Bounds on Ultra Heavy HNLs

BLV 2024

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in collaboration with Inar Timiryasov and Oleg Ruchayskiy

Based on [2206.04540] and [2409.13412]

2024

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UNIVERSITY OF COPENHAGEN

Type-I Seesaw

Mikowski (1977), Gell-Mann, et al. (1979), Mohapatra and Senjanović (1979), Yanagida (1980), Glashow (1980), …

Neutral fermion singlets can explain the origin of neutrino masses, BAU, and dark matter.

New interactions before SSB

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + i\,\bar{N}_R\,\partial\!\!\!/\,N_R - \bar{L}_L \cdot \tilde{H}\,Y\,N_R - \frac{1}{2}\,\bar{N}_R^C\,M_M\,N_R + \text{H.c.}\,,$$

After SSB, N and ν mix in their mass terms

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\begin{pmatrix}\bar{\nu}_L & \bar{N}_R^C\end{pmatrix}\begin{pmatrix}0 & M_D \\ M_D^T & M_M\end{pmatrix}\begin{pmatrix}\nu_L^C \\ N_R\end{pmatrix} + \text{H.c.}\,.$$

$M_D = \frac{v}{\sqrt{2}}\,Y$

Diagonalization gives mass spectrum

$$M_\nu \simeq -M_D^T\,\frac{1}{M_N}\,M_D\,, \qquad\qquad M_N \simeq M_M\,.$$

Seesaw parameters

Interactions between N and the rest of SM particles is proportional to the mixing angle Θ

$$\Theta = M_D \, \frac{1}{M_N} \; .$$

Naively, the mixing angle should be proportional to $\Theta \propto \sqrt{m_\nu / M_N}$. However, we can choose parametrizations of that preserve small m_ν and large Θ.

Casas-Ibarra parametrization [Casas and Ibarra (2001)]

$$\Theta = i \, V^{\text{PMNS}} \, \sqrt{m_\nu} \, O \, \frac{1}{\sqrt{M_N}} \; ,$$

where O is an arbitrary (semi-)orthogonal matrix.

Seesaw parameters

Interac ... the
mixing

Naively ... owever,
we can ... .

Casas-

where

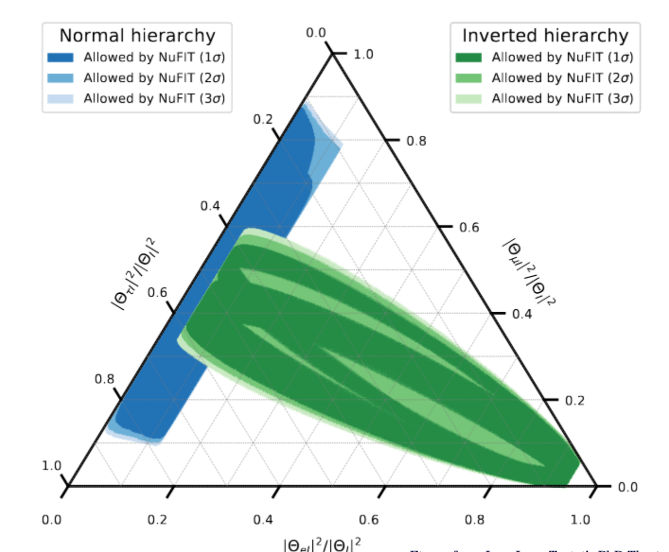
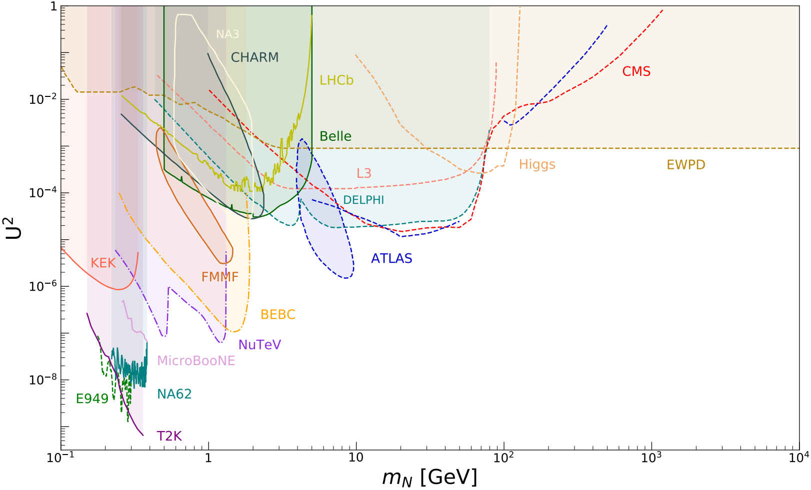


