2024 P3H school - From amplitudes to cross sections and events - Exercises

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September 30 - October 4, 2024

1 The QCD infrared limits

The ratio between the amplitude for the process $\gamma^* \to q\,\bar{q}\,g$ and that for the process $\gamma^* \to q\,\bar{q}$ can be written as

$$\frac{|M_R|^2}{|M_B|^2} = \frac{2g_{\mathrm{S}}^2 C_{\mathrm{F}}}{Q^2}\,\frac{x_1^2 + x_2^2 - \epsilon\,(2 - x_1 - x_2)^2}{(1-x_1)\,(1-x_2)}, \tag{1}$$

where

$$x_1 = \frac{2p_1 \cdot q}{q^2} \qquad x_2 = \frac{2p_2 \cdot q}{q^2}, \tag{2}$$

p_1, p_2, p_3 and q are the momenta of the quark, antiquark, gluon and photon, respectively, and $Q^2 = q^2$. Since the ratio of the two phase spaces reads

$$\begin{aligned}\frac{d\Phi_3}{d\Phi_2} = {} & \frac{\left(Q^2\right)^{1-\epsilon} dx_1\,dx_2}{(4\pi)^{2-\epsilon}\,\Gamma(1-\epsilon)}\left[(1-x_1)\,(1-x_2)\,(x_1+x_2-1)\right]^{-\epsilon}\theta(x_1)\,\theta(x_2) \\ & \times \theta(2-x_1-x_2)\,\theta((1-x_1)\,(1-x_2)\,(x_1+x_2-1))\,,\end{aligned} \tag{3}$$

we can write the decay ratio

$$dK_{\mathrm{R}} = \frac{\displaystyle\int d\Phi_3\,|M_R|^2}{\displaystyle\int d\Phi_2\,|M_B|^2} \tag{4}$$

as

$$\begin{aligned}dK_{\mathrm{R}} = {} & \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{C_{\mathrm{F}}}{Q^{2\epsilon}}\,dx_1\,dx_2\,\frac{x_1^2 + x_2^2 - \epsilon\,(2-x_1-x_2)^2}{(1-x_1)^{1+\epsilon}\,(1-x_2)^{1+\epsilon}\,(x_1+x_2-1)^{\epsilon}} \\ & \times \theta(x_1)\,\theta(x_2)\,\theta(2-x_1-x_2)\,\theta((1-x_1)\,(1-x_2)\,(x_1+x_2-1))\,.\end{aligned} \tag{5}$$

We now want to study the limit where the momentum p_3 of the gluon becomes collinear to the momentum p_1 of the quark or the momentum p_2 of the antiquark. The first limit is parametrized by $(p_1+p_3)^2$ becoming much smaller than the hard scale Q^2 of the decay while being smaller than $(p_2+p_3)^2$, which is instead the small scale of the limit where the momentum of the gluon becomes collinear to the momentum p_2 of the antiquark. Since the condition

$$(p_1+p_3)^2 < (p_2+p_3)^2 \tag{6}$$

is equivalent to the condition $x_1 < x_2$, we can exploit the symmetry of the integrand under the exchange $x_1 \leftrightarrow x_2$ to rewrite dK_{R} as

$$\begin{aligned} dK_{\mathrm{R}} = {} & 2\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{C_{\mathrm{F}}}{Q^{2\epsilon}}\,dx_1\,dx_2\,\frac{x_1^2+x_2^2-\epsilon\,(2-x_1-x_2)^2}{(1-x_1)^{1+\epsilon}\,(1-x_2)^{1+\epsilon}\,(x_1+x_2-1)^\epsilon} \\ & \times \theta(x_1)\,\theta(x_2)\,\theta(2-x_1-x_2)\,\theta((1-x_1)\,(1-x_2)\,(x_1+x_2-1)) \\ & \times \theta(x_2-x_1)\,, \end{aligned} \tag{7}$$

such that we can study both the regions at the same time.

