Summer School 2024 on Particle Physics after the Higgs Discovery

Modern methods for scattering amplitudes

Sheet 02: Canonical basis and Iterated integrals

Introduction

In Sheet 1 we derived the system of differential equations for the basis

$$\mathbf{I} = \{I(0,0,1), I(1,0,1), I(1,1,1)\}. \tag{1}$$

We also determined a set of boundary conditions which is sufficient to fix the solution, provided the knowledge of the tadpole integral

$$I(0,0,1) = -e^{\epsilon\gamma}\Gamma(\epsilon-1)\left(m^2\right)^{1-\epsilon}. \tag{2}$$

. In this exercise sheet we will look into the explicit solution of the differential equation system.

1 Boundary conditions

1. To determine the unique physical solution to these differential equations we need to specify suitable boundary terms. Focus on the differential equation of m_H^2 and identify its poles. Using the fact that the chosen master integrals (1) are finite in the limit $m_H^2 \to 0$, argue that at that limit the relevant boundary terms for $\{I(1,0,1), I(1,1,1)\}$ are

$$I(1,0,1)|_{m_H^2\to 0} = -\frac{(\epsilon-1)\mathrm{I}(0,0,1)}{m_t^2} \tag{3}$$

$$I(1,1,1)|_{m_H^2\to 0} = \frac{(\epsilon-1)\epsilon\mathrm{I}(0,0,1)}{2\left(m_t^2\right)^2} \tag{4}$$

where

$$I(0,0,1) = -e^{\epsilon\gamma}\Gamma(\epsilon-1)\left(m_t^2\right)^{1-\epsilon}. \tag{5}$$

Canonical basis and integral solutions

1. Instead of working with the integrals basis (1), switch to the following basis of master integrals,

$$\begin{aligned} f_1 =& \epsilon(m_t^2)^\epsilon I(2,0,0) \\ f_2 =& \epsilon m_H^2(m_t^2)^\epsilon\sqrt{1-\frac{4m_t^2}{m_H^2}}I(1,0,2) \\ f_3 =& \epsilon^2 m_H^2(m_t^2)^\epsilon I(1,1,1). \end{aligned} \tag{6}$$

Use IBP identities to express $I(2,0,0), I(1,0,2)$ in terms of $I(0,0,1), I(1,0,1)$, so that $\mathbf{f} = \mathbf{TI}$, where $\mathbf{f} = \{f_1, f_2, f_3\}$ and $\mathbf{T}$ is the transformation matrix that connects the two bases.

2. The square root which appears in the differential equation of the previous step can be rationalised by the change of variables

$$\frac{m_t^2}{-m_H^2} = \frac{x}{(1-x)^2}. \tag{7}$$

Perform this change of variables assuming for $0 < x < 1$: what does this correspond in terms of the original variable m_H^2, which analytical region does it represent? Verify that in this new variable the differential equation takes the form

$$M_x = \epsilon \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{x} & \frac{1}{x} - \frac{2}{x+1} & 0 \\ 0 & -\frac{1}{x} & 0 \end{pmatrix} \tag{8}$$

Hint: To avoid having to deal with square root, start by changing variables and then rotate the basis. Notice that the new basis is dimensionless, and the variable x has no mass dimension: use this to check your result.

3. Assuming a power series solution for (8), i.e. $f_i = \sum_{n\geq 0} \epsilon^n f_i^{[n]}$, solve (8) using the boundary conditions that you obtained in previous steps. Express your solution in terms of Harmonic Polylogarithms. You should find

$$f_1 = 1 + \frac{\pi^2\epsilon^2}{12} \tag{9}$$

$$f_2 = -\epsilon\, G(0;x) + \epsilon^2 \left(\frac{\pi^2}{6} + 2G(-1,0;x) - G(0,0;x) \right) \tag{10}$$

$$f_3 = \epsilon^2 G(0,0;x) \tag{11}$$

where $G(-1,0,1) = -\pi^2/12$. The following expansion might be useful

$$e^{\epsilon\gamma_E}\,\Gamma(\epsilon+1) = 1 + \frac{\epsilon^2\pi^2}{12} - \frac{1}{3}\zeta(3)\epsilon^3 + \mathcal{O}(\epsilon^4) \tag{12}$$

4. **Bonus** What value of the variable x correspond to the region above the threshold $m_H^2 > 4m_t^2$? Perform analytic continuation of your result using Feynman prescription.