Figure from Jean-Loup Tastet's PhD Thesis

Current constrains

Plot adapted from Bolton, et al. (2019)



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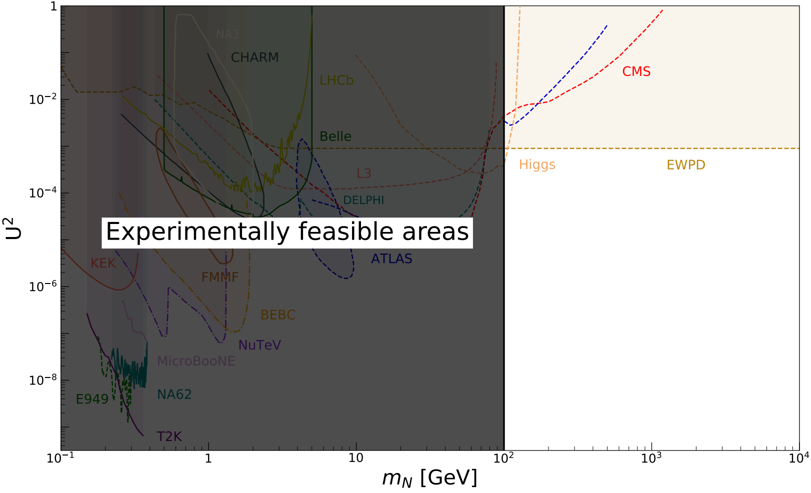

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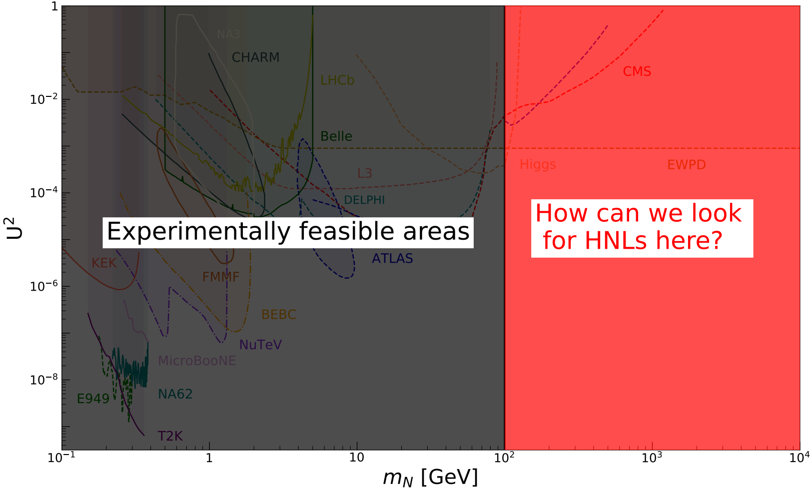

How to search for heavier HNLs?

- Not feasible to directly search for heavy HNLs, can only place bounds indirectly
- HNLs can mediate cLFV processes that are not allowed in the SM, such as
  - $\mu \to e\gamma$
  - $\mu \to eee$
  - $\mu \to e$ conversion in nucleus
- The non-observation of such processes places bounds on HNL parameters
- Not a new idea, decay rates have been known for years

Petcov (1976), Bilenky, et al (1977), Marciano and Sanda (1977), Minkowski (1977), Cheng and Li (1980), Lim and Inami (1981), Langacker and London (1988), Pilaftsis (1992), Ilakovac and Pilaftsis (1994), Chang, et al. (1994), Pilaftsis (1998), Ioannisian and Pilaftsis (1999), Illana, et al. (1999), Illana and Riemann (2000), Pascoli, et al. (2003), Pascoli, et al. (2003), Pilaftsis and Underwood (2005), Deppisch, et al. (2005), …