1.1 The Sudakov parametrization

In the collinear limit, it is useful to introduce the variable

$$t = (p_1+p_3)^2 \tag{8}$$

and parametrize the momenta p_1, p_2 and p_3 as

$$\begin{cases} p_1 = z\,\bar{p}_1 + A\,p_{\mathrm{T}} + B\,\bar{p}_2 \\ p_2 = C\,\bar{p}_2 \\ p_3 = D\,\bar{p}_1 + E\,p_{\mathrm{T}} + F\,\bar{p}_2, \end{cases} \tag{9}$$

where $\bar{p}_1$ and $\bar{p}_2$ are two massless vectors defined such that $q = \bar{p}_1 + \bar{p}_2$ and p_{T} is a vector orthogonal to both of them.

Exercise 1: *Express the parameters A, B, C, D, E and F in term of z, t, Q^2 and $|\vec{p}_{\mathrm{T}}|$.*

Solution: The values of the parameters A, B, C, D, E and F can be fixed by imposing that $p_1+p_2+p_3 = q$, $p_1^2 = p_3^2 = 0$ ($p_2^2 = 0$ is automatically satisfied) and $(p_1+p_3)^2 = t$. We obtain

$$\begin{cases} p_1 = z\,\bar{p}_1 + \sqrt{z\,(1-z)\,t}\,\dfrac{p_{\mathrm{T}}}{|\vec{p}_{\mathrm{T}}|} + (1-z)\,\dfrac{t}{Q^2}\,\bar{p}_2 \\ p_2 = \left(1-\dfrac{t}{Q^2}\right)\bar{p}_2 \\ p_3 = (1-z)\,\bar{p}_1 - \sqrt{z\,(1-z)\,t}\,\dfrac{p_{\mathrm{T}}}{|\vec{p}_{\mathrm{T}}|} + z\,\dfrac{t}{Q^2}\,\bar{p}_2. \end{cases} \tag{10}$$

1.2 The leading-power result

Substituting the expressions for p_1, p_2 and p_3 found above in the definitions of x_1 and x_2, we find

$$x_1 = z + (1-z)\,\frac{t}{Q^2} \qquad x_2 = 1 - \frac{t}{Q^2}. \tag{11}$$

Exercise 2: *Substitute the above expressions for x_1 and x_2 in the expression for $dK_{\rm R}$ and truncate it at leading power in the t/Q^2 expansion. NB: Be careful in the truncation: $1-z$ is not guaranteed to be much larger than t/Q^2.*

Solution: Using the above expressions for x_1 and x_2, we can rewrite $dK_{\rm R}$ as

$$\begin{aligned} dK_{\rm R} = {} & 2\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\rm S}}{2\pi}\,C_{\rm F}\,\frac{dt}{t^{1+\epsilon}}\left(1-\frac{t}{Q^2}\right)^{-2\epsilon}\frac{dz}{z^\epsilon\,(1-z)^{1+\epsilon}}\Bigg\{1+z^2 \\ & - 2\,\frac{t}{Q^2}\,[1-z\,(1-z)] + \left(\frac{t}{Q^2}\right)^2\left[1+(1-z)^2\right] - \epsilon\left(1-z+z\,\frac{t}{Q^2}\right)^2\Bigg\} \\ & \times\theta\!\left((1-z)\left(1-\frac{t}{Q^2}\right)-\frac{t}{Q^2}\right). \end{aligned} \tag{12}$$

Note that the other θ functions appearing in the expression for $dK_{\rm R}$ constrain the integration limits of t and z to be

$$0 < \frac{t}{Q^2} < 1 \qquad 0 < z < 1. \tag{13}$$

If we are only interested in the t/Q^2 leading power behavior of $dK_{\rm R}$, we can truncate it as

$$\begin{aligned} dK_{\rm R} = {} & 2\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\rm S}}{2\pi}\,\frac{dt}{t^{1+\epsilon}}\Bigg\{\frac{dz}{z^\epsilon\,(1-z)^\epsilon}\,C_{\rm F}\left[\frac{1+z^2}{1-z}-\epsilon\,(1-z)\right]\theta\!\left(1-z-\frac{t}{Q^2}\right) \\ & + \mathcal{O}\!\left(\frac{t}{Q^2}\right)\Bigg\}. \end{aligned} \tag{14}$$

