

Recursion relations in multiplet bases

Johan Thorén

In collaboration with Yi-Jian Du and Malin Sjödhahl

11:th MCnet meeting, Karlsruhe, October 2014

Table of contents

- 1 Introduction
- 2 Multiplet bases
- 3 Decomposition into multiplet bases
- 4 Recursion relations
- 5 Conclusions and outlook

Section 1

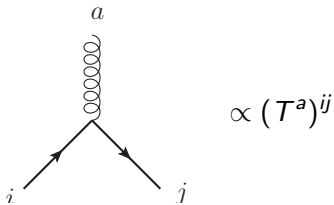
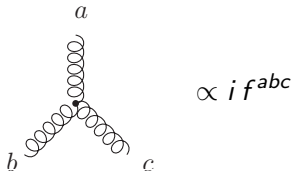
Introduction

Introduction

- The high energy of the LHC gives events with many colored partons in the perturbative regime.
- It is a challenge to deal with the color structure exactly for many partons, due to the non-Abelian structure of QCD.

Local symmetry of QCD

- The symmetry of QCD is $SU(3)$
- It enters the calculations from the vertices

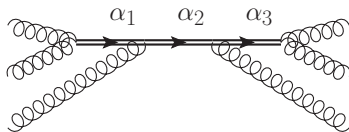


Section 2

Multiplet bases

Multiplet bases

- Orthogonal and minimal basis!
[Keppeler and Sjödaahl, JHEP 1209, arXiv:1207.0609](#)
- Number of vectors grow exponentially, not factorially.
- Orthogonality makes squaring easier!
- Downside is that decomposition and recursion are not as straightforward as with traditional bases.



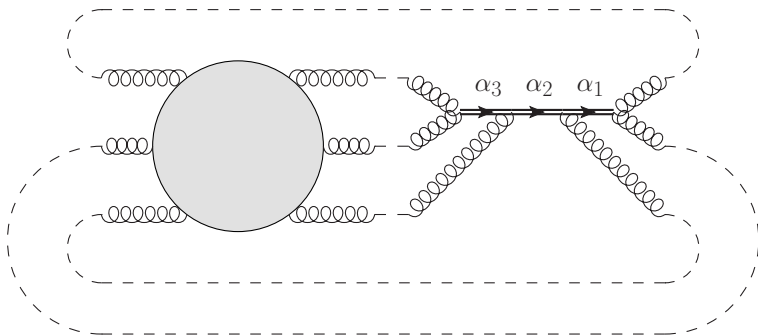
Quarks can be handled by combining the external quarks and antiquarks into pairs, which can then be in either a singlet or an octet.

Section 3

Decomposition into multiplet bases

Decomposition

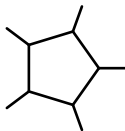
- The decomposition is just an evaluation of the scalar product between the basis vectors and the color structure



- This is just the evaluation of vacuum bubbles.

Vacuum bubbles

- Any vacuum bubble can be rewritten as sums over factors of simpler vacuum bubbles.
- This is achieved by finding loops in the bubble

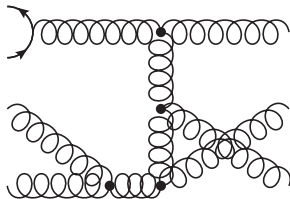


and repeatedly applying completeness relations, resulting in a reduction of the number of vertices in the bubble

$$\begin{array}{c} \mu \\ \hline \hline \longrightarrow \\ \hline \hline \\ \nu \end{array} = \sum_{\alpha \in \mu \otimes \nu} \frac{d_\alpha}{\begin{array}{c} \mu \\ \circlearrowleft \\ \alpha \\ \nu \end{array}} \begin{array}{c} \mu \quad \mu \\ \diagdown \quad \diagup \\ \alpha \\ \diagup \quad \diagdown \\ \nu \quad \nu \end{array}$$

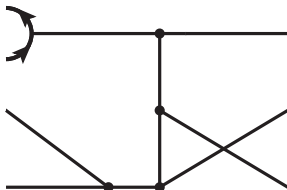
Example with a $q\bar{q}$ pair and 5 gluons