Non-decoupling and new bounds

However, in presenting these bounds, the recent literature have not properly taken into consideration the effect of non-decoupling diagrams. The shape of the decay width of some cLFV should be

$$\Gamma \propto \left| \Theta^2 + \Theta^4 \left(\frac{M_N}{M_W} \right)^2 \right|^2 ,$$

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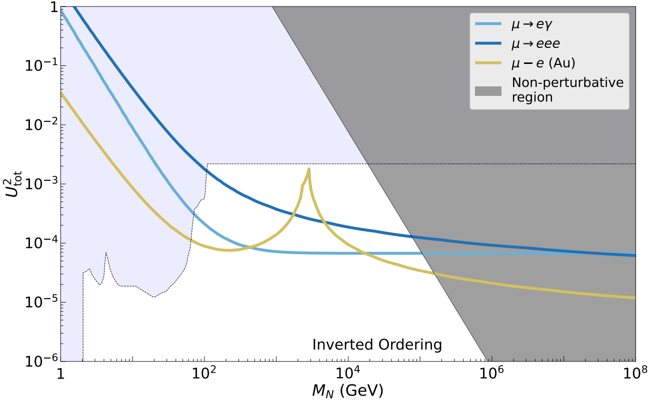

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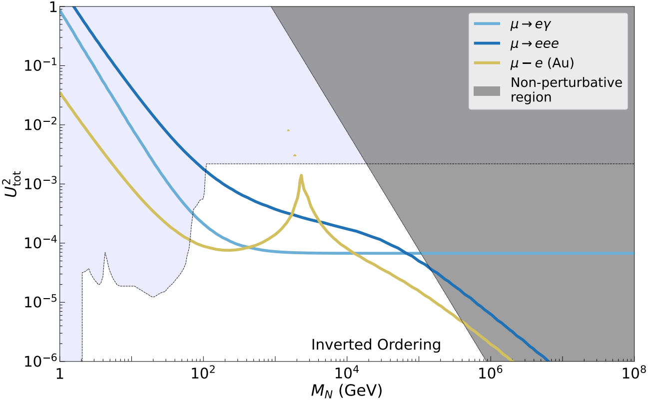

Where does the perturbativity line come from?

We can get a measure of the perturbativity of a theory by using perturbative unitarity. The unitarity condition of the S matrix, brings certain condition to scattering amplitudes

$$\mathcal{M} = 16\pi \sum_J (2J+1) d^J_{\mu_i,\mu_f}\, a^J \,,$$

a^J are the partial waves (or the scattering amplitude with transferred J angular momentum). On $2 \to 2$ elastic scatterings, unitarity demands the inequalities

$$\left|a^J\right| \leq 1\,, \qquad 0 \leq \left|\mathrm{Im}(a^J)\right| \leq 1\,, \qquad \left|\mathrm{Re}(a^J)\right| \leq \frac{1}{2}\,.$$

Any scattering amplitude should automatically satisfy it. However, for tree-level computations alone cannot properly satisfy them for all coupling constants.

Yukawa interactions in the limit $s \to \infty$

We shall do the same analysis on the minimal type-I seesaw model. There are a few theoretical caveats

- In the limit $s \to \infty$, we can take advantage of the *Goldstone equivalence theorem*, and only consider interactions with scalars

$$\mathcal{M}(W_L^{\pm}, Z_L, \dots) = (i\,C)^n\,\mathcal{M}(\phi^{\pm}, \phi_Z, \dots)$$

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Only interactions that matter:

$$\mathcal{L}_{\text{int.}} = -\bar{\nu}_\alpha\, Y_{\alpha,i}\, P_R\, N_i\, (h - i\phi^Z) + \bar{\ell}_\alpha\, Y_{\alpha,i}\, P_R\, N_i\, \phi^- + \text{H.c.}$$

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- Multiple flavors of leptons and generations of HNLs complicate things 😦

