Automating the calculation of jet functions in SCET

Guido Bell, Kevin Brune, Goutam Das, Marcel Wald

Young Scientist Session
07.10.2020

Motivation

- Starting point: Factorization theorems in SCET

$$\mathrm{d}\sigma \simeq H(\mu_F) \cdot J_n(\mu_F) \otimes J_{\bar{n}}(\mu_F) \otimes S(\mu_F)$$

- Resummation of Sudakov logarithms by solving RGEs
- Extracting observable dependent ingredients

  - Anomalous Dimension: $\Gamma_{Cusp}, \gamma_H, \gamma_J, \gamma_S$
  - Matching corrections: c_H, c_J, c_S

- Two loop ingredients enter at NNLL' accuracy of SCET resummation
- Automation computation in the same spirit as soft functions

[Bell,Rahn,Talbert,18]

Jet functions

Definitions:

$$J_q(\tau,\mu) \sim \sum_{i\in X} \delta(Q - \textstyle\sum_i \bar{n}\cdot k_i)\delta^{(\mathrm{d}-2)}(\textstyle\sum_i k_\perp^i)\mathcal{M}(\tau,\{k_i\}) \langle 0|\chi|X\rangle \langle X|\bar{\chi}|0\rangle$$

collinear field operators $\chi = W^\dagger \frac{\not{\bar{n}}\not{n}}{4}\psi$

generic measurement function $\mathcal{M}(\tau,\{k_i\})$

Light cone coordinates

- $n_\mu = (1,0,0,1)$
- $n^2 = \bar{n}^2 = 0$
- $\bar{n}_\mu = (1,0,0,1)$
- $n\cdot\bar{n} = 2$

- $k^\mu = (\bar{n}\cdot k)\frac{n^\mu}{2} + (n\cdot k)\frac{\bar{n}^\mu}{2} + k_\perp^\mu = (k_-, k_+, \mathbf{k}_\perp)$
- $(2k\cdot l) = k_- l_+ + k_+ l_- + 2\mathbf{k}_\perp \mathbf{l}_\perp = k_- l_+ + k_+ l_- - 2\mathbf{k_T l_T}$

Jet functions

Definitions:

$$J_q(\tau,\mu) \sim \sum_{i\in X} \delta(Q - \textstyle\sum_i \bar{n}\cdot k_i)\delta^{(\mathrm{d}-2)}(\textstyle\sum_i k_\perp^i)\mathcal{M}(\tau,\{k_i\}) \langle 0|\,\chi\,|X\rangle \langle X|\,\bar{\chi}\,|0\rangle$$

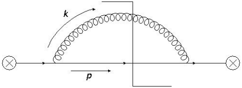
collinear field operators $\chi = W^\dagger \frac{\not{\bar{n}}\not{n}}{4}\psi$

generic measurement function $\mathcal{M}(\tau,\{k_i\})$

Status:

- formalism exists for NLO calculation [KB Master Thesis]
- formalism extended to NNLO real-virtual interference
- currently looking into NNLO real-real contribution

NLO calculation



- Matrix element: LO splitting function $P^0_{gq}(z_k)$

$$P^0_{gq}(z_k) \sim \frac{(1-\epsilon)z_k^2+2(1-z_k)}{z_k}$$

[Altarelli,Parisi,77]

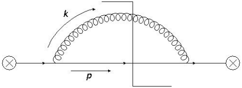
- Phase space parametrization that factorizes all divergences

$$k_T = \sqrt{k_+ k_-}, \quad z_k = \frac{k_-}{Q}, \quad t_k = \frac{1-\cos(\theta_k)}{2}$$

- Find generic parametrization of measurement function such that $f(z_k, t_k)$ finite and non-zero for $z_k \to 0$

$$\mathcal{M}^0(\tau,\mu) = exp\Big(- \tau k_T \Big(\frac{k_T}{z_k \bar{z}_k Q}\Big)^n f(z_k, t_k)\Big)$$

NLO calculation



- Find generic parametrization of measurement function such that $f(z_k, t_k)$ finite and non-zero for $z_k \to 0$

$$\mathcal{M}^0(\tau,\mu) = exp\Big(- \tau k_T \Big(\frac{k_T}{z_k \bar{z}_k Q}\Big)^n f(z_k, t_k)\Big)$$

