linear relations among them, that allow us to reduce them to MASTER INTEGRALS

Clearly, if we consider a sector, and derive IBPs for it, we will naturally produce scalar products that might cancel some propagators, generating integrals that belong to subsectors

So when we derive IBPs, we deal in general with the FULL SUBTOPOLOGY TREE

IMPORTANT

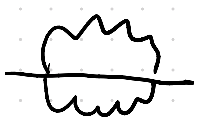
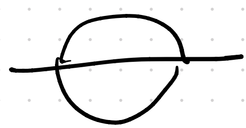
- at 1 loop every graph can contribute **AT MOST one master integral** (can be proven)
- at L-loops not true in general!;



QED electron self-energy

No MI   No MIs   1 MI

$\Downarrow$

...  ;  sunrise

$\Uparrow$ 2 MIs on top sector! $\Uparrow$ 2 MIs on top sector!

How do we compute all

these Integrals ?

THE "SIMPLEST" FEYNMAN INTEGRAL :

$$= \int \frac{d^D k}{k^2+m^2} = \Omega(D) \int_0^\infty \frac{dk\, k^{D-1}}{(k^2+m^2)}$$

$$= \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)} (m^2)^{\frac{D-2}{2}} \int_0^\infty \frac{dt\, t^{D-1}}{(t^2+1)}$$

$$= \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)} (m^2)^{\frac{D-2}{2}} \frac{1}{2} \int_0^\infty \frac{dx\, x^{\frac{D}{2}-1}}{(1+x)}$$

Use

$$B(x,y) = \int_0^\infty dt \frac{t^{x-1}}{(1+t)^{x+y}}$$

$$= \frac{\pi^{D/2}}{\cancel{\Gamma\left(\frac{D}{2}\right)}} (m^2)^{\frac{D-2}{2}} \frac{\cancel{\Gamma\left(\frac{D}{2}\right)}\, \Gamma\left(\frac{2-D}{2}\right)}{\Gamma(1)}$$

$$= \pi^{D/2} (m^2)^{\frac{D-2}{2}} \left\{ \Gamma\left(\frac{2-D}{2}\right) = \frac{2}{2-D} \frac{2}{4-D} \Gamma\left(\frac{6-D}{2}\right) \right\}$$

$$= \frac{4\pi^{D/2}\, \Gamma\left(\frac{6-D}{2}\right)}{(2-D)(4-D)} (m^2)^{\frac{D-2}{2}}$$

poles

expand series in $D \simeq 4$ for physical results

Tadpole is the only integral that can be computed so easily directly in loop momentum parametrization

In general, one has to integrate over angles among loop and external momenta, very cumbersome

(at least) TWO SOLUTIONS

Feynman Parameter representations

Direct integration method

Differential Equations method

"Indirect" method

extremely powerful

nice connection to Leading Singularities and Special Functions

DIFFERENTIAL EQUATIONS METHOD

- it's a direct consequence of IBPs
- it allows us to trivialise angular integration – derive diff eq wrt kinematical invariants!

Let's build this in general:

1] Feynman Integrals are homogeneous functions of the <u>external invariants</u> and <u>internal masses</u>

$$= f(\underbrace{s_{12}, s_{13}, \dots, s_{1N}, \dots}_{s_{ij}}, \underbrace{m_1^2, \dots m_M^2}_{m_j^2})$$

$s_{ij} = (p_i + p_j)^2 =$ MANDELSTAM INVARIANTS by dimensional analysis we have

$$f(\lambda^{2} s_{ij}, \lambda^{2} m_j^2) = \lambda^{+\alpha} f(s_{ij}, m_j^2)$$

so λ has dimension of $[m]$

homogeneous of degree $+\alpha$

2] Feynman integrals fulfil Integration-by-parts identities

$$\int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \left(\frac{S_1^{a_1} \cdots S_\sigma^{a_\sigma}}{D_1^{b_1} \cdots D_\tau^{b_\tau}} \right) = I(b_1, \dots, b_\tau, -a_1, \dots, -a_\sigma)$$