One of the color structures for $q\bar{q}2g \rightarrow 3g$ is



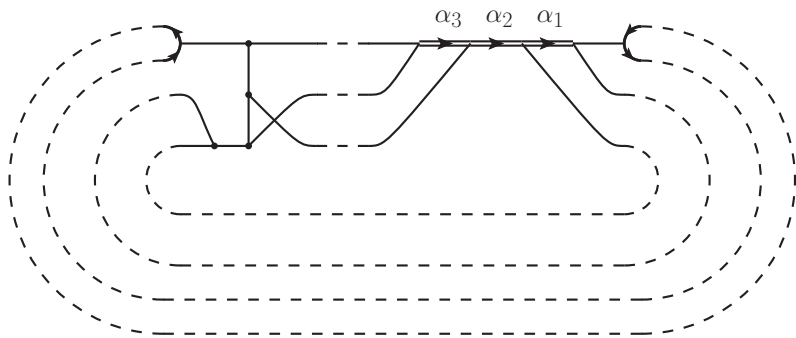
Example with a $q\bar{q}$ pair and 5 gluons

Using straight lines instead of curly lines for gluons gives



Example with a $q\bar{q}$ pair and 5 gluons

Decomposition into the basis vectors is equivalent to determining the scalar product between the basis vector and the color structure



Example with a $q\bar{q}$ pair and 5 gluons

Two-vertex loops gives a factor

$$\begin{array}{c} \alpha \quad \beta \\ \text{---} \bigcirc \text{---} \\ \delta \quad \gamma \end{array} = \frac{\begin{array}{c} \beta \\ \text{---} \bigcirc \text{---} \\ \alpha \quad \gamma \end{array}}{d_\alpha} \text{---} \alpha ,$$

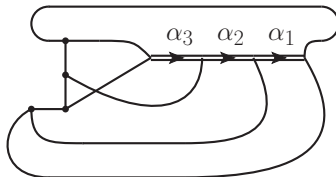
hence the quark loop gives

$$\text{---} \bigcirc \text{---} = T_R \text{---} .$$

Example with a $q\bar{q}$ pair and 5 gluons

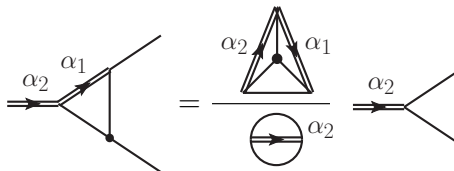
Removing the quark loop then gives

$$A(\alpha_1, \alpha_2, \alpha_3) = T_R$$



Reducing the loop

A vertex correction only gives a factor (given by small bubbles)

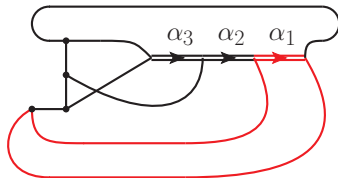


with a possible sum over vertices.

Example with a $q\bar{q}$ pair and 5 gluons

Using the vertex correction result on

$$A(\alpha_1, \alpha_2, \alpha_3) = T_R$$



Example with a $q\bar{q}$ pair and 5 gluons

Gives

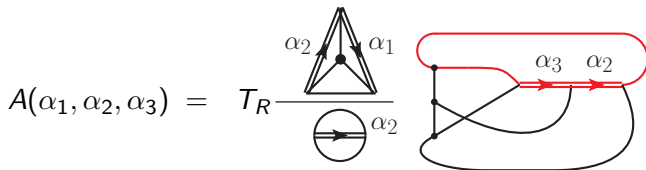
$$A(\alpha_1, \alpha_2, \alpha_3) = T_R \frac{\text{Diagram 1}}{\text{Diagram 2}} \text{Diagram 3}$$

The diagram on the left is a tetrahedron with a central black dot. Four lines connect the vertices to the center. The top-left edge is labeled α_2 and the top-right edge is labeled α_1 . The diagram below it is a circle with a horizontal line through the center and an arrow pointing to the right, labeled α_2 .

The diagram on the right is a complex graph with four vertices on the left and two on the right. It features several loops and lines. The top horizontal line is labeled α_3 and α_2 .

Example with a $q\bar{q}$ pair and 5 gluons

Now we must pick a 4-vertex loop



Reducing the second loop

A 4-vertex loop is less trivial than a vertex correction

The diagrammatic equation illustrates the reduction of a 4-vertex loop. On the left, a square loop with four external lines is shown, with the top and bottom edges labeled α_2 and the left and right edges labeled α_3 . This is equal to a sum over ψ of a vertex correction term (a circle with a horizontal line and a loop) multiplied by a diagram with two triangles sharing a central vertex, each triangle having a ψ line and an α_3 line. This is further equal to a sum over ψ of a vertex correction term multiplied by two triangle diagrams (each with a ψ line and an α_3 line) and a final diagram with a double line labeled ψ connecting two vertices.