Note that the term $1-z-t/Q^2$ cannot be truncated because $1-z$ is not guaranteed to be much larger than t/Q^2. In the above expression, we can recognize the D-dimensional unregularized **Altarelli-Parisi splitting function**

$$P_{qq}(z) = \frac{1+z^2}{1-z} - \epsilon\,(1-z)\,. \tag{15}$$

1.3 The integration over z

The leading power result obtained in the above section reads

$$dK_{\mathrm{R}} = 2\,\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{dt}{t^{1+\epsilon}}\left\{\frac{dz}{z^{\epsilon}\,(1-z)^{\epsilon}}\,C_{\mathrm{F}}\left[\frac{1+z^2}{1-z}-\epsilon\,(1-z)\right]\theta\!\left(1-z-\frac{t}{Q^2}\right)\right.$$
$$\left.+\,\mathcal{O}\!\left(\frac{t}{Q^2}\right)\right\}. \tag{16}$$

In order to integrate it over z, we start by writing its integral as

$$K_{\mathrm{R}} = 2\,\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{dt}{t^{1+\epsilon}}\left[K_{\mathrm{R}}^{(1)}+K_{\mathrm{R}}^{(2)}+\mathcal{O}\!\left(\frac{t}{Q^2}\right)\right], \tag{17}$$

where the two integrals $K_{\mathrm{R}}^{(1)}$ and $K_{\mathrm{R}}^{(2)}$ read respectively

$$K_{\mathrm{R}}^{(1)} = 2\,C_{\mathrm{F}}\int_0^1 \frac{dz}{z^{\epsilon}\,(1-z)^{1+\epsilon}}\,\theta\!\left(1-z-\frac{t}{Q^2}\right) \tag{18}$$

and

$$K_{\mathrm{R}}^{(2)} = C_{\mathrm{F}}\int_0^1 \frac{dz}{z^{\epsilon}\,(1-z)^{\epsilon}}\,\left[-1-z-\epsilon\,(1-z)\right]\theta\!\left(1-z-\frac{t}{Q^2}\right). \tag{19}$$

Exercise 3: *Compute the integrals $K_{\mathrm{R}}^{(1)}$ and $K_{\mathrm{R}}^{(2)}$ and use the equation*

$$\frac{1}{x^{1-\epsilon}} = \frac{\delta(x)}{\epsilon}+\mathcal{L}_0(x)+\epsilon\,\mathcal{L}_1(x)+\mathcal{O}(\epsilon^2)\,, \tag{20}$$

where

$$\mathcal{L}_n(x) = \lim_{\beta\to 0^+}\left[\frac{\log^n x}{x}\,\theta(x-\beta)+\frac{\log^{n+1}\beta}{n+1}\,\delta(x-\beta)\right] = \left(\frac{\log^n x}{x}\right)_+\,, \tag{21}$$

to expand K_{R} up to the finite part in ϵ (at leading power in t/Q^2).

Solution: The results for $K_{\mathrm{R}}^{(1)}$ and $K_{\mathrm{R}}^{(2)}$ read

$$K_{\mathrm{R}}^{(1)} = \frac{2\,C_{\mathrm{F}}}{1-\epsilon}\left(1-\frac{t}{Q^2}\right)^{1-\epsilon}{}_2F_1\!\left(1+\epsilon,\,1-\epsilon;\,2-\epsilon;\,1-\frac{t}{Q^2}\right) \tag{22}$$

and

$$K_{\mathrm{R}}^{(2)} = C_{\mathrm{F}}\left(1-\frac{t}{Q^2}\right)^{1-\epsilon}\left[{}_2F_1\!\left(-1+\epsilon,\,1-\epsilon;\,2-\epsilon;\,1-\frac{t}{Q^2}\right)\right.$$
$$\left.-\,\frac{2}{1-\epsilon}\,{}_2F_1\!\left(\epsilon,\,1-\epsilon;\,2-\epsilon;\,1-\frac{t}{Q^2}\right)\right]. \tag{23}$$

