Technical aspects of B meson mixing at NNLO

Young Scientists Meeting of the CRC TRR 257

Pascal Reeck | Karlsruhe, 27th September 2024

INSTITUTE FOR THEORETICAL PARTICLE PHYSICS, KARLSRUHE INSTITUTE OF TECHNOLOGY

based on (Reeck, Shtabovenko, and Steinhauser 2024)

Motivation

Quantum mechanics of B mesons

Time evolution of B_s mesons

Relation between self-energy and scattering matrix elements

$$-i(2\pi)^4\delta^{(4)}(p_i - p_j)\Sigma_{ij} = \frac{1}{2M_B}\langle B_i|S|B_j\rangle \tag{1}$$

provides a way of calculating the mixing as described by the Schrödinger equation (Nierste 2009; Weisskopf and Wigner 1930; Lee, Oehme, and Yang 1957):

$$i\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}|B(t)\rangle\\|\bar{B}(t)\rangle\end{pmatrix} = \Sigma\begin{pmatrix}|B(t)\rangle\\|\bar{B}(t)\rangle\end{pmatrix}. \tag{2}$$

Mass vs flavour eigenstates

- Diagonalising $\Sigma \longrightarrow$ eigenstates B_L and B_H
- $\Delta M = M_H - M_L$ and $\Delta\Gamma = \Gamma_L - \Gamma_H$ related to off-diagonal elements

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Introduction to B meson observables

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Lifetime difference

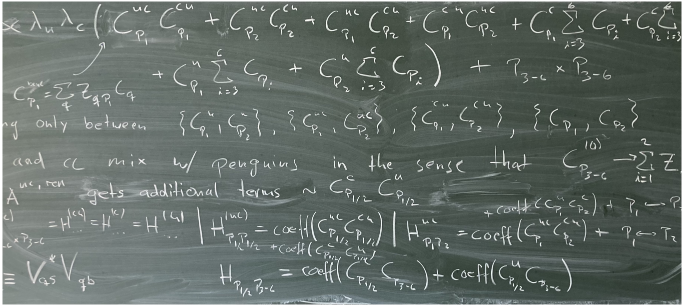
- Off-diagonal matrix element $\longrightarrow$ width difference

$$\begin{aligned} \Delta\Gamma &\equiv \Gamma_L - \Gamma_H \\ &= -2|\Gamma_{12}|\cos(\phi_\Gamma - \phi_M) + \mathcal{O}\left(\frac{|\Gamma_{12}|^2}{|M_{12}|^2}\right) \end{aligned} \tag{3}$$

- Absorptive part of self-energy $\longrightarrow$ off-diagonal matrix element

$$-\frac{\Gamma_{12}}{2} = -i\frac{\Sigma_{12} - \Sigma_{21}^*}{2} \tag{4}$$

Calculation

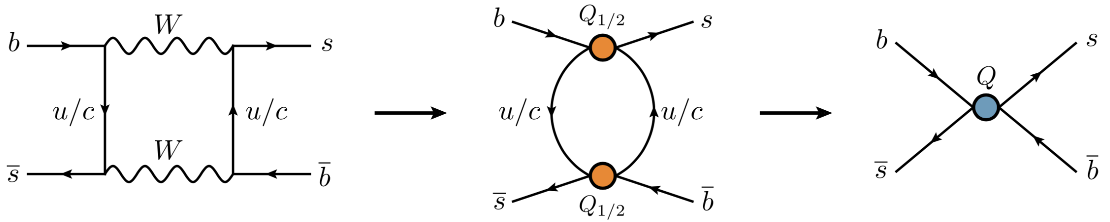


Operator product expansion

- Basic idea:

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \to \sum_n C_{12}^n(x)\mathcal{O}_n(0) \tag{5}$$

- B mixing to leading order:



Matching condition for Γ_{12}

- Calculating Σ_{12} in the $|\Delta B| = 1$ theory:

$$\Sigma_{12} = \frac{-i}{2M_B}\langle B| \left(\frac{1}{2}\int \mathrm{d}^4x\, T\, \mathcal{H}^{\Delta B=1}(x)\mathcal{H}^{\Delta B=1}(0)\right)|\overline{B}\rangle \tag{6}$$