- Examples of NLO measurement functions $\mathcal{M}^0(\tau,\mu)$

Observable	n	$f(z_k, t_k)$
Thrust	1	1
Angularities	$1-A$	$z_k^{1-A} + \bar{z}_k^{1-A}$
Transverse Thrust	1	$\frac{4 t_k \bar{t}_k}{\sin(\theta)}$

NLO calculation

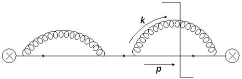
- Master formula for NLO jet function

$$J_q^1(\tau,\mu) \sim \Gamma\left(\frac{-2\epsilon}{1+n}\right)\int_0^1 \mathrm{d}z_k\, z_k^{-1-2\frac{n}{1+n}\epsilon}\,\bar{z}_k^{-2\frac{n}{1+n}\epsilon}(z_k P_{qg}^0(z_k))$$
$$\times \int_0^1 \mathrm{d}t_k\,(4t_k\bar{t}_k)^{-1/2-\epsilon} f(z_k,t_k)^{\frac{-2\epsilon}{1+n}}$$

- ▶ Collinear divergence($k_T \to 0$): $\Gamma\left(\frac{-2\epsilon}{1+n}\right)$
- ▶ Soft divergence($z_k \to 0$): $z_k^{-1-2\frac{n}{1+n}\epsilon}$

⇒ Both singularities factorized

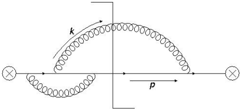
NNLO real-virtual interference



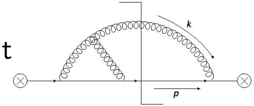
- Matrix element: NLO splitting function $P^1_{gq}(z_k)$

$$P^1_{gq}(z_k) \sim f(\epsilon)\Big[P^0_{gq}(z_k)\Big\{P^{C_F}_{gq}(z_k) + P^{C_A}_{gq}(z_k)\Big\} + P^{C_F,C_A}_{gq}(z_k)\Big]$$
$$= f(\epsilon)\tilde{P}^1_{gq}(z_k)$$

[Sborlini,de Florian, Rodrigo,14]



- Phase space is the same as in the NLO case
  - $\Rightarrow$ No new parametrization required
  - $\Rightarrow$ Measurement function $\mathcal{M}^0(\tau, \mu)$ still sufficient



NNLO real-virtual interference

- Master formula for NNLO real-virtual jet function

$$J_q^{2,RV}(\tau,\mu) \sim f(\epsilon)\Gamma\left(\frac{-4\epsilon}{1+n}\right)\int_0^1 \mathrm{d}z_k\, z_k^{-1-4\frac{n}{1+n}\epsilon}\bar{z}_k^{-4\frac{n}{1+n}\epsilon}(z_k\tilde{P}^1_{gq}(z_k)) \times \int_0^1 \mathrm{d}t_k\,(4t_k\bar{t}_k)^{-1/2-\epsilon} f(z_k,t_k)^{\frac{-4\epsilon}{1+n}}$$

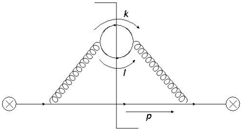
- ▶ Loop divergence: $f(\epsilon) \sim \frac{1}{\epsilon^2} + \mathcal{O}(\epsilon^{-1})$
- ▶ Collinear divergence($k_T \to 0$): $\Gamma\left(\frac{-4\epsilon}{1+n}\right)$
- ▶ Soft divergence($z_k \to 0$): $z_k^{-1-4\frac{n}{1+n}\epsilon}$

⇒ Both phase space singularities factorized

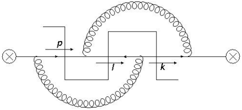
NNLO real-real contribution

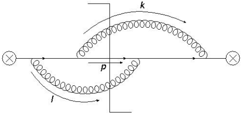
- Matrix element: LO triple collinear splitting functions

$$P^{0}_{q'\bar{q}'q},\ P^{0,\mathrm{id}}_{q\bar{q}q},\ P^{0,C_F^2}_{ggq},\ P^{0,C_F C_A}_{ggq}$$ [Catani,Grazzini,99]