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$|Y_{\text{tot}}|^2 = \sum_{\alpha,i} |Y_{\alpha i}|^2$

tions that matter:

$$\mathcal{L}_{\text{int.}} = -\bar{\nu}\, Y_{\text{tot}}\, P_R\, N\, \phi^{0*} + \bar{\ell}\, Y_{\text{tot}}\, P_R\, N\, \phi^{-} + \text{H.c.}$$

- Multiple flavors of leptons and generations of HNLs complicate things 😦

  However, choosing a parametrization of the Yukawa that is rank-one, makes interactions as if only one HNL and one lepton flavor interact 😄

$J = 0$ results

For $J = 0$, we have the elastic scatterings

$$N_\pm \, \ell^\pm_\pm \leftrightarrow N_\pm \, \ell^\pm_\pm \, ,$$
$$N_\pm \, \nu_\pm \leftrightarrow N_\pm \, \nu_\pm \, .$$

Both processes have the same partial wave

$$a^{J=0} = -\frac{|Y_{\text{tot}}|^2}{16\pi} \, ,$$

Remember that $|\text{Re}(a)| \leq \frac{1}{2}$

for the unitarity of the S matrix to be maintained, we demand that

$$|Y_{\text{tot}}|^2 \leq 8\pi \, .$$

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This replicates a result widely used in different literature (up to a factor of 2)

$$\frac{\Gamma_N}{M_N} \leq \frac{1}{2} \implies |Y_{\text{tot}}|^2 \leq 4\pi$$

$J = 1$ results

For $J = 1$ we can have the set of scatterings

$$\{N_- N_+, \nu_- \nu_+, \ell_-^- \ell_+^+, \phi_0^0 \phi_0^{0*}, \phi_0^+ \phi_0^- \} \leftrightarrow \{N_- N_+, \nu_- \nu_+, \ell_-^- \ell_+^+, \phi_0^0 \phi_0^{0*}, \phi_0^+ \phi_0^- \} ,$$

we can write all the partial amplitudes in a matrix

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$$\mathbf{a}^{J=1} = \frac{|Y_{\text{tot}}|^2}{32\,\pi} \left(\qquad \right)$$

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$$\mathbf{a}^{J=1} = \frac{|Y_{\text{tot}}|^2}{32\,\pi}\left(\quad\right)\begin{matrix} N_-\,N_+ \\ \nu_-\,\nu_+ \\ \ell_-^-\,\ell_+^+ \\ \phi_0^0\,\phi_0^{0*} \\ \phi_0^+\,\phi_0^- \end{matrix}$$

Initial states

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Final states $N_- N_+$ $\nu_- \nu_+$ $\ell^-_- \ell^+_+$ $\phi^0_0 \phi^{0*}_0$ $\phi^+_0 \phi^-_0$

$$\mathbf{a}^{J=1} = \frac{|Y_{\text{tot}}|^2}{32\,\pi} \left(\quad \right) \begin{matrix} N_- N_+ \\ \nu_- \nu_+ \\ \ell^-_- \ell^+_+ \\ \phi^0_0 \phi^{0*}_0 \\ \phi^+_0 \phi^-_0 \end{matrix}$$

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We can get bounds by diagonalizing the matrix. Strongest bound comes from the largest eigenvalue.

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$$\mathbf{a}^{J=1} = \frac{\left|Y_{\text{tot}}\right|^2}{32\,\pi} \begin{pmatrix} 1+\sqrt{5} & & & & \\ & 1-\sqrt{5} & & & \\ & & \sqrt{2} & & \\ & & & -\sqrt{2} & \\ & & & & -2 \end{pmatrix}$$

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Best bound:

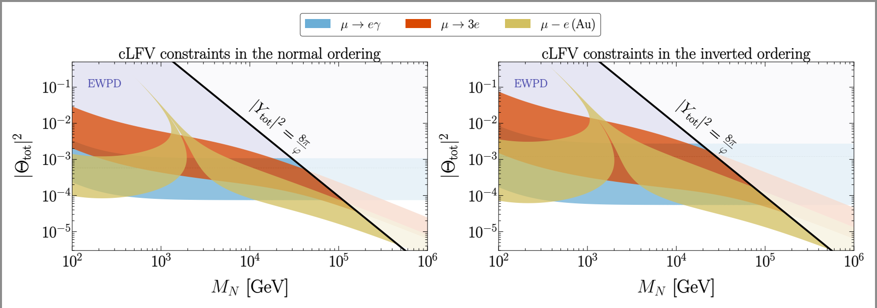
$$|Y_{\text{tot}}|^2 \leq \frac{8\pi}{\varphi}\ .$$