Using the properties of the hypergeometric functions, we can rewrite them as

$$K_{\mathrm{R}}^{(1)} = \frac{2\,C_{\mathrm{F}}}{\epsilon}\left(1-\frac{t}{Q^2}\right)^{1-\epsilon}\Bigg[\left(\frac{t}{Q^2}\right)^{-\epsilon}{}_2F_1\!\left(1,\,1-2\epsilon;\,1-\epsilon;\,\frac{t}{Q^2}\right) - \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,{}_2F_1\!\left(1-\epsilon,\,1+\epsilon;\,1+\epsilon;\,\frac{t}{Q^2}\right)\Bigg] \tag{24}$$

and

$$\begin{aligned}K_{\mathrm{R}}^{(2)} = C_{\mathrm{F}}\left(1-\frac{t}{Q^2}\right)^{1-\epsilon}\Bigg[&-\left(\frac{t}{Q^2}\right)^{2-\epsilon}\frac{1-\epsilon}{2-\epsilon}\,{}_2F_1\!\left(1,\,3-2\epsilon;\,3-\epsilon;\,\frac{t}{Q^2}\right)\\ &+\left(\frac{t}{Q^2}\right)^{1-\epsilon}\frac{2}{1-\epsilon}\,{}_2F_1\!\left(1,\,2-2\epsilon;\,2-\epsilon;\,\frac{t}{Q^2}\right)\\ &+\frac{1-\epsilon}{2\,(1-2\epsilon)}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,{}_2F_1\!\left(-1+\epsilon,\,1-\epsilon;\,-1+\epsilon;\,\frac{t}{Q^2}\right)\\ &-\frac{2}{1-2\epsilon}\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,{}_2F_1\!\left(\epsilon,\,1-\epsilon;\,\epsilon;\,\frac{t}{Q^2}\right)\Bigg].\end{aligned} \tag{25}$$

The above expressions are more useful because they expose the $\left(t/Q^2\right)^{-\epsilon}$ factors that have to be combined with the $t^{-1-\epsilon}$ factor appearing in eq. (17), such that they can be expanded using eq. (20). At this point, we can safely truncate the above results at the t/Q^2 leading power and write

$$K_{\mathrm{R}}^{(1)} = \frac{2\,C_{\mathrm{F}}}{\epsilon}\left\{\left(\frac{t}{Q^2}\right)^{-\epsilon}\left[1+\mathcal{O}\!\left(\frac{t}{Q^2}\right)\right] - \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} + \mathcal{O}\!\left(\frac{t}{Q^2}\right)\right\} \tag{26}$$

and

$$K_{\mathrm{R}}^{(2)} = C_{\mathrm{F}}\left\{\left(\frac{t}{Q^2}\right)^{-\epsilon}\mathcal{O}\!\left(\frac{t}{Q^2}\right) - \frac{3+\epsilon}{2\,(1-2\epsilon)}\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} + \mathcal{O}\!\left(\frac{t}{Q^2}\right)\right\}. \tag{27}$$

After expanding with respect to ϵ, the final result for K_{R} reads

$$\begin{aligned}K_{\mathrm{R}} = 2\,\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,C_{\mathrm{F}}\,\frac{dt}{(Q^2)^{1+\epsilon}}\Bigg[&\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}+\frac{7}{2}-\frac{\pi^2}{3}\right)\delta\!\left(\frac{t}{Q^2}\right)\\ &-\frac{3}{2}\,\mathcal{L}_0\!\left(\frac{t}{Q^2}\right) - 2\,\mathcal{L}_1\!\left(\frac{t}{Q^2}\right)\Bigg] + \mathcal{O}(\epsilon) + \mathcal{O}\!\left(\frac{t}{Q^2}\right).\end{aligned} \tag{28}$$

1.4 The method of regions

Instead of computing the exact integrals and then expanding them with respect to t/Q^2 as we did above, we can adopt a different approach that allows us to expand the integrand functions before performing the integration. This is generally convenient since the expanded integrand functions are typically easier to integrate than the original ones. Such an approach is usually called method of regions and consists in writing the original integrals as the sum of a set of new integrals, each one obtained expanding its integrand in a certain limit of the integration variables and the external parameters that we will call phase space region or simply region. In this case, if we introduce the small parameter

$$\lambda = \frac{t}{Q^2}, \tag{29}$$

there are two regions that contribute to the integral over z. We will call soft region the region where $1 - z \sim \lambda$ and collinear region the one where $1 - z \sim 1$.