- Splitting functions contain different divergence structures
  - ▶ $P^{0}_{q'\bar{q}'q} \sim \frac{1}{s_{123}^2 s_{12}^2 (z_1+z_2)^2}$
  - ▶ $P^{0,\mathrm{id}}_{q\bar{q}q} \sim \frac{1}{s_{123}^2 s_{12}}, \frac{1}{s_{123} s_{12} (1-z_2)(1-z_3)}, \frac{1}{s_{13} s_{12} (1-z_2)(1-z_3)}$
  - ▶ $P^{0,C_F^2}_{ggq} \sim \frac{1}{s_{123}^2 s_{13}}, \frac{1}{s_{123} s_{13} (z_1)(z_2)}, \frac{1}{s_{13} s_{23} (z_1)(z_2)}$





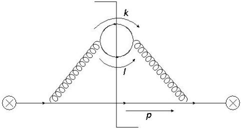
$s_{123} = s_{12}+s_{13}+s_{23},\ s_{12} = 2(k\cdot l),\ s_{13} = 2(k\cdot p),\ s_{23} = 2(l\cdot p)$
$z_1 = \frac{k_-}{Q},\ z_2 = \frac{l_-}{Q},\ z_3 = \frac{p_-}{Q}$

NNLO real-real contribution

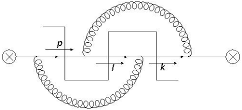
- Matrix element: LO triple collinear splitting functions

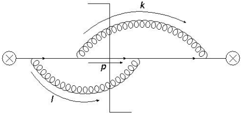
$$P^0_{q'\bar{q}'q},\ P^{0,\text{id}}_{q\bar{q}q},\ P^{0,C_F^2}_{ggq},\ P^{0,C_F C_A}_{ggq}$$

[Catani,Grazzini,99]



- Splitting functions contain different divergence structures
  - $P^0_{q'\bar{q}'q} \sim \frac{1}{s_{123}^2 s_{12}^2 (z_1+z_2)^2}$
  - $P^{0,\text{id}}_{q\bar{q}q} \sim \frac{1}{s_{123}^2 s_{12}},\ \frac{1}{s_{123} s_{12}(1-z_2)(1-z_3)},\ \frac{1}{s_{13} s_{12}(1-z_2)(1-z_3)}$
  - $P^{0,C_F^2}_{ggq} \sim \frac{1}{s_{123}^2 s_{13}},\ \frac{1}{s_{123} s_{13}(z_1)(z_2)},\ \frac{1}{s_{13} s_{23}(z_1)(z_2)}$





$s_{123} = s_{12}+s_{13}+s_{23},\ s_{12} = 2(k\cdot l),\ s_{13} = 2(k\cdot p),\ s_{23} = 2(l\cdot p)$

$z_1 = \frac{k_-}{Q},\ z_2 = \frac{l_-}{Q},\ z_3 = \frac{p_-}{Q}$

$P^0_{q'\bar{q}'q}$

- Identify kinds of divergence in $\frac{1}{s_{123}^2 s_{12}^2 (z_1+z_2)^2}$

$$\frac{1}{s_{123}} \to \infty \text{ if } k \parallel l \parallel p \text{ (triple collinear limit)}$$
$$\frac{1}{z_1+z_2} \to \infty \text{ if } k^\mu \text{ and } l^\mu \to 0 \text{ (double soft limit)}$$
$$\frac{1}{s_{12}} \to \infty \text{ if } \begin{cases} k^\mu \text{ or } l^\mu \to 0 & \text{(single soft limit)} \\ k \parallel l & \text{(double collinear limit)} \end{cases}$$

$P^0_{q'\bar{q}'q}$

- Identify kinds of divergence in $\frac{1}{s_{123}^2 s_{12}^2 (z_1+z_2)^2}$

$$\frac{1}{s_{123}} \to \infty \text{ if } k \parallel l \parallel p \text{ (triple collinear limit)}$$
$$\frac{1}{z_1+z_2} \to \infty \text{ if } k^\mu \text{ and } l^\mu \to 0 \text{ (double soft limit)}$$
$$\frac{1}{s_{12}} \to \infty \text{ if } \begin{cases} k^\mu \text{ or } l^\mu \to 0 & \text{(single soft limit)} \\ k \parallel l & \text{(double collinear limit)} \end{cases}$$