$J = 1$ results

For $J = 1$ we can have the set of scatterings

$\{N_- N_+, \nu_-$ **Where is the new line now?** $\phi_0^{0*}, \phi_0^+ \phi_0^-\}$,

we can write all the partial amplitudes in a matrix

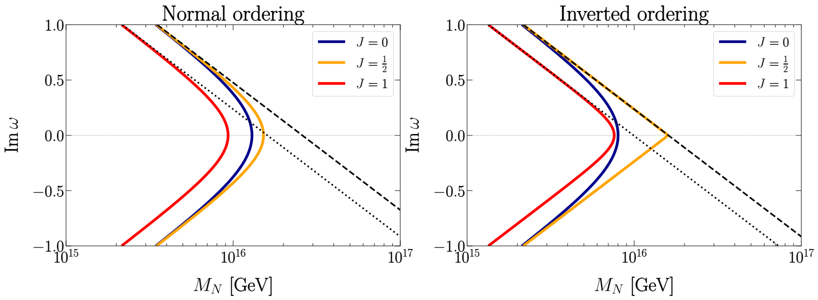


$$|Y_{\mathrm{tot}}|^2 \leq \frac{8\pi}{\varphi} \, .$$

We can do the same analysis at the seesaw line, it is interesting since it gives us an "upper-bound of the HNL mass". At the seesaw line:

$$Y = i\,\frac{\sqrt{2}}{v}\,V^{\mathrm{PMNS}}\sqrt{m_\nu}\,\sqrt{M_N}\,.$$

previous bounds are not valid, Yukawa matrix is not rank-one.



Results at the Seesaw line

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	$M_N < \ldots$ [GeV]	
	Normal ordering	Inverted ordering
$J = 0$	1.30×10^{16}	8.01×10^{15}
$J = \frac{1}{2}$	1.52×10^{16}	1.59×10^{16}
$J = 1$	9.33×10^{15}	7.60×10^{15}

Conclusions

- Charged lepton flavour violating processes allow us to probe HNLs with masses that experiments will never be capable of probing
- These are further enhanced by the non-decoupling behaviour of the processes, which makes the bounds more sensitive to heavier HNL masses
- Perturbative unitarity tells us that $|Y_{\rm tot}|^2 \leq 8\pi/\varphi$ as long as we want tree-level unitarity to hold

Backup slides

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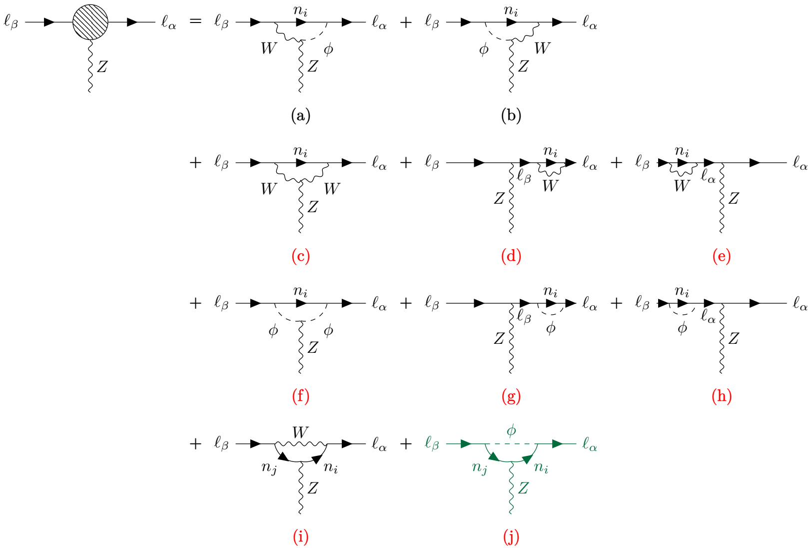
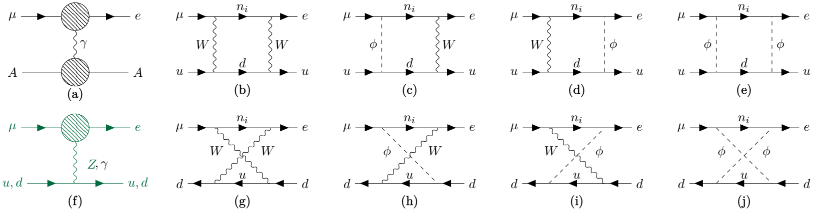
Diagrams