$$\begin{aligned}
 & \text{4-vertex loop} = \sum_{\psi} \frac{d\psi}{\psi} \text{[vertex correction]} \text{[triangle diagram]} \\
 & = \sum_{\psi} \frac{d\psi}{\psi} \text{[vertex correction]} \frac{\text{[triangle diagram]} \text{[triangle diagram]}}{\text{[circle]} \text{[circle]}} \text{[double line diagram]}
 \end{aligned}$$

Reducing the second loop

A 4-vertex loop is less trivial than a vertex correction

The diagram shows an equation between two Feynman diagrams. On the left is a 4-vertex loop diagram: a square loop with four external lines. The top-left vertex is a black dot. The right edge of the loop is labeled α_2 with an upward arrow, and the bottom edge is labeled α_3 with a rightward arrow. The loop itself is drawn with red lines. On the right is a vertex correction diagram: a central vertex with four external lines, where the top two lines are connected by a double line labeled ψ with a rightward arrow. The equation is represented as:

$$\text{4-vertex loop} = \sum_{\psi} C(\psi, \alpha_2, \alpha_3) \text{ vertex correction}$$

Example with a $q\bar{q}$ pair and 5 gluons

Now we are to remove the 4-vertex loop

$$A(\alpha_1, \alpha_2, \alpha_3) = T_R \frac{\text{Diagram 1}}{\text{Diagram 2}} \text{Diagram 3}$$

The diagram shows the decomposition of the amplitude $A(\alpha_1, \alpha_2, \alpha_3)$. The numerator is a triangle diagram with a central vertex and three external lines labeled α_1 , α_2 , and α_3 . The denominator is a circle with a horizontal line and an arrow labeled α_2 . The result is a more complex diagram with a red oval highlighting a specific loop structure involving lines α_3 and α_2 .

Example with a $q\bar{q}$ pair and 5 gluons

Giving us the final expression

$$A(\alpha_1, \alpha_2, \alpha_3) = \sum_{\psi} C'(\psi, \alpha_1, \alpha_2, \alpha_3) \text{ (diagram)} = \\
 = T_R \sum_{\psi} \frac{\text{(triangle diagram with } \alpha_2, \alpha_1 \text{)} \cdot \frac{d\psi}{\text{(circle diagram with } \alpha_2, \psi, \alpha_3 \text{)}}}{\text{(circle diagram with } \psi \text{)} \cdot \text{(circle diagram with } \psi \text{)}} \cdot \text{(triangle diagram with } \psi, \alpha_2 \text{)} \cdot \text{(triangle diagram with } \psi \text{)}$$

Decomposing Feynman diagrams

- Any vacuum bubble can be rewritten as a sum of factors of smaller vacuum bubbles, called Wigner 3j and 6j coefficients.

$$\sum_{\psi_1} \sum_{\psi_2} \dots \sum_{\psi_n} \left(\frac{\begin{array}{ccc} \text{triangle} & \dots & \text{triangle} \\ \text{circle} & \dots & \text{circle} \end{array}}{\dots} \right)$$

The diagram shows a large vacuum bubble (a triangle with three internal lines) being decomposed into a sum of products of smaller vacuum bubbles (a circle with a horizontal line). The decomposition is represented by a large fraction where the numerator contains two triangle diagrams and the denominator contains two circle diagrams, with ellipses indicating more terms in the sum.

- Rewriting into smaller bubbles can be done without specifying for which basis vector it is.
- These smaller bubbles can be calculated once and then be stored.

Wigner coefficients

An upper limit of the number of required Wigner coefficients are

$n = N_g + N_{q\bar{q}}$	4	5	6	7	8	9	10	11	12
LO $N_c = 3$	21	39	106	152	254	318	452	536	705
NLO $N_c = 3$	29	55	120	176	272	350	476	576	733
LO $N_c \rightarrow \infty$	28	68	313	636	1777	3095	7289	12009	25487
NLO $N_c \rightarrow \infty$	44	108	389	808	2023	3693	8077	13783	27613

I have calculated the coefficients required for gluons for LO up to $N_g = 6$.

M. Sjödaahl and J. Thorén, [Decomposing color structure into multiplet bases, to be submitted very soon.](#)

Section 4

Recursion relations

Recursion

- Avoid Feynman diagrams, the number grows factorially!
- Recursion relation in external particles.
- Has been explored for traditional bases (trace, DDM [1], color flow [2]).
- The orthogonality of the multiplet bases is promising.

[1] V. Del Duca, L. Dixon and F. Maltoni, [hep-ph/9910563](#).

[2] F. Maltoni, K. Paul, T. Stelzer and S. Willenbrock, [hep-ph/0209271](#).