3] Feynman Integrals fulfil LORENTZ IDENTITIES

Feynman Ints are scalar functions, they should be invariant under a Lorentz transf of external momenta

Consider problem with "E" external momenta

Infinitesimally: $p^\mu \to p^\mu + \delta p^\mu = p^\mu + \delta\varepsilon^\mu{}_\nu p^\nu$

$$\underbrace{\delta\varepsilon^\mu{}_\nu = -\delta\varepsilon^\nu{}_\mu}_{\text{infinitesimal generators Lorentz group}}$$

$I(p_i + \delta p_i) \equiv I(p_i)$ Lorentz scalar

Expanding

$$I(p_i + \delta p_i) = I(p_i) + \sum_{j=1}^{E} \delta p_j{}^\mu \frac{\partial}{\partial p_j{}^\mu} I(p_i)$$

which using $\delta p^\mu = \delta\varepsilon^\mu{}_\nu p^\nu$ & equating $I(p_i)$ gives:

$$\delta\varepsilon^\mu{}_\nu \left[p_1^\nu \frac{\partial}{\partial p_1{}^\mu} + \ldots + p_E^\nu \frac{\partial}{\partial p_E{}^\mu} \right] I(p_i) = 0$$

all external momenta

now $\delta\varepsilon^{\mu}{}_{\nu}$ has 6 components (antisymmetric tensor)

there are up to 6 LIs, in fact using antisymm of $\delta\varepsilon^{\mu}{}_{\nu}$ we can write explicitly

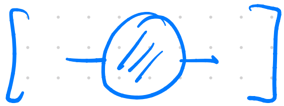
$$\Rightarrow \left[p_1^{\nu}\frac{\partial}{\partial p_1^{\mu}} - p_1^{\mu}\frac{\partial}{\partial p_1^{\nu}} + \dots + p_E^{\nu}\frac{\partial}{\partial p_E^{\mu}} - p_E^{\mu}\frac{\partial}{\partial p_E^{\nu}} \right] I(p_i) = 0 \qquad (*)$$

$$\Downarrow$$

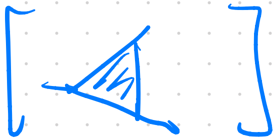
can be projected to scalar identities, by contracting it with all antisymmetric combinations of $p_{i\mu}\, p_{j\nu}$

$$(p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu}) \text{ etc}$$

EXAMPLES

- 2-point integrals 

  1 momentum p^μ, zero LIs

- 3-point functions 

  2 momenta, p_1^μ, p_2^μ, 1 LIs $(p_1^\mu p_2^\nu - p_1^\nu p_2^\mu) = p_{[1}^\mu p_{2]}^\nu$

- 4-point functions 

  3 momenta $p_1^\mu, p_2^\mu, p_3^\mu$; 3 LIs $\left\{ p_{[1}^\mu p_{2]}^\nu,\; p_{[1}^\mu p_{3]}^\nu,\; p_{[2}^\mu p_{3]}^\nu \right\}$

- 5-point functions 

  4 momenta $p_1^\mu \dots p_4^\mu$, $\underline{\underline{6 \text{ LIs}}}$! $\left\{ p_{[1}^\mu p_{2]}^\nu,\; p_{[1}^\mu p_{3]}^\nu,\; p_{[1}^\mu p_{4]}^\nu,\; p_{[2}^\mu p_{3]}^\nu,\; p_{[2}^\mu p_{4]}^\nu,\; p_{[3}^\mu p_{4]}^\nu \right.$

So only starting at 5-point, we can PROJECT OUT ALL LIs !

In practice, we act with (*) on the INTEGRAND and contract it with all combinations of $p^{\mu}_{[i}, p^{\nu}_{j]}$ and in this way obtain new identities between Feynman integrals

- one can prove that LIs are NOT linear independent from IBPs

- they are nevertheless helpful to find all relations among integrals "more easily"

- they help to understand how to derive differential equations for Feynm. Integrals in general

4] Differential Equations

Scalar Feynman Ints only depend on $s_{ij} = (p_i + p_j)^2$

not on the momenta themselves $= 2 p_i \cdot p_j$

massless case!