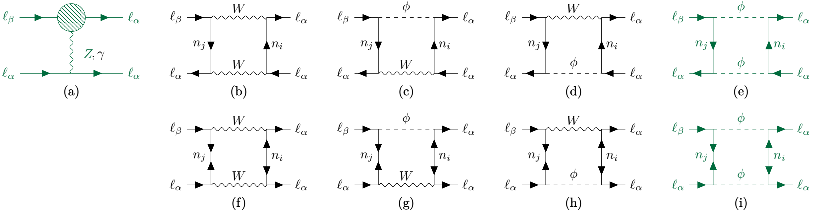


Diagrams

$\mu - e$ conversion



$\mu \to 3e$



Yukawa is rank-one?

Casas-Ibarra parametrization

$$Y = i\,\frac{\sqrt{2}}{v}\,V^{\rm PMNS}\sqrt{m_\nu}\,O\sqrt{M_N}\,.$$

for 2 HNLs, and in the case of normal ordering

$$O = \begin{pmatrix} 0 & 0 \\ \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \simeq e^{-i\omega}\begin{pmatrix} 0 & 0 \\ 1 & -i \\ i & 1 \end{pmatrix}$$

if $\mathrm{Im}(\omega) \gg 1$.

Other popular parametrization for 3 HNLs:

$$Y = \begin{pmatrix} Y_e & i\,Y_e & 0 \\ Y_e & i\,Y_\mu & 0 \\ Y_e & i\,Y_\tau & 0 \end{pmatrix}\,,$$

is also rank-one.

Results for general shape of Yukawa

For $J = 0$ the results hold for any shape of the Yukawa matrix. This is because the partial wave matrix will have the shape

$$\mathbf{a}^{J=0} = -\frac{1}{16\pi} \begin{pmatrix} Y_{e1}^* \\ Y_{\mu 1}^* \\ \vdots \\ Y_{\tau \mathcal{N}}^* \end{pmatrix} \begin{pmatrix} Y_{e1} & Y_{\mu 1} & \cdots & Y_{\tau \mathcal{N}} \end{pmatrix} ,$$

is rank-one. Only non-zero eigenvalue is the trace.

$J = \frac{1}{2}$ general results give

$$\mathbf{a}^{J=\frac{1}{2}} = -\frac{1}{16\pi} Y Y^\dagger ,$$

whose eigenvalues in general do not have a nice shape. However, regardless of the number of additional HNLs, the matrix only has three non-zero eigenvalues.

$J = 1$ general results

For $\mathcal{N}$ HNLs, the $J = 1$ matrix becomes a $(\mathcal{N}^2 + 20) \times (\mathcal{N}^2 + 20)$ matrix

$$\mathbf{a}^{J=1} = \frac{1}{32\pi} \begin{pmatrix} 0 & \mathbf{Y} & \mathbf{Y} & -\sqrt{2}\,\mathscr{Y} & -\sqrt{2}\,\mathscr{Y} \\ \mathbf{Y}^\dagger & 0 & 0 & -\sqrt{2}\,\tilde{\mathscr{Y}} & 0 \\ \mathbf{Y}^\dagger & 0 & 0 & 0 & -\sqrt{2}\,\tilde{\mathscr{Y}} \\ -\sqrt{2}\,\mathscr{Y}^\dagger & -\sqrt{2}\,\tilde{\mathscr{Y}}^\dagger & 0 & 0 & 0 \\ -\sqrt{2}\,\mathscr{Y}^\dagger & 0 & -\sqrt{2}\,\tilde{\mathscr{Y}}^\dagger & 0 & 0 \end{pmatrix},$$

