2024 P3H school - From amplitudes to cross sections and events- Exercises

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September 30 - October 4, 2024

1 The QCD infrared limits

The ratio between the amplitude for the process $\gamma^* \to q\,\bar{q}\,g$ and that for the process $\gamma^* \to q\,\bar{q}$ can be written as

$$\frac{|M_R|^2}{|M_B|^2} = \frac{2g_{\rm S}^2 C_{\rm F}}{Q^2}\,\frac{x_1^2 + x_2^2 - \epsilon\,(2 - x_1 - x_2)^2}{(1 - x_1)\,(1 - x_2)}, \tag{1}$$

where

$$x_1 = \frac{2p_1 \cdot q}{q^2} \qquad x_2 = \frac{2p_2 \cdot q}{q^2}, \tag{2}$$

p_1, p_2, p_3 and q are the momenta of the quark, antiquark, gluon and photon, respectively, and $Q^2 = q^2$. Since the ratio of the two phase spaces reads

$$\begin{aligned}\frac{d\Phi_3}{d\Phi_2} = \frac{\left(Q^2\right)^{1-\epsilon} dx_1\,dx_2}{(4\pi)^{2-\epsilon}\,\Gamma(1-\epsilon)} &\left[(1 - x_1)\,(1 - x_2)\,(x_1 + x_2 - 1)\right]^{-\epsilon}\theta(x_1)\,\theta(x_2)\\ &\times \theta(2 - x_1 - x_2)\,\theta((1 - x_1)\,(1 - x_2)\,(x_1 + x_2 - 1))\,,\end{aligned} \tag{3}$$

we can write the decay ratio

$$dK_{\rm R} = \frac{\displaystyle\int d\Phi_3\,|M_R|^2}{\displaystyle\int d\Phi_2\,|M_B|^2} \tag{4}$$

as

$$\begin{aligned}dK_{\rm R} = \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\rm S}}{2\pi}\,\frac{C_{\rm F}}{Q^{2\epsilon}}\,dx_1\,dx_2 &\,\frac{x_1^2 + x_2^2 - \epsilon\,(2 - x_1 - x_2)^2}{(1 - x_1)^{1+\epsilon}\,(1 - x_2)^{1+\epsilon}\,(x_1 + x_2 - 1)^\epsilon}\\ &\times \theta(x_1)\,\theta(x_2)\,\theta(2 - x_1 - x_2)\,\theta((1 - x_1)\,(1 - x_2)\,(x_1 + x_2 - 1))\,.\end{aligned} \tag{5}$$

We now want to study the limit where the momentum p_3 of the gluon becomes collinear to the momentum p_1 of the quark or the momentum p_2 of the antiquark. The first limit is parametrized by $(p_1+p_3)^2$ becoming much smaller than the hard scale Q^2 of the decay while being smaller than $(p_2+p_3)^2$, which is instead the small scale of the limit where the momentum of the gluon becomes collinear to the momentum p_2 of the antiquark. Since the condition

$$(p_1+p_3)^2 < (p_2+p_3)^2 \tag{6}$$

is equivalent to the condition $x_1 < x_2$, we can exploit the symmetry of the integrand under the exchange $x_1 \leftrightarrow x_2$ to rewrite dK_{R} as

$$\begin{aligned} dK_{\mathrm{R}} = {} & 2\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{C_{\mathrm{F}}}{Q^{2\epsilon}}\,dx_1\,dx_2\,\frac{x_1^2+x_2^2-\epsilon\,(2-x_1-x_2)^2}{(1-x_1)^{1+\epsilon}\,(1-x_2)^{1+\epsilon}\,(x_1+x_2-1)^\epsilon} \\ & \times\theta(x_1)\,\theta(x_2)\,\theta(2-x_1-x_2)\,\theta((1-x_1)\,(1-x_2)\,(x_1+x_2-1)) \\ & \times\theta(x_2-x_1)\,, \end{aligned} \tag{7}$$

such that we can study both the regions at the same time.

1.1 The Sudakov parametrization

In the collinear limit, it is useful to introduce the variable

$$t = (p_1+p_3)^2 \tag{8}$$

and parametrize the momenta p_1, p_2 and p_3 as

$$\begin{cases} p_1 = z\,\bar{p}_1 + A\,p_{\mathrm{T}} + B\,\bar{p}_2 \\ p_2 = C\,\bar{p}_2 \\ p_3 = D\,\bar{p}_1 + E\,p_{\mathrm{T}} + F\,\bar{p}_2, \end{cases} \tag{9}$$

where $\bar{p}_1$ and $\bar{p}_2$ are two massless vectors defined such that $q = \bar{p}_1 + \bar{p}_2$ and p_{T} is a vector orthogonal to both of them.