CHAIN RULE

$$\frac{\partial}{\partial p_{i\mu}} = \sum_j 2(p_{i\mu} + p_{j\mu}) \frac{\partial}{\partial s_{ij}}$$

substituting this into general Lorentz Id

$$\underbrace{\left(p_1^\nu \frac{\partial}{\partial p_1^\mu} - p_1^\mu \frac{\partial}{\partial p_1^\nu} + \dots + p_E^\nu \frac{\partial}{\partial p_E^\mu} - p_E^\mu \frac{\partial}{\partial p_E^\nu} \right)}_{} I = 0$$

this bracket is identically zero

the LIs become trivial identities !

BUT remember, non-trivial at the

INTEGRAND Level !

the crucial point is that, if I want to express now $\frac{\partial}{\partial s_{ij}} = f\left(p^\mu_\nu \frac{\partial}{\partial p_j^\mu}\right)$ there is a redundancy hidden in Lorentz ids! I can always add a combination of LIs to $\frac{\partial}{\partial s_{ij}}$, it will not change final result, since any LIs applied on Feynman Ints gives zero!

EXAMPLES

- 2 point : p^μ ; $p^\mu p_\mu = s$ $\Rightarrow$ $\frac{\partial}{\partial p^\mu} = \frac{\partial s}{\partial p^\mu} \frac{\partial}{\partial s} = 2 p_\mu \frac{\partial}{\partial s}$

$$\text{so } p^\mu \frac{\partial}{\partial p^\mu} = 2s \frac{\partial}{\partial s} \Rightarrow \frac{\partial}{\partial s} = \frac{1}{2s} p^\mu \frac{\partial}{\partial p^\mu}$$

zero LIs, no ambiguity!

- 3 point: p_1^μ, p_2^μ 2 momenta

$$\left\{\begin{array}{l} 3\ s_{ij} \Rightarrow p_1^2,\ p_2^2,\ s_{12} = (p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1\cdot p_2 \\ 1\ \text{LI} \Rightarrow (p_{1\mu}p_{2\nu} - p_{1\nu}p_{2\mu})\ (*) \end{array}\right.$$

Invertible modulo $\underline{\underline{\text{LIs!}}}$

$=$ 4 combinations $p_i^\mu \dfrac{\partial}{\partial p_{j\mu}}$ $\quad i,j = \{1,2\}$

$$\frac{\partial}{\partial p_{1\mu}} = \frac{\partial p_1^2}{\partial p_{1\mu}}\frac{\partial}{\partial p_1^2} + \frac{\partial s_{12}}{\partial p_{1\mu}}\frac{\partial}{\partial s_{12}}$$

$$\underline{= 2p_1^\mu \frac{\partial}{\partial p_1^2} + 2(p_1^\mu + p_2^\mu)\frac{\partial}{\partial s_{12}}}$$

$$\boxed{\begin{aligned} p_{1\mu}\frac{\partial}{\partial p_{1\mu}} &= 2p_1^2\frac{\partial}{\partial p_1^2} + \left(s_{12} + p_1^2 - p_2^2\right)\frac{\partial}{\partial s_{12}} \qquad (1) \\ p_{2\mu}\frac{\partial}{\partial p_{1\mu}} &= \left(s_{12} - p_1^2 - p_2^2\right)\frac{\partial}{\partial p_1^2} + \left(s_{12} + p_2^2 - p_1^2\right)\frac{\partial}{\partial s_{12}} \qquad (2) \end{aligned}}$$

$$\frac{\partial}{\partial p_{2\mu}} = 2 p_{2\mu} \frac{\partial}{\partial p_2^2} + 2(p_1^\mu + p_2^\mu) \frac{\partial}{\partial s_{12}}$$