- Equating the absorptive part with the $|\Delta B| = 2$ transition operator:

$$\Gamma_{12} = -2\mathrm{Abs}(\Sigma_{12}) = \frac{1}{M_B}\langle B| \left(\frac{1}{2}\,\mathrm{Abs}\, i\int \mathrm{d}^4x\, T\, \mathcal{H}^{\Delta B=1}(x)\mathcal{H}^{\Delta B=1}(0)\right)|\overline{B}\rangle \tag{7}$$

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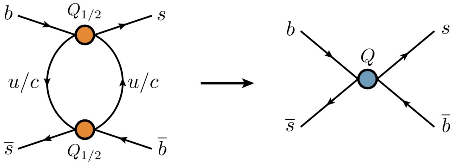
3 loops



2 loops

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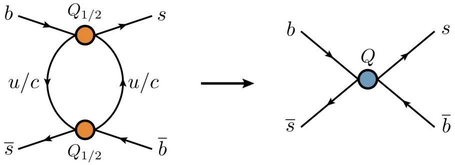

2 loops

Factorisation of scales in Γ_{12}

$$\Gamma_{12} = -\sum_{\alpha,\beta}\lambda_\alpha\lambda_\beta\Gamma_{12}^{\alpha\beta} = -\sum_{\alpha,\beta}\lambda_\alpha\lambda_\beta\frac{G_F^2 m_b^2}{24\pi M_B}\left[H^{\alpha\beta}\langle B|Q|\bar{B}\rangle + \widetilde{H}_S^{\alpha\beta}\langle B|\widetilde{Q}_S|\bar{B}\rangle\right] + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{m_b}\right) \quad (8)$$

Current numerical status

Contributions to Γ_{12}

NB: $z \equiv \frac{m_c^2}{m_b^2}$

Contribution	Previous results	(Gerlach, Nierste, Shtabovenko, Steinhauser 2022)
$P_{1,2} \times P_{3-6}$	2 loops, z-exact, n_f-part only[1]	2 loops, $\mathcal{O}(z)$, full
$P_{1,2} \times P_8$	2 loops, z-exact, n_f-part only[1]	2 loops, $\mathcal{O}(z)$, full
$P_{3-6} \times P_{3-6}$	1 loop, z-exact, full[2]	2 loops, $\mathcal{O}(z)$, full
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$P_8 \times P_8$	1 loop, z-exact, n_f-part only[1]	2 loops, $\mathcal{O}(z)$, full
$P_{1,2} \times P_{1,2}$	3 loops, $\mathcal{O}(\sqrt{z})$, n_f-part only[1]	3 loops, $\mathcal{O}(z)$, full

[1] (Asatrian et al. 2020)
[2] (Beneke, Buchalla, and Dunietz 1996)

Numerical results

ΔΓ to NNLO (Gerlach, Nierste, Shtabovenko, Steinhauser 2022)

$$\left.\frac{\Delta\Gamma_s}{\Delta M_s}\right|_{\text{pole}} = \left(3.79\,^{+0.53}_{-0.58}{}_{\text{scale}}\,^{+0.09}_{-0.19}{}_{\text{scale},1/m_b} \pm 0.11_{B\tilde{B}_S} \pm 0.78_{1/m_b} \pm 0.05_{\text{input}}\right) \times 10^{-3}, \quad (9)$$

$$\left.\frac{\Delta\Gamma_s}{\Delta M_s}\right|_{\overline{\text{MS}}} = \left(4.33\,^{+0.23}_{-0.44}{}_{\text{scale}}\,^{+0.09}_{-0.19}{}_{\text{scale},1/m_b} \pm 0.12_{B\tilde{B}_S} \pm 0.78_{1/m_b} \pm 0.05_{\text{input}}\right) \times 10^{-3}, \quad (10)$$

$$\left.\frac{\Delta\Gamma_s}{\Delta M_s}\right|_{\text{PS}} = \left(4.20\,^{+0.36}_{-0.39}{}_{\text{scale}}\,^{+0.09}_{-0.19}{}_{\text{scale},1/m_b} \pm 0.12_{B\tilde{B}_S} \pm 0.78_{1/m_b} \pm 0.05_{\text{input}}\right) \times 10^{-3}. \quad (11)$$