- Phase space parametrization

$$z_{12} = \frac{k_- + l_-}{Q} \text{ (handles double soft limit)}$$
$$b = \frac{k_T}{l_T} \text{ (handles single soft limit)}$$
$$a = \frac{k_- l_T}{k_T l_-},\; t_{kl} = \frac{1-\cos(\theta_{kl})}{2} \text{ (handle double collinear limit)}$$
$$q_T = \sqrt{(k_- + l_-)(k_+ + l_+)} \text{ (handles triple collinear limit)}$$

$P^0_{q'\bar{q}'q}$

- Under previous assumptions and IRC measurement function can be written as

$$\mathcal{M}^1(\tau,\mu) = exp\Big(-\tau q_T \Big(\frac{q_T}{z_{12}\bar{z_{12}}Q}\Big)^n F(z_{12}, a, b, t_{kl}, t_l, t_k)\Big)$$

- Variables a and b have no upper bound
  - Remap to unit hypercube $\rightarrow$ 4 different contributions
- Parametization factorizes all divergences except double collinear divergence
  - $s_{12} \sim ((1-a)^2 + 4at_{kl})$: overlapping singularity at $a \rightarrow 1, t_{kl} \rightarrow 0$
- Introduce NLT $a = 1 - u(1-v)$, $t_{kl} = \frac{u^2 v}{1-u(1-v)}$
  - $s_{12} \sim u^2(1+v)^2$

$P^0_{q'\bar{q}'q}$

- Master formula for NNLO real-real jet function $P^0_{q'\bar{q}'q}$

$$J^{2,RR}_{q'\bar{q}'q}(\tau,\mu) \sim \Gamma\left(\frac{-4\epsilon}{1+n}\right)\int_0^1 \mathrm{d}z_{12} \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}b \int_0^1 \mathrm{d}v \; z_{12}^{-1-4\frac{n}{1+n}\epsilon} \bar{z}_{12}^{-4\frac{n}{1+n}\epsilon} \times u^{-1-2\epsilon}\mathcal{W}(z_{12},u,v,b)\int_0^1 \mathrm{d}t_l \int_0^1 \mathrm{d}t_5' \, \mathcal{T}(t_l,t_5') \times F(z_{12},u,b,v,t_l,t_k^{\pm}(t_l,t_5'))^{\frac{-4\epsilon}{1+n}}$$

- Triple collinear divergence ($q_T \to 0$): $\Gamma\left(\frac{-4\epsilon}{1+n}\right)$
- Double collinear divergence ($u \to 0$): $u^{-1-2\epsilon}$
- Double soft divergence ($z_{12} \to 0$): $z_{12}^{-1-4\frac{n}{1+n}\epsilon}$

$\Rightarrow$ All singularities factorized

NNLO real-real contribution

- Around half of the divergence structures can be computed by the parametrization
- Problems arise in divergence structures with either two different invariant masses

  Such as $\frac{1}{s_{12}s_{13}(1-z_2)(1-z_3)}$

- Or with incompatible combination of soft and collinear divergences

  Such as $\frac{1}{s_{123}s_{13}(z_1)(z_2)}$

- Workaround through phase space projections

Phase space projection

- Goal of phase space projection:
  Make specific divergence explicit and suppress undesired divergences without creating new complicated overlapping divergences
  - In the same spirit as selector functions [Czakon,Heymes,14]
- Example: $\frac{1}{s_{12}s_{13}(1-z_2)(1-z_3)}$
  - Phase space projection: $S_{13,12} = \frac{s_{13}}{s_{12}+s_{13}}$

    $$\frac{1}{s_{12}s_{13}(1-z_2)(1-z_3)}S_{13,12} = \frac{1}{s_{12}(1-z_2)(1-z_3)(s_{12}+s_{13})}$$
  - $(s_{12} + s_{13}) \to 0$ if $k \parallel l \parallel p$ handled by q_T
- Similar phase space projection required for the other cases