$$\boxed{\begin{aligned} p_{1\mu} \frac{\partial}{\partial p_{2\mu}} &= (s_{12} - p_1^2 - p_2^2) \frac{\partial}{\partial p_2^2} + (s_{12} + p_1^2 - p_2^2) \frac{\partial}{\partial s_{12}} \quad (3) \\ p_{2\mu} \frac{\partial}{\partial p_{2\mu}} &= 2 p_2^2 \frac{\partial}{\partial p_2^2} + (s_{12} + p_2^2 - p_1^2) \frac{\partial}{\partial s_{12}} \quad (4) \end{aligned}}$$

system not invertible for $\left\{ \frac{\partial}{\partial s_{12}}, \frac{\partial}{\partial p_1^2}, \frac{\partial}{\partial p_2^2} \right\}$!

LI eqs

$$(p_{1\mu} p_{2\nu} - p_{2\mu} p_{1\nu}) \left(p_1^\mu \frac{\partial}{\partial p_1{}_\nu} - p_1^\nu \frac{\partial}{\partial p_1{}_\mu} + p_2^\mu \frac{\partial}{\partial p_2{}_\nu} - p_2^\nu \frac{\partial}{\partial p_2{}_\mu} \right) =$$

$$= 2 p_1^2\, p_{2\mu} \frac{\partial}{\partial p_{1\mu}} - 2\, p_1 \cdot p_2\, p_{1\mu} \frac{\partial}{\partial p_{1\mu}} + 2\, p_1 \cdot p_2\, p_{2\mu} \frac{\partial}{\partial p_{2\mu}} - 2 p_2^2\, p_{1\mu} \frac{\partial}{\partial p_{2\mu}}$$

LI becomes

$$2p_1^2\, p_{2\mu}\frac{\partial}{\partial p_{1\mu}} - 2p_2^2\, p_{1\mu}\frac{\partial}{\partial p_{2\mu}} +$$

$$+\underbrace{2p_1\cdot p_2}_{s_{12}-p_1^2-p_2^2}\left[p_{2\mu}\frac{\partial}{\partial p_{2\mu}} - p_{1\mu}\frac{\partial}{\partial p_{1\mu}}\right] = 0$$

when applied on a Feynman (SCALAR!) integral

One of the $p_{i\mu}\frac{\partial}{\partial p_{j\mu}}$ can always be removed in terms of the others !

and the system becomes invertible, we can express all $\left\{\frac{\partial}{\partial s_{12}}, \frac{\partial}{\partial p_1^2}, \frac{\partial}{\partial p_2^2}\right\}$ in terms of the $p_{i\mu}\frac{\partial}{\partial p_{j\mu}}$

In particular, I can choose to remove difference and keep only

$$p_{1\mu}\frac{\partial}{\partial p_{2\mu}}\,,\quad p_{2\mu}\frac{\partial}{\partial p_{1\mu}}\,,\quad \left(p_{1\mu}\frac{\partial}{\partial p_{1\mu}}+p_{2\mu}\frac{\partial}{\partial p_{2\mu}}\right)$$

which means using 2, 3, 1 + 4

$$p_{2\mu}\frac{\partial}{\partial p_{1\mu}} = \left(s_{12}-p_1^2-p_2^2\right)\frac{\partial}{\partial p_1^2} + \left(s_{12}+p_2^2-p_1^2\right)\frac{\partial}{\partial s_{12}} \quad (2)$$

$$p_{1\mu}\frac{\partial}{\partial p_{2\mu}} = \left(s_{12}-p_1^2-p_2^2\right)\frac{\partial}{\partial p_2^2} + \left(s_{12}+p_1^2-p_2^2\right)\frac{\partial}{\partial s_{12}} \quad (3)$$

$$\left(p_{1\mu}\frac{\partial}{\partial p_{1\mu}}+p_{2\mu}\frac{\partial}{\partial p_{2\mu}}\right) = 2\left(p_1^2\frac{\partial}{\partial p_1^2}+p_2^2\frac{\partial}{\partial p_2^2}\right)+2s_{12}\frac{\partial}{\partial s_{12}} \quad (1)+(4)$$

