Summer School 2024 on Particle Physics after the Higgs Discovery

Modern methods for scattering amplitudes

Sheet 01: Integration by part identities and differential equations for Feynman integrals

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Introduction

In the lecture you have seen that Feynman Integrals satisfy Integration-By-Part (IBP) identities, relating integrals with different power of numerators and denominators via lienar relations with coefficient which are rational functions of the kinematic invariants and of the space time dimensions D. As a consequence, only a small and finite set of *master integrals* needs to be computed. In this exercise you will study the case of a one loop, three point diagrams, with internal massive lines.

1 $gg \to H$ via a top quark loop

We consider the production of a Higgs boson in gluon-gluon annihilation at 1 loop in QCD. We write

$$g(p_1) + g(p_2) \to H(q)$$

with $p_1^2 = p_2^2 = 0$ and $q^2 = 2p_1 \cdot p_2 = m_H^2$. The quark running in the loop has mass m_t. We assume that the gluons are in the cyclic axial gauge

$$\epsilon_1 \cdot p_1 = \epsilon_2 \cdot p_2 = \epsilon_1 \cdot p_2 = \epsilon_2 \cdot p_1 = 0\,.$$

It can be shown that the amplitude, $\mathcal{S}$, for this process, takes the form

$$\mathcal{S} = \delta_{a_1 a_2} g_H \alpha_S\, \mathcal{A} \tag{1}$$

and

$$\mathcal{A} = 2\left\{\int \frac{d^Dk}{(2\pi)^D}\left[\frac{4(\epsilon_1\cdot k)(\epsilon_2\cdot k)}{D_1D_2D_3}\right] - \epsilon_1\cdot\epsilon_2\left[\frac{m_H^2}{2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{D_1D_2D_3} + \int\frac{d^Dk}{(2\pi)^D}\frac{1}{D_1D_3}\right]\right\}, \tag{2}$$

where $D_1 = k^2 - m_t^2, D_2 = (k-p_1)^2 - m_t^2, D_3 = (k-p_1-p_2)^2 - m_t^2$. Moreover, it can be shown that

$$\int \frac{d^Dk}{(2\pi)^D}\frac{(\epsilon_1\cdot k)(\epsilon_2\cdot k)}{D_1D_2D_3} = \epsilon_1\cdot\epsilon_2 F, \tag{3}$$

where

$$F = I(1,1,1)\left(\frac{8m_t^2}{D-2} - m_H^2\right) + \frac{8I(1,-1,1)}{(D-2)m_H^2} - \frac{8I(1,0,0)}{(D-2)m_H^2} + \left(\frac{8}{D-2} - 2\right)I(1,0,1) \tag{4}$$

Now let's consider the rank 1 integral

$$I(1,-1,1) = \int\frac{d^Dk}{(2\pi)^D}\frac{(k-p_1)^2 - m_t^2}{D_1D_3} \tag{5}$$

1. perform the shift $k \to -k + p_1 + p_2$ and show that

$$I(1,-1,1) = I(1,0,0) - \frac{m_H^2}{2} I(1,0,1)\,.$$

Substituting this into (4) and using $D = 4 - 2\epsilon$ we finally obtain

$$F = -\left(m_H^2 - \frac{4m_t^2}{1-\epsilon}\right) I(1,1,1) + \frac{2\epsilon}{1-\epsilon} I(1,0,1) \tag{6}$$

1.1 The master integrals

The scalar integrals we identified in the first part of the exercise can be incorporated into a single *integral family*

$$I(a_1, a_2, a_3) = \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{\mathrm{e}^{\epsilon\gamma_E}}{[k^2 - m_t^2]^{a_1}[(k-p_1)^2 - m_t^2]^{a_2}[(k-p_1-p_2)^2 - m_t^2]^{a_3}}. \tag{7}$$

All external particles are considered incoming and the kinematics is specified as follows:

$$p_1^2 = 0,\ p_2^2 = 0,\ p_1 \cdot p_2 = m_H^2/2\,.$$

Notice the slight change of notation with respect to the first part of the exercise.

1. Starting from (7) and the general IBP formula

$$\int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \left[\frac{\partial}{\partial k^\mu} \frac{u^\mu}{[k^2 - m_t^2]^{a_1}[(k-p_1)^2 - m_t^2]^{a_2}[(k-p_1-p_2)^2 - m_t^2]^{a_3}}\right] = 0 \tag{8}$$

derive the relevant IBP identities for $u^\mu = k^\mu, p_1^\mu, p_2^\mu$. You should find

$$u^\mu = p_1^\mu: \quad (-a_1 + a_2)\mathbb{1} + a_1 2^- 1^+ - a_2 1^- 2^+ - a_3 1^- 3^+ + a_3 2^- 3^+ + a_3 m_H^2 3^+ \tag{9}$$

$$u^\mu = p_2^\mu: \quad (-a_2 + a_3)\mathbb{1} + a_2 3^- 2^+ - a_1 2^- 1^+ + a_1 3^- 1^+ - a_3 2^- 3^+ - a_1 m_H^2 1^+ \tag{10}$$