Exercise 4: *Expand the integrals $K_{\mathrm{R}}^{(1)}$ and $K_{\mathrm{R}}^{(2)}$ in each of the regions and compute them. Get the final result by summing the sub-results from the different regions. NB: The two regions may have a non-zero overlapping that should not be double-counted.*

Solution: The integral $K_{\mathrm{R}}^{(1)}$, in the $\epsilon \to 0$ limit, scales as λ^0 both in the soft and collinear regions (notice that $|dz| = |d\,(1-z)|$ scales as λ in the soft region). In the soft region, we can take advantage of the fact that $1 - z \sim \lambda$ to substitute z^ϵ with 1, such that the integral becomes

$$K_{\mathrm{R}}^{(1,s)} = 2\,C_{\mathrm{F}} \int_0^1 \frac{dz}{(1-z)^{1+\epsilon}}\, \theta\!\left(1 - z - \frac{t}{Q^2}\right) = \frac{2\,C_{\mathrm{F}}}{\epsilon} \left[\left(\frac{t}{Q^2}\right)^{-\epsilon} - 1\right]. \tag{30}$$

In the collinear region instead we can take advantage of the fact that $1 - z \sim 1$ to get rid of the θ function, such that the integral becomes

$$K_{\mathrm{R}}^{(1,c)} = 2\,C_{\mathrm{F}} \int_0^1 \frac{dz}{z^\epsilon\,(1-z)^{1+\epsilon}} = -\frac{2\,C_{\mathrm{F}}}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}. \tag{31}$$

Finally, we have to subtract the overlapping contribution obtained either from the expansion of the soft integrand of eq. (30) in the collinear limit or the expansion of the collinear integrand of eq. (31) in the soft limit, which reads

$$K_{\mathrm{R}}^{(1,sc)} = 2\,C_{\mathrm{F}} \int_0^1 \frac{dz}{(1-z)^{1+\epsilon}} = -\frac{2\,C_{\mathrm{F}}}{\epsilon}. \tag{32}$$

It is also possible to entirely get rid of the overlapping contribution by expressing the integral $K_{\mathrm{R}}^{(1)}$ given in eq. (18) as

$$\tilde{K}_{\mathrm{R}}^{(1)} = K_{\mathrm{R}}^{(1)} = 2\,C_{\mathrm{F}} \int_{-\infty}^{\infty} \frac{dz}{z^\epsilon\;(1-z)^{1+\epsilon}}\, \theta\!\left(1 - z - \frac{t}{Q^2}\right) \theta(z) \tag{33}$$

and expanding $\theta(z)$ according to the scaling of $1 - z$. In the collinear region, $\theta(z)$ does not get expanded, such that $\tilde{K}_{\mathrm{R}}^{(1,c)} = K_{\mathrm{R}}^{(1,c)}$. However, in the soft region, $\theta(z)$ at leading power gets truncated to $\theta(1) = 1$, such that we get

$$\tilde{K}_{\mathrm{R}}^{(1,s)} = 2\,C_{\mathrm{F}} \int_{-\infty}^{\infty} \frac{dz}{(1-z)^{1+\epsilon}}\, \theta\!\left(1 - z - \frac{t}{Q^2}\right) = \frac{2\,C_{\mathrm{F}}}{\epsilon} \left(\frac{t}{Q^2}\right)^{-\epsilon}. \tag{34}$$

If we tried to compute the new expression for the overlapping region, we would now consistently get

$$\tilde{K}_{\mathrm{R}}^{(1,sc)} = 2\,C_{\mathrm{F}} \int_{-\infty}^{\infty} \frac{dz}{(1-z)^{1+\epsilon}}\,\theta(1-z) = 0. \tag{35}$$

The integral $K_{\mathrm{R}}^{(2)}$ instead, in the $\epsilon \to 0$ limit, scales as λ^1 in the soft region and as λ^0 in the collinear region. This means that its leading power expansion takes contribution only from the collinear region and we can write it as

$$K_{\mathrm{R}}^{(2,c)} = C_{\mathrm{F}} \int_0^1 \frac{dz}{z^{\epsilon}\,(1-z)^{\epsilon}} \left[-1 - z - \epsilon\,(1-z)\right] = -C_{\mathrm{F}}\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{3+\epsilon}{2\,(1-2\epsilon)}. \tag{36}$$