if $p_1^2=0$, $p_2^2=0$ recover case of exercise

· 4 Point , 3 momenta $p_1^\mu, p_2^\mu, p_3^\mu \Longrightarrow$

$p_1^2, p_2^2, p_3^2, s_{12}, s_{13}, s_{23}$ 6 variables in total

$p_{i\mu} \frac{\partial}{\partial p_{j\mu}}$ = 9 combinations

there are indeed 3 LIs $[p_{1\mu} p_{2\nu}]$ $[p_{1\mu} p_{3\nu}]$ $[p_{2\mu} p_{3\nu}]$ etc

[etc for higher point]

EULER SCALING RELATION

Since Feyn Ints are homogeneous functions, the derivatives are not all independent

$$I(\lambda^2 s_{ij}, \lambda^2 m_j^2) = f(\lambda, s_{ij}, m_j^2)$$

$$\shortparallel$$

$$\lambda^a I(s_{ij}, m_j^2)$$ homogeneity

Differentiating wrt λ we find:

$$\frac{\partial}{\partial\lambda} I(\lambda^2 s_{ij}, \lambda^2 m_j^2) = \frac{\partial}{\partial\lambda}\left(\lambda^\alpha I(s_{ij}, m_j^2)\right)$$

$\Longrightarrow$

$$\sum_{kn} s_{kn} \frac{\partial}{\partial s_{kn}} I(\lambda^2 s_{ij}, \lambda^2 m_j^2) + \sum_k m_k^2 \frac{\partial}{\partial m_k^2} I(\lambda^2 s_{ij}, \lambda^2 m_j^2)$$

$$= \frac{\alpha}{2} \lambda^\alpha I(s_{ij}, m_j^2)$$

For $\lambda = 1$ it gives SCALING RELATION

$$\left[\sum_{kn} s_{kn} \frac{\partial}{\partial s_{kn}} + \sum_k m_k^2 \frac{\partial}{\partial m_k^2}\right] I(s_{ij}, m_j^2) = \frac{\alpha}{2} I(s_{ij}, m_j^2)$$

Dimensional scaling

$$\alpha = LD + 2S - 2R$$

\# Loops

powers scal prod

tot powers of propagators

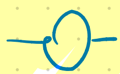
this reflects the fact that we can always put one scale $= 1$ and only consider dimensionless ratios!

$\lambda^2 = \frac{1}{m_1^2}$ for example removes dependence on one of the masses.

EXAMPLE 1 loop bubble

Consider $I(a,b) = \int \frac{d^D k}{(k^2+m^2)^a\,((k-p)^2+m^2)^b}$

two masters $I(1,1)$ & $I(1,0)$

$I(1,1) =$ 

$I(1,0) =$ 

use $\frac{\partial}{\partial p^2} = \frac{1}{2p^2}\left[p^\mu \frac{\partial}{\partial p^\mu}\right]$

$$\frac{\partial}{\partial p^2}\int \frac{d^Dk}{\pi^{D/2}}\frac{1}{D_1 D_2} = \frac{1}{2p^2}\int \frac{d^Dk}{\pi^{D/2}}\left\{\frac{1}{D_2^2} - \frac{1}{D_1 D_2} - p^2\frac{1}{D_1 D_2^2}\right\}$$

which means we can write

$\Rightarrow$

$$\frac{\partial}{\partial p^2}\underbrace{I(1,1)}_{B(p^2)} = \frac{1}{2p^2}\left[I(0,2) - I(1,1)\right]$$

$$-\frac{1}{2}\,I(1,2)$$

and using

$$I(0,2) = -\frac{(D-2)}{2m^2}\,I(1,0)$$

$$I(1,2) = -\frac{(D-2)}{2m^2(p^2+4m^2)}\,I(1,0) - \frac{(D-3)}{p^2+4m^2}\,I(1,1)$$