Exercise 1: *Express the parameters A, B, C, D, E and F in term of z, t, Q^2 and $|\vec{p}_{\mathrm{T}}|$.*

1.2 The leading-power result

Substituting the expressions for p_1, p_2 and p_3 found above in the definitions of x_1 and x_2, we find

$$x_1 = z + (1-z)\,\frac{t}{Q^2} \qquad x_2 = 1 - \frac{t}{Q^2}. \tag{10}$$

Exercise 2: *Substitute the above expressions for x_1 and x_2 in the expression for dK_{R} and truncate it at leading power in the t/Q^2 expansion. NB: Be careful in the truncation: $1-z$ is not guaranteed to be much larger than t/Q^2.*

1.3 The integration over z

The leading power result obtained in the above section reads

$$
\begin{aligned}
dK_{\mathrm{R}} = 2\,\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{dt}{t^{1+\epsilon}}\Bigg\{&\frac{dz}{z^{\epsilon}\,(1-z)^{\epsilon}}\,C_{\mathrm{F}}\left[\frac{1+z^2}{1-z}-\epsilon\,(1-z)\right]\theta\!\left(1-z-\frac{t}{Q^2}\right)\\
&+\mathcal{O}\!\left(\frac{t}{Q^2}\right)\Bigg\}.
\end{aligned}
\tag{11}
$$

In order to integrate it over z, we start by writing its integral as

$$
K_{\mathrm{R}} = 2\,\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{dt}{t^{1+\epsilon}}\left[K_{\mathrm{R}}^{(1)}+K_{\mathrm{R}}^{(2)}+\mathcal{O}\!\left(\frac{t}{Q^2}\right)\right],
\tag{12}
$$

where the two integrals $K_{\mathrm{R}}^{(1)}$ and $K_{\mathrm{R}}^{(2)}$ read respectively

$$
K_{\mathrm{R}}^{(1)} = 2\,C_{\mathrm{F}}\int_0^1 \frac{dz}{z^{\epsilon}\;(1-z)^{1+\epsilon}}\,\theta\!\left(1-z-\frac{t}{Q^2}\right)
\tag{13}
$$

and

$$
K_{\mathrm{R}}^{(2)} = C_{\mathrm{F}}\int_0^1 \frac{dz}{z^{\epsilon}\;(1-z)^{\epsilon}}\left[-1-z-\epsilon\,(1-z)\right]\theta\!\left(1-z-\frac{t}{Q^2}\right).
\tag{14}
$$

Exercise 3: *Compute the integrals $K_{\mathrm{R}}^{(1)}$ and $K_{\mathrm{R}}^{(2)}$ and use the equation*

$$
\frac{1}{x^{1-\epsilon}} = \frac{\delta(x)}{\epsilon}+\mathcal{L}_0(x)+\epsilon\,\mathcal{L}_1(x)+\mathcal{O}\big(\epsilon^2\big)\,,
\tag{15}
$$

where

$$
\mathcal{L}_n(x) = \lim_{\beta\to 0^+}\left[\frac{\log^n x}{x}\,\theta(x-\beta)+\frac{\log^{n+1}\beta}{n+1}\,\delta(x-\beta)\right] = \left(\frac{\log^n x}{x}\right)_+,
\tag{16}
$$

to expand K_{R} up to the finite part in ϵ (at leading power in t/Q^2).

1.4 The method of regions

Instead of computing the exact integrals and then expanding them with respect to t/Q^2 as we did above, we can adopt a different approach that allows us to expand the integrand functions before performing the integration. This is generally convenient since the expanded integrand functions are typically easier to integrate than the original ones. Such an approach is usually called method of regions and consists in writing the original integrals as the sum of a set of new integrals, each one obtained expanding its integrand in a certain limit of the integration variables and the external parameters that we will call phase space region or simply region. In this case, if we introduce the small parameter

$$
\lambda = \frac{t}{Q^2},
\tag{17}
$$

there are two regions that contribute to the integral over z. We will call soft region the region where $1 - z \sim \lambda$ and collinear region the one where $1 - z \sim 1$.

Exercise 4: *Expand the integrals* $K_{\mathrm{R}}^{(1)}$ *and* $K_{\mathrm{R}}^{(2)}$ *in each of the regions and compute them. Get the final result by summing the sub-results from the different regions. NB: The two regions may have a non-zero overlapping that should not be double-counted.*