BCFW

A recursion relation for on-shell amplitudes is the BCFW recursion relation,

$$\mathcal{M}(g_1, \dots, g_n) = \sum_{\mathcal{I}, \mathcal{J}} \sum_{g_{\mathcal{I}}, h_{\mathcal{I}}} \mathcal{I} \left\{ \begin{array}{c} \hat{1} \\ \vdots \\ \text{---} \text{---} \text{---} \end{array} \right. \left. \begin{array}{c} \hat{n} \\ \vdots \\ \text{---} \text{---} \text{---} \end{array} \right\} \mathcal{J}$$

$\begin{array}{c} \text{---} \hat{p}_{\mathcal{I}, \mathcal{J}}^{g_{\mathcal{I}}, h_{\mathcal{I}}} \quad \hat{p}_{\mathcal{I}, \mathcal{J}}^{g_{\mathcal{I}}, -h_{\mathcal{I}}} \\ \text{---} \end{array}$

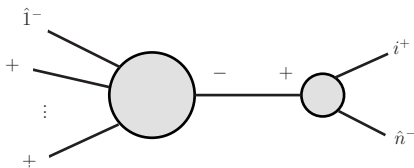
 $\frac{i}{P_{\mathcal{I}, \mathcal{J}}^2}$

R. Britto, F. Cachazo, B. Feng and E. Witten, hep-th/0501052

BCFW for MHV amplitudes

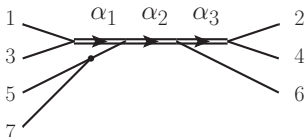
If the amplitude is maximally helicity violating, the relation simplifies.

It is possible to choose the momentum shift such that only one factorization channel contributes



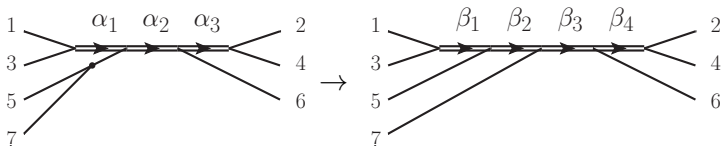
Color structures for MHV

The color structures of interest are then multiplet basis vectors emitting a gluon,

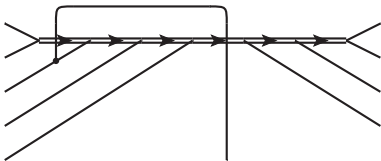


Color structures for MHV

The color structures of interest are then multiplet basis vectors emitting a gluon,