with

$$\mathbf{Y} = \begin{pmatrix} |Y_{e1}|^2 & Y_{e1}\,Y^*_{\mu 1} & Y_{e1}\,Y^*_{\tau 1} & |Y_{\mu 1}|^2 & Y_{\mu 1}\,Y^*_{e1} & Y_{\mu 1}\,Y^*_{\tau 1} & |Y_{\tau 1}|^2 & Y_{\tau 1}\,Y^*_{e} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{e\mathcal{N}}\,Y^*_{e1} & Y_{e\mathcal{N}}\,Y^*_{\mu 1} & Y_{e\mathcal{N}}\,Y^*_{\tau 1} & Y_{\mu\mathcal{N}}\,Y^*_{\mu 1} & \cdots & \cdots & \cdots & \cdots \\ Y_{e1}\,Y^*_{e2} & Y_{e1}\,Y^*_{\mu 2} & Y_{e1}\,Y^*_{\tau 2} & Y_{\mu 1}\,Y^*_{\mu 2} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ |Y_{e\mathcal{N}}|^2 & Y_{e\mathcal{N}}\,Y^*_{\mu\mathcal{N}} & Y_{e\mathcal{N}}\,Y^*_{\tau\mathcal{N}} & |Y_{\mu\mathcal{N}}|^2 & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

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$$\mathbf{a}^{J=1} = \frac{1}{32\pi} \begin{pmatrix} 0 & \mathbf{Y} & \mathbf{Y} & -\sqrt{2}\,\mathscr{Y} & -\sqrt{2}\,\mathscr{Y} \\ \mathbf{Y}^\dagger & 0 & 0 & -\sqrt{2}\,\tilde{\mathscr{Y}} & 0 \\ \mathbf{Y}^\dagger & 0 & 0 & 0 & -\sqrt{2}\,\tilde{\mathscr{Y}} \\ -\sqrt{2}\,\mathscr{Y}^\dagger & -\sqrt{2}\,\tilde{\mathscr{Y}}^\dagger & 0 & 0 & 0 \\ -\sqrt{2}\,\mathscr{Y}^\dagger & 0 & -\sqrt{2}\,\tilde{\mathscr{Y}}^\dagger & 0 & 0 \end{pmatrix},$$

with

$$\mathscr{Y} = \sum_\alpha \begin{pmatrix} |Y_{\alpha 1}|^2 \\ \vdots \\ Y_{\alpha \mathcal{N}}\, Y^*_{\alpha 1} \\ Y_{\alpha 1}\, Y^*_{\alpha 2} \\ \vdots \\ |Y_{\alpha \mathcal{N}}|^2 \end{pmatrix}, \qquad \tilde{\mathscr{Y}} = \sum_i \begin{pmatrix} |Y_{ei}|^2 \\ Y_{\mu i}\, Y^*_{ei} \\ Y_{\tau i}\, Y^*_{ei} \\ |Y_{\mu i}|^2 \\ Y_{ei}\, Y^*_{\mu i} \\ Y_{\tau i}\, Y^*_{\mu i} \\ |Y_{\tau i}|^2 \\ Y_{ei}\, Y^*_{\tau i} \\ Y_{\mu i}\, Y^*_{\tau i} \end{pmatrix}.$$

Results beyond $s \to \infty$

For $J = 0$, we have HNLs in the final and initial state. Conditions on partial waves change

$$\left|a^J\right| \leq \frac{\sqrt{s}/2}{|\vec{p}_f|}, \qquad 0 \leq \text{Im}[a^J] \leq \frac{\sqrt{s}/2}{|\vec{p}_f|}, \qquad \left|\text{Re}[a^J]\right| \leq \frac{1}{2}\frac{\sqrt{s}/2}{|\vec{p}_f|}.$$

for $J = \frac{1}{2}$ the bounds change because we have a resonance.

$J = 1$ states have both HNLs in the final and initial state, as well as resonances. Not clear how to proceed.