$$\begin{aligned} u^\mu = k^\mu: \quad & (\mathrm{D} - 2a_1 - a_2 - a_3)\,\mathbb{1} - a_2 1^- 2^+ - a_3 1^- 3^+ - 2a_1 1^+ m_t^2 \\ & - 2a_2 2^+ m_t^2 - 2a_3 3^+ m_t^2 + a_3 m_H^2 3^+ \end{aligned} \tag{11}$$

where the three equations should be interpreted as operator equations acting on a generic integral o f the family under consideration. In particular, $\mathbb{1}$ is the identity operator, while $j^\pm$ increases or reduces the power of the corresponding propagator

$$\begin{aligned} \mathbb{1} I(a_1, a_2, a_3) &= I(a_1, a_2, a_3) \\ j^\pm I(a_1, ..., a_j, ..., a_3) &= I(a_1, ..., a_j \pm 1, ..., a_3) \quad \text{with} \quad j = 1,2,3\,. \end{aligned} \tag{12}$$

2. Using the ibps derived above, convince yourself that the integral family defined in (7) has three master integrals, which can be chosen to be

$$\mathbf{I} = \{I(0,0,1), I(1,0,1), I(1,1,1)\}. \tag{13}$$

For example: try to reduce triangles with a dotted propagator $\{I(2,1,1), I(1,2,1), I(1,1,2)\}$ to the scalar triangle $I(1,1,1)$ and sub-topologies.

3. In general, since the master integrals are scalar, they can only depend on all possible scalar products $p_i \cdot p_j$ between the external momenta. Show that

$$\frac{\partial}{\partial m_H^2} = \frac{1}{2m_H^2}\left(p_1^\mu \frac{\partial}{\partial p_1^\mu} + p_2^\mu \frac{\partial}{\partial p_2^\mu}\right). \tag{14}$$

4. Apply the differential operator from the previous step to the master integrals (13) and compute the derivative with respect to m_H^2. Compute also the derivative of the master integrals with respect to m^2. Show that the result of the differentiation can be written in terms of integrals that belong to family (7),

$$\partial_{m_H^2} I(0,0,1) = \frac{I(-1,0,2)}{2m_H^2} - \frac{I(0,0,1)}{2m_H^2} - \frac{I(0,0,2)}{2} \tag{15}$$

$$\partial_{m_H^2} I(1,0,1) = \frac{I(0,0,2)}{2m_H^2} - \frac{I(1,0,1)}{2m_H^2} - \frac{I(1,0,2)}{2} \tag{16}$$

$$\partial_{m_H^2} I(1,1,1) = \frac{I(0,1,2)}{2m_H^2} + \frac{I(0,2,1)}{2m_H^2} - \frac{I(1,1,1)}{m_H^2} - \frac{\mathrm{I}(1,1,2)}{2} \tag{17}$$

$$\partial_{m_t^2} I(0,0,1) = \mathrm{I}(0,0,2) \tag{18}$$

$$\partial_{m_t^2} I(1,0,1) = \mathrm{I}(1,0,2) + \mathrm{I}(2,0,1) \tag{19}$$

$$\partial_{m_t^2} I(1,1,1) = \mathrm{I}(1,1,2) + \mathrm{I}(1,2,1) + \mathrm{I}(2,1,1). \tag{20}$$

5. Use IBP identities (11), (9), (10) to recast the result of the differentiation in terms of the master integrals (13). The goal is to obtain an expression of the form

$$\partial_{m_H^2} \mathbf{I} = M_{m_H^2} \mathbf{I}, \quad \partial_{m_t^2} \mathbf{I} = M_{m_t^2} \mathbf{I}. \tag{21}$$

You should get

$$M_{s_{12}} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{d-2}{(4m_t^2 - m_H^2)m_H^2} & \frac{4m_H^2 - 4m_t^2 - d\,m_H^2}{2(4m_t^2 - m_H^2)m_H^2} & 0 \\ \frac{2-d}{2m_t^2(4m_t^2 - m_H^2)m_H^2} & \frac{d-3}{(4m_t^2 - m_H^2)m_H^2} & -\frac{1}{m_H^2} \end{pmatrix}, \quad M_{m_t^2} = \begin{pmatrix} \frac{d-2}{2m_t^2} & 0 & 0 \\ \frac{2-d}{m_t^2(4m_t^2 - m_H^2)} & \frac{2(d-3)}{4m_t^2 - m_H^2} & 0 \\ \frac{d-2}{2m_t^4(4m_t^2 - m_H^2)} & \frac{(3-d)}{m_t^2(4m_t^2 - m_H^2)} & \frac{d-4}{2m_t^2} \end{pmatrix} \tag{22}$$

Hint: You can use the reduction identities for the bubble derived in the lecture and the following reduction

6. Compute $(m_H^2 \partial_{m_H^2} + m_t^2 \partial_{m_t^2})\mathbf{I}$. What does the result tell you?