If we now define

$$S^{(1)} = 2\,\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{dt}{t^{1+\epsilon}}\left(K_{\mathrm{R}}^{(1,s)} - K_{\mathrm{R}}^{(1,sc)}\right) \tag{37}$$

and

$$J^{(1)} = \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{dt}{t^{1+\epsilon}}\left(K_{\mathrm{R}}^{(1,c)} + K_{\mathrm{R}}^{(2,c)}\right), \tag{38}$$

the final result is now given by

$$K_{\mathrm{R}} = S^{(1)} + 2J^{(1)} + \mathcal{O}\!\left(\frac{t}{Q^2}\right). \tag{39}$$

The expansion of $S^{(1)}$ can be computed using eq. (20) and reads

$$\begin{aligned} S^{(1)} &= 2\,\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{dt}{(Q^2)^{1+\epsilon}}\left[-\frac{1}{\epsilon^2}\,\delta\!\left(\frac{t}{Q^2}\right) + \frac{2}{\epsilon}\,\mathcal{L}_0\!\left(\frac{t}{Q^2}\right) - 4\,\mathcal{L}_1\!\left(\frac{t}{Q^2}\right)\right] \\ &\quad + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{40}$$

It corresponds to the first order in the α_{S} expansion of the unrenormalized **soft function**. The expansion of $J^{(1)}$ instead reads

$$\begin{aligned} J^{(1)} &= \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{dt}{(Q^2)^{1+\epsilon}}\left[\left(\frac{2}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{7}{2} - \frac{\pi^2}{3}\right)\delta\!\left(\frac{t}{Q^2}\right)\right. \\ &\quad \left. + \left(-\frac{2}{\epsilon} - \frac{3}{2}\right)\mathcal{L}_0\!\left(\frac{t}{Q^2}\right) + 2\,\mathcal{L}_1\!\left(\frac{t}{Q^2}\right)\right] + \mathcal{O}(\epsilon) \end{aligned} \tag{41}$$

and corresponds to the first order in the α_{S} expansion of the unrenormalized **jet function**. Combining the two above results, we find again that

$$\begin{aligned} K_{\mathrm{R}} &= 2\,\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,C_{\mathrm{F}}\,\frac{dt}{(Q^2)^{1+\epsilon}}\left[\left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{7}{2} - \frac{\pi^2}{3}\right)\delta\!\left(\frac{t}{Q^2}\right)\right. \\ &\quad \left. - \frac{3}{2}\,\mathcal{L}_0\!\left(\frac{t}{Q^2}\right) - 2\,\mathcal{L}_1\!\left(\frac{t}{Q^2}\right)\right] + \mathcal{O}(\epsilon) + \mathcal{O}\!\left(\frac{t}{Q^2}\right), \end{aligned} \tag{42}$$

which is coherent with the result of eq. (28).

1.5 The extension to the next-to-leading power

Exercise 5: *Extend the computation of the integral K_{R} to the next-to-leading power in the expansion with respect to t/Q^2. Show that the exact integration and the method of regions give the same result.*

Solution: We start by extending the result from the exact integration at next-to-leading power. Including one power order more in the expansion, eqs. (26) and (27) become

$$K_{\mathrm{R}}^{(1)} = \frac{2\,C_{\mathrm{F}}}{\epsilon}\left\{\left(\frac{t}{Q^2}\right)^{-\epsilon}\left[1-\frac{\epsilon^2}{1-\epsilon}\,\frac{t}{Q^2}+\mathcal{O}\!\left(\left(\frac{t}{Q^2}\right)^2\right)\right] - \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}+\mathcal{O}\!\left(\left(\frac{t}{Q^2}\right)^2\right)\right\} \tag{43}$$