Renormalization

- Jet function renormalises multiplicatively in Laplace space

$$\frac{\mathrm{d}J(\tau,\mu)}{\mathrm{d}\ln\mu} = \left[\frac{2(1+n)}{n}\Gamma_{Cusp}(\alpha_s)L + 2\gamma^J(\alpha_s)\right]J(\tau,\mu)$$

$$L = \ln\left(\frac{\mu\bar{\tau}^{\frac{1}{1+n}}}{Q^{\frac{n}{1+n}}}\right),\ \left(\Gamma_{Cusp}(\alpha_s),\gamma^J(\alpha_s)\right) = \sum_{m=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^{1+m}(\Gamma_m,\gamma_m^J)$$

$$J(\tau,\mu) = 1 + \left(\frac{\alpha_s}{4\pi}\right)\left[\left(\frac{1+n}{n}\right)\Gamma_0 L^2 + 2\gamma_0 L + c_1\right] + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{2}\left(\frac{1+n}{n}\right)^2\Gamma_0^2 L^4 + \left(\frac{2}{3}\left(\frac{1+n}{n}\right)\beta_0\Gamma_0 + 2\left(\frac{1+n}{n}\right)\Gamma_0\gamma_0\right)L^3 + \left(2\gamma_0^2 + \left(\frac{1+n}{n}\right)\Gamma_0 c_1 + 2\gamma_0\beta_0 + \left(\frac{1+n}{n}\right)\Gamma_1\right)L^2 + \left(2\gamma_0 c_1 + 2\beta_0 c_1 + 2\gamma_1\right)L + c_2\right] + \mathcal{O}(\alpha_s^3)$$

- Similar function for the counterterm (same RGE)

Preliminary Results

- Expansion and integration performed by pySecDec [Heinrich,et al.,17]
- Bare coefficient comparison of NNLO Thrust contributions

RV: C_F^2	$\frac{1}{\epsilon^4}$	$\frac{1}{\epsilon^3}$	$\frac{1}{\epsilon^2}$	$\frac{1}{\epsilon}$	ϵ^0
[1]	0	0	8.1595	28.0823	32.9427
	0	0.0004	8.1548	28.0772	32.9572
RV: $C_F C_A$					
[1]	-2	-3	-4.7101	-24.4285	-36.5671
	-1.9998	-2.9998	-4.7074	-24.4236	-36.5669
$P^{0,id}_{\bar{q}qq}$					
[1]	0	0	0	-1.4386	-9.2410
	0	0	-0.0001	-1.4361	-9.2196

[1]:[Ritzmann,Waalewijn, 14]

Preliminary Results

- Thrust $C_F n_F$-piece

Thrust	γ_1^{J,n_f}	$c_2^{n_f}$
[1]	-13.3495	-10.7871
	-13.3471	-10.7892

[1]:[Ritzmann,Waalewijn, 14]

- Angularities $C_F n_F$-piece

A	$\gamma_1^{J,n_f,[2]}$	$r_2^{n_f,[2]}$	γ_1^{J,n_f}	$r_2^{n_f}$
-1	-11.1553	$6.76^{+0.08}_{-0.03}$	-11.153	6.928
-0.5	-12.0092	$6.03^{+0.07}_{-0.03}$	-12.007	6.132
0.25	-14.4878	$-1.42^{+0.02}_{-0.06}$	-14.486	-1.480
0.5	-16.5559	$-9.92^{+0.23}_{-0.87}$	-16.547	-11.551

[2]:[Bell,Hornig,Lee,Talbert, 19]

Conclusion

- Automation procedure shown for NLO and NNLO real-virtual interference
- Difficulties in NNLO real-real contribution shown
- Prelimary result for thrust and angularities shown

Outlook

- Completing C_F^2 and $C_F C_A$ color structure
- Including SCET-2 observables
- Similar approach for the gluon jet function