$$\boxed{\frac{d}{dp^2} I(1,1) = \frac{1}{2}\left(\frac{D-3}{p^2+4m^2} - \frac{1}{p^2}\right) I(1,1) - \frac{D-2}{p^2(p^2+4m^2)} I(1,0)}$$

the other derivative is very simple

$$\frac{\partial}{\partial m^2} I(1,1) = -I(1,2) - I(2,1) = -2I(1,2)$$

$$= -\frac{(D-2)}{m^2(p^2+4m^2)} I(1,0) - \frac{2(D-3)}{p^2+4m^2} I(1,1)$$

such that we can easily verify scaling relation

$$\boxed{\left(p^2\frac{\partial}{\partial p^2} + m^2\frac{\partial}{\partial m^2}\right) I(1,1) = \left(\frac{D-4}{2}\right) I(1,1)}$$

SCALING RELATION

CONCLUSION

Given a family of L-loop integrals

- Use IBPs (and Lorentz Ids) to reduce them to N master integrals M_i

- Differentiate wrt all independent s_{ij} and m_j^2

$x_k = (s_{ij}, m_j^2)$ all variables

$$\frac{\partial M_i}{\partial x_k} = A_{ij}^{(x_k)} M_j \Rightarrow \frac{\partial}{\partial x_k} \vec{M} = [A_k] \vec{M}$$

$N \times N$ matrix with rational coefficients in x & D

$\Rightarrow$ obvious because IBPs (and LIs) only produce rational coefficients!

- Useful to recollect all equations in form of **total differential** (language of differential forms)

$$d\vec{M} = \sum_{k=1}^{N} \frac{\partial \vec{M}}{\partial x_k} dx_k$$

$$A = \sum_{k=1}^{N} A_k \, dx_k$$

matrix-valued one form !

so we can write

$$\boxed{d\vec{M} = A \vec{M}} \Rightarrow (d - A)\vec{M} = 0$$

the system of differential Eqs must be integrable

$\Rightarrow$ the result $(\vec{M})$ must be a "proper function"

which means $\dfrac{\partial \vec{M}}{\partial x_i \partial x_j} = \dfrac{\partial \vec{M}}{\partial x_j \partial x_i} \Leftrightarrow d^2\vec{M} = 0$

start from $\frac{\partial \vec{M}}{\partial x_i} = A_i \vec{M}$ & $\frac{\partial \vec{M}}{\partial x_j} = A_j \vec{M}$

$$\frac{\partial^2 \vec{M}}{\partial x_j \partial x_i} = \frac{\partial A_i}{\partial x_j} \vec{M} + A_i \overbrace{\frac{\partial \vec{M}}{\partial x_j}}^{A_j \vec{M}} \qquad (1)$$

$$\frac{\partial^2 \vec{M}}{\partial x_i \partial x_j} = \frac{\partial A_j}{\partial x_i} \vec{M} + A_j \underbrace{\frac{\partial \vec{M}}{\partial x_i}}_{A_i \vec{M}} \qquad (2)$$

- $(1)-(2)$ gives

$$\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right)\vec{M} + (A_i A_j - A_j A_i)\vec{M} = 0$$

$$\underbrace{\left[\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right) + [A_i, A_j]\right]}_{\text{INTEGRABILITY CONDITION ON MATRICES } \underline{A_i} \, !} \vec{M} = 0$$

can also be rewritten in language of DIFFERENTIAL FORMS

A one form $\quad A = A_i \, dx_i \quad A_i = \frac{\partial A}{\partial x_i}$

then its total differential = Exterior derivative

$$dA = \frac{\partial A}{\partial x_j \partial x_i} \underbrace{dx_j \wedge dx_i}_{\text{antisymmetric}}$$

antisymmetrize:

$$= \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right) dx_j \wedge dx_i$$

similarly

$$A \wedge A = A_i A_j \underbrace{dx_i \wedge dx_j}_{\text{antisymmetric}}$$

$$= [A_i, A_j] \, dx_i \wedge dx_j$$

So in diff. forms language INTEGRABILITY

$$\boxed{dA - A \wedge A = 0}$$