Overall result:

$$\Delta\Gamma^{th} = (0.076 \pm 0.017)\,\text{ps}^{-1} \quad (12)$$

Comparison to experiment

Results from ((HFLAV) 2020; Aad et al. 2021; Sirunyan et al. 2021):

$$(\Delta\Gamma)^{\text{exp}} = (0.085 \pm 0.005)\,\text{ps}^{-1} \quad (13)$$

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Renormalisation scale dependence of result

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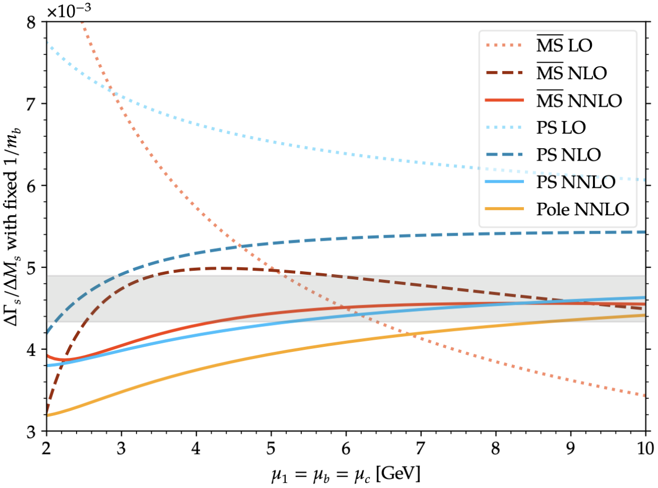

- Need to improve convergence by including contributions involving penguins at 3-loop
- Need to improve accuracy of charm mass dependence

Diagrams needed to complete the NNLO calculation

Contribution	Number of 3-loop diagrams	Maximum number of gamma matrices on each spin line
$P_{1,2} \times P_{1,2}$	$\approx 18,000$	7
$P_{1,2} \times P_{3-6}$	$\approx 200,000$	9
$P_{3-6} \times P_{3-6}$	$\approx 400,000$	11

- Resulting spinor space structures are made up of up to 11 gamma matrices (cf. up to 7 with only current-current)
- Inclusion of more evanescent operators at LO and NLO

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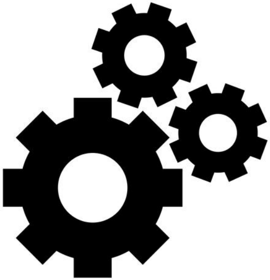
Technical challenge



Key challenge: Projector methodology

- Need to resolve many permutations of tensor structures in amplitude like

$$\Xi = \not{p}_1 \dots \not{p}_m \gamma^{\mu_1} \dots \gamma^{\mu_n} \otimes \not{p}_{m+1} \dots \not{p}_k \gamma_{\mu_n} \dots \gamma^{\mu_1} \tag{14}$$

- Apply projectors, in general:

$$P_i(e_k) = \sum_j \lambda_{ij} \langle e_j, e_k \rangle = \sum_j \lambda_{ij} G_{jk} \stackrel{!}{=} \delta_{ik}, \tag{15}$$

where the Gram matrix is $G_{ij} \equiv \langle e_i, e_j \rangle$

- Projectors yield operator matrix elements times scalar integrals

Choosing the right scalar product

- Freedom of choice for scalar product in (15)
- Any bilinear map works if Gram matrix G_{ij} invertible