and

$$K_{\mathrm{R}}^{(2)} = C_{\mathrm{F}}\left\{\left(\frac{t}{Q^2}\right)^{-\epsilon}\left[\frac{2}{1-\epsilon}\,\frac{t}{Q^2}+\mathcal{O}\!\left(\left(\frac{t}{Q^2}\right)^2\right)\right] - \frac{3+\epsilon}{2\,(1-2\epsilon)}\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}+\mathcal{O}\!\left(\left(\frac{t}{Q^2}\right)^2\right)\right\}, \tag{44}$$

such that the next-to-leading power correction to eq. (28) reads

$$K_{\mathrm{R}}^{(\mathrm{NLP})} = 2\,\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{dt}{\left(Q^2\right)^{1+\epsilon}}\,2\,C_{\mathrm{F}}+\mathcal{O}(\epsilon)\,. \tag{45}$$

The next step is to obtain the above result using the method of regions. After noticing that, in the soft region, we can expand

$$z^{-\epsilon} = 1+\epsilon\,(1-z)+\mathcal{O}\big(\lambda^2\big)\,, \tag{46}$$

and

$$\theta(z) = \theta(1)-(1-z)\,\delta(1)+\mathcal{O}\big(\lambda^2\big) = 1+\mathcal{O}\big(\lambda^2\big)\,, \tag{47}$$

starting from eq. (18), we can write the next-to-leading power correction to the soft contribution to $K_{\mathrm{R}}^{(1)}$ as

$$K_{\mathrm{R}}^{(1,s,\mathrm{NLP})} = 2\,C_{\mathrm{F}}\,\epsilon\int_{-\infty}^{\infty}\frac{dz}{(1-z)^{\epsilon}}\,\theta\!\left(1-z-\frac{t}{Q^2}\right) = -\frac{2\,C_{\mathrm{F}}\,\epsilon}{1-\epsilon}\left(\frac{t}{Q^2}\right)^{1-\epsilon}. \tag{48}$$

The next-to-leading power correction to the collinear contribution to $K_{\mathrm{R}}^{(1)}$ would instead come from the expansion of the θ function

$$\theta\!\left(1-z-\frac{t}{Q^2}\right) = \theta(1-z)-\frac{t}{Q^2}\,\delta(1-z)+\mathcal{O}\big(\lambda^2\big)\,. \tag{49}$$

However the product

$$(1-z)^{-\epsilon}\,\delta(1-z) = 0 \tag{50}$$

makes such a correction vanish, such that we can write

$$K_{\mathrm{R}}^{(1,c,\mathrm{NLP})} = 0. \tag{51}$$

As for the integral $K_{\mathrm{R}}^{(2)}$ given in eq. (19), we proved in the previous section that the soft region starts contributing at next-to-leading power, where we can write

$$K_{\mathrm{R}}^{(2,s,\mathrm{NLP})} = -2\,C_{\mathrm{F}} \int_{-\infty}^{\infty} \frac{dz}{(1-z)^{\epsilon}}\,\theta\left(1-z-\frac{t}{Q^2}\right) = \frac{2\,C_{\mathrm{F}}}{1-\epsilon}\left(\frac{t}{Q^2}\right)^{1-\epsilon} \tag{52}$$

and, using eq. (49) again,

$$K_{\mathrm{R}}^{(2,c,\mathrm{NLP})} = 0. \tag{53}$$

Summing together $K_{\mathrm{R}}^{(1,s,\mathrm{NLP})}$, $K_{\mathrm{R}}^{(1,c,\mathrm{NLP})}$, $K_{\mathrm{R}}^{(2,s,\mathrm{NLP})}$ and $K_{\mathrm{R}}^{(2,c,\mathrm{NLP})}$ and expanding with respect to ϵ up to the finite part, we finally find that the next-to-leading power correction to the integral given in eq. (17) is

$$K_{\mathrm{R}}^{(\mathrm{NLP})} = 2\,\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{dt}{(Q^2)^{1+\epsilon}}\,2\,C_{\mathrm{F}} + \mathcal{O}(\epsilon) \tag{54}$$

which is coherent with the result of eq. (45) obtained with the exact integration.

We notice that the above result is not the full next-to-leading power correction to the decay amplitude since we have not included in the calculation the next-to-leading power terms dropped when eq. (12) was truncated to eq. (14).