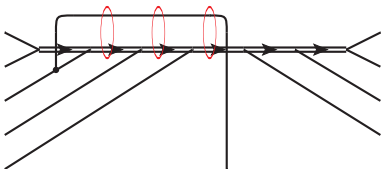


Basis vectors radiating gluons



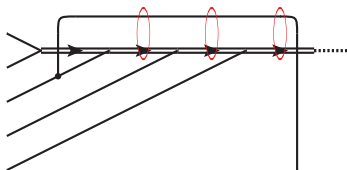
Basis vectors radiating gluons

Applying completeness relations



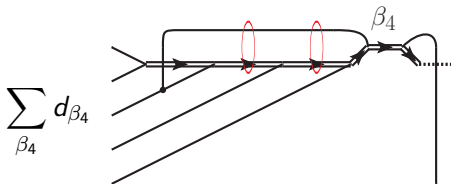
Basis vectors radiating gluons

Applying completeness relations



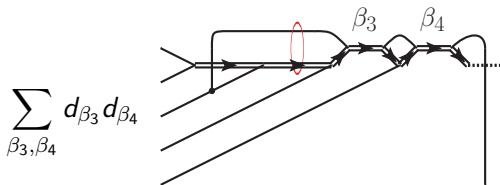
Basis vectors radiating gluons

Applying completeness relations



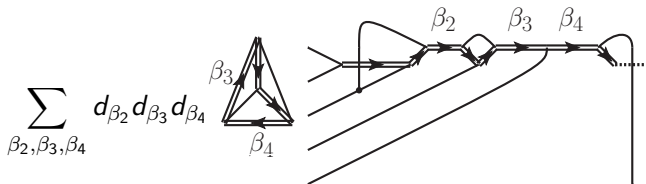
Basis vectors radiating gluons

Applying completeness relations



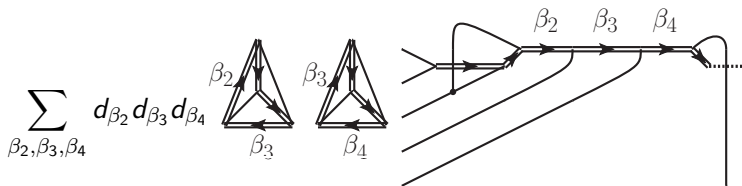
Basis vectors radiating gluons

Removing vertex corrections



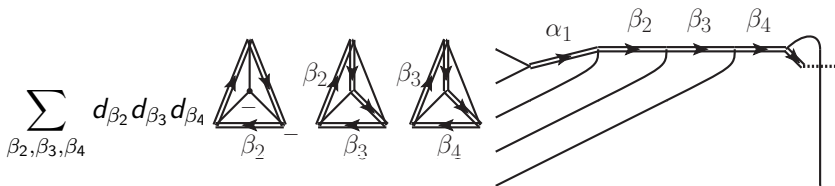
Basis vectors radiating gluons

Removing vertex corrections

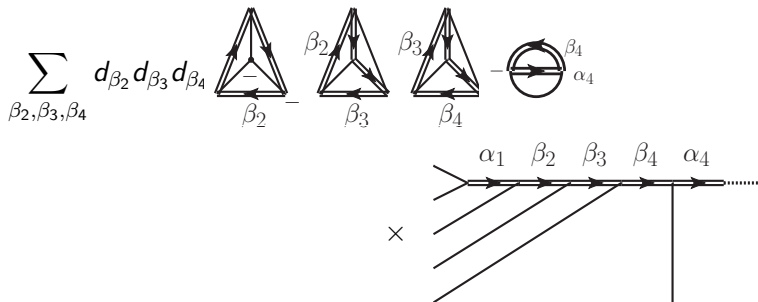


Basis vectors radiating gluons

Removing vertex corrections



Basis vectors radiating gluons



a specific vertex normalization has been chosen to simplify the expression.

Radiation matrices

Preliminary numbers for multiplet basis vectors radiating a gluon.

	n	5	6	7	8	9	10	11	12
Avg over $2, \dots, n-1$	$N_c \geq n$	6.8	14.4	24.6	57.9	109	299	593	1 775
	QCD	6.0	10.8	17.5	32.5	54.6	106	185	268
Max	$N_c \geq n$	9	44	44	400	400	4 006	4 006	41 256
	QCD	8	33	33	178	178	962	962	5 220
Vectors (all orders)	$N_c \geq n$	44	265	1 854	14 833	133 496	1 334 961	14 684 570	176 214 841
	QCD	32	145	702	3 598	19 280	107 160	614 000	3 609 760
Trace (LO)	any N_c	24	120	720	5 040	40 320	362 880	3 628 800	39 916 800
DDM (LO)	any N_c	6	24	120	720	5 040	40 320	362 880	3 628 800

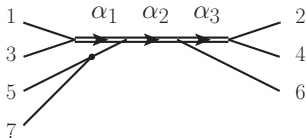
For 10 gluons, $(9!)^2 \sim 10^{11}$ for the trace basis, $(8!)^2 \sim 10^9$ for DDM, and $19\,280 \cdot 8 \cdot 106 \sim 10^7$ for multiplet bases.

Y-J. Du, M. Sjö Dahl and J. T., Recursion in multiplet bases for tree-level MHV gluon amplitudes, to be submitted soon.

Representative also for parton showers.

Radiation matrices

Color structures of interest for parton showers are of the same form,



Section 5

Conclusions and outlook

Conclusions

- The multiplet bases are minimal and orthogonal.
- Decomposing color structures into them is non-trivial.
- A manageable number of vacuum bubbles has to be calculated and stored (even for general N_C).
- Recursion in multiplet bases requires fewer terms when squaring than traditional bases.
- The number of terms required for recursion is also representative for parton showers.

Outlook

- Calculate $6j$ coefficients for more than 6 external gluons.
- Implement recursion relations efficiently.
- Handle recursion with quarks.
- Go beyond MHV amplitudes in recursion.
- Handle NLO amplitudes.

Backup slide: momentum shift

In the spinor helicity formalism the chosen momentum shift is

$$\begin{aligned}\lambda_1 &\rightarrow \lambda_1, & \tilde{\lambda}_1 &\rightarrow \tilde{\lambda}_1 - z\tilde{\lambda}_n, \\ \tilde{\lambda}_n &\rightarrow \tilde{\lambda}_n, & \lambda_n &\rightarrow \lambda_n + z\lambda_1,\end{aligned}$$

this causes the $(3, n-1)$ factorization channel to vanish.