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Traditional scalar product

- Defined by a map acting on each length of gamma matrices separately,

$$\begin{aligned} \phi_t : (\mathbb{C}^d)^m \oplus \tilde{V}^{(k)} \times (\mathbb{C}^d)^m \oplus \tilde{V}^{(k)} &\to \mathbb{C} \\ (x_1 \otimes x_2,\ y_1 \otimes y_2) &\mapsto \mathrm{Tr}\left[x_1^\dagger y_1\right] \times \mathrm{Tr}\left[x_2^\dagger y_2\right] \end{aligned} \quad (16)$$

- Explicitly for two tensor structures x, y of the same length:

$$\begin{aligned} \langle x, y\rangle = &\mathrm{Tr}\left[\left(\not{p}_1 \cdots \not{p}_{m_x} \gamma^{\mu_1} \cdots \gamma^{\mu_{n_x}}\right)^\dagger \not{p}_1 \cdots \not{p}_{m_y} \gamma^{\nu_1} \cdots \gamma^{\nu_{n_y}}\right] \times \\ &\mathrm{Tr}\left[\left(\not{p}_{m_x+1} \cdots \not{p}_{k_x} \gamma_{\mu_{\sigma(1)}} \cdots \gamma_{\mu_{\sigma(n_x)}}\right)^\dagger \not{p}_{m_y+1} \cdots \not{p}_{k_y} \gamma^{\nu_{\sigma'(1)}} \cdots \gamma^{\nu_{\sigma'(n_y)}}\right], \end{aligned} \quad (17)$$

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- All Lorentz indices except for loop momenta are contracted $\implies$ quicker computation and parallelisation possible
- Single core calculation of the same number of gamma matrices takes only $\approx$ 1 week, but parallelisation reduces this significantly

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- Single core calculation of the same number of gamma matrices takes only $\approx$ 1 week, but parallelisation reduces this significantly

Key challenge: Projector methodology

Optimised projector algorithm

1. Split off slashed momenta on both spin lines $\Longrightarrow$ reduce number of different structures

$$\text{e.g. } \not{p}_1 \cdots \not{p}_m \gamma^{\mu_1} \cdots \gamma^{\mu_n} \otimes \not{p}_{m+1} \cdots \not{p}_k \gamma_{\mu_{\sigma(1)}} \cdots \gamma_{\mu_{\sigma(n)}} \tag{20}$$

2. Contract Lorentz indices on the same spin line $\Longrightarrow$ reduce number of gamma matrices
3. Canonically order gamma matrices using a lookup table $\Longrightarrow$ reduce number of different structures; pre-computed table reduces runtime

$$\text{e.g. } \not{p}_1 \cdots \not{p}_m \gamma^{\mu_1} \cdots \gamma^{\mu_n} \otimes \not{p}_{m+1} \cdots \not{p}_k \gamma_{\mu_n} \cdots \gamma_{\mu_1} \tag{21}$$

4. Choose projector based on number of gamma matrices and slashed momenta
5. Apply projectors using a lookup table

$$\text{e.g. } E_3^{(2)} \tag{22}$$

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Karlsruhe Institute of Technology

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Outlook

Contribution	(Gerlach, Nierste, Shtabovenko, Steinhauser 2022)	WIP (Nierste, Reeck, Shtabovenko, Steinhauser)
$P_{1,2} \times P_{3-6}$	2 loops, $\mathcal{O}(x^2)$	3 loops, semianalytic
$P_{1,2} \times P_8$	2 loops, $\mathcal{O}(x^2)$	3 loops, semianalytic
$P_{3-6} \times P_{3-6}$	2 loops, $\mathcal{O}(x^2)$	3 loops, semianalytic
$P_{3-6} \times P_8$	2 loops, $\mathcal{O}(x^2)$	3 loops, semianalytic
$P_8 \times P_8$	2 loops, $\mathcal{O}(x^2)$	3 loops, semianalytic
$P_{1,2} \times P_{1,2}$	3 loops, $\mathcal{O}(x^2)$	3 loops, semianalytic

Challenges

- Long gamma traces and projector optimisation
- High-rank tensor integrals
- 3-loop master integrals with 2 mass scales

Thank you for your attention!

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