Backup Slide 1: $P^1_{gq}(z_k)$

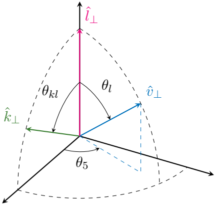
$$P^1_{gq}(z_k) \sim \frac{\pi\Gamma(1-\epsilon)}{\epsilon\tan(\pi\epsilon)\Gamma(1-2\epsilon)}\left\{\left[\frac{(1-\epsilon)z_k^2+2(1-z_k)}{z_k}\right]\left[(C_F-C_A)\left(1-\frac{\epsilon^2}{1-2\epsilon}\right)\right.\right.$$

$$+(C_A-2C_F)\,{}_2F_1\left(1,-\epsilon;1-\epsilon;\frac{z_k}{1-z_k}\right)-C_A\,{}_2F_1\left(1,-\epsilon;1-\epsilon;\frac{1-z_k}{z_k}\right)$$

$$\left.\left.+C_F\right]+(C_F-C_A)\left(\frac{(1-z_k)(2-z_k)}{z_k}\frac{\epsilon^2}{1-2\epsilon}\right)\right\}$$

Backup Slide 2: Angular Parametrization

- Three dimensional subspace spanned by angles $\{\theta_l, \theta_{kl}, \theta_5\}$
  - $\cos(\theta_k) = \cos(\theta_{kl})\cos(\theta_l) + \sin(\theta_{kl})\sin(\theta_l)\cos(\theta_5)$
  - Map θ_i onto unit hypercube via $\cos(\theta_i) = 1 - 2t_i$
  - Introduce t_5' to deal with spurious divergence of t_5
- $t_k^{\pm} = t_l + t_{kl} - 2t_l t_{kl} \pm 2\sqrt{t_l \bar{t}_l t_{kl} \bar{t}_{kl}}(1 - t_5')$



Backup Slide 3: Bare jet function

$$J(\tau) = 1 + \frac{\alpha_s}{4\pi}\left(\frac{\mu\bar{\tau}^{\frac{1}{1+n}}}{Q^{\frac{n}{1+n}}}\right)^{2\epsilon}\left(\frac{x_2}{\epsilon^2} + \frac{x_1}{\epsilon^1} + x_0 + x_{-1}\epsilon + x_{-2}\epsilon^2 + \mathcal{O}(\epsilon^3)\right)$$

$$+\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\mu\bar{\tau}^{\frac{1}{1+n}}}{Q^{\frac{n}{1+n}}}\right)^{4\epsilon}\left(\frac{y_4}{\epsilon^4} + \frac{y_3}{\epsilon^3} + \frac{y_2}{\epsilon^2} + \frac{y_1}{\epsilon} + y_0 + \mathcal{O}(\epsilon)\right) + \mathcal{O}(\alpha_s^3)$$

Bare jet function relations

- $\Gamma_0 = 2\left(\frac{n}{1+n}\right)x_2$
- $\gamma_0 = x_1$
- $c_1 = x_0$
- $\Gamma_1 = 4\left(\frac{n}{1+n}\right)\left[2y_2 - 2x_0x_2 - (\beta_0 + x_1)x_1\right]$
- $\gamma_1 = 2(y_1 - (\beta_0 + x_1)x_0 - x_{-1}x_2)$
- $c_2 = y_0 - (\beta_0 + x_1)x_{-1} - x_{-2}x_2$

Backup Slide 4: Angularities $r_2^{n_f}$ Value

- $r_2^{n_f} = \sigma_{\text{tot},2}^{n_f} - c_2^{n_f}$
- $c_2^{n_f} = \frac{c_2^J}{2} + \frac{c_2^H + c_2^S}{4} + \frac{2\zeta_3\Gamma_0}{(2-a)^2}\left(1 - \frac{a}{3}\right)\beta_0 + \frac{\pi^2}{12}\frac{1}{2-a}\left(\Gamma_1 - \frac{\gamma_0^J\beta_0}{2-a}\right)$
  - $\sigma_{\text{tot},2}^{n_f} = \frac{-11}{2} + 4\zeta_3$
  - $\Gamma_0 = 4$
  - $\Gamma_1 = -\frac{80}{9}$
  - $\gamma_0^J = 6$
  - $\beta = -\frac{4}{3}$
  - $c_2^H = \frac{4085}{81} - \frac{182}{27}\pi^2 + \frac{8}{9}\zeta_3$
- All values proportioned to $C_F n_f$