Matching coefficients in nonrelativistic QCD to two-loop accuracy

[Phys. Rev. D 100, 054016]

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Young Scientists Forum of the SFB TRR 257 09.06.2020
Motivation

- Main focus: Describing heavy quarkonia (quark-antiquark state) dynamics
- Difficult in full QCD, both in
  - weakly coupling regime (below $\Lambda_{\text{QCD}}$)
  - strong coupling regime (above $\Lambda_{\text{QCD}}$)
- One approach: Using Effective Field Theory (EFT)
- Heavy quarks in the low energy regime can be treated as being non-relativistic. The effective field theory must incorporate this.
- Definition of Non-Relativistic QCD (NRQCD) in 1997 by Bodwin, Braaten and Lepage\(^1\).
- Later improvement to potential NRQCD (pNRQCD) by Pineda and Soto in 1998. Describing the interaction as a semi-classical Schrödinger-style potential.

\(^1\)[hep-ph/9407339], see also [hep-ph/9711391]
Starting from QCD, describe a pair of heavy quarks $Q$ with mass $m$. Three relevant energy regimes:

- **hard**: $m \gg |p|, E, \Lambda_{\text{QCD}}$
- **soft**: $mv \approx |p|$
- **ultra-soft**: $mv^2 \approx E$

with

- $v$: relative velocity between heavy quark and antiquark
- $p$: relative (3-)momentum between heavy quark and antiquark
- $E$: Binding energy of the quarkonium state

$\Rightarrow$ First step in low energy approximation: \textit{Integrating out} hard degrees of freedom, i.e. expand QCD action in $1/m$
\section*{QCD and NRQCD}

Non-relativistic expansion in $1/m$:

$$\mathcal{L}_{\text{NRQCD}} = \mathcal{L}_\psi + \mathcal{L}_\chi + \mathcal{L}_{\psi\chi} + \mathcal{L}_G + \mathcal{L}_{\text{light}} + \mathcal{O}\left(\frac{1}{m^3}\right)$$

where

$$\mathcal{L}_\psi = \psi^\dagger \left( iD^0 + \frac{D^2}{2m} \right) \psi - \frac{d_1 g_s}{2m} \psi^\dagger (\sigma \cdot B) \psi$$

$$+ \psi^\dagger \left( \frac{d_2 g_s}{8m^2} [D, E] + i \frac{d_3 g_s}{8m^2} \sigma^{ij} \{ D^i, E^j \} \right) \psi + \mathcal{O}\left(\frac{1}{m^3}\right)$$

$$\mathcal{L}_\chi = - \mathcal{L}_\psi (\psi \rightarrow \chi)$$

- $\psi$: Pauli particle spinor
- $\chi$: Pauli antiparticle spinor
- $d_i$: Wilson coefficients
- $E$: Chromo-electric field
- $B$: Chromo-magnetic field

⇒ Propagation of heavy quarks is treated perturbatively, since $\nu \ll 1$. 

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QCD and NRQCD

Especially relevant for bound states at $O \left( \frac{1}{m^2} \right)$: four-quark interactions

$$\mathcal{L}_{\psi \chi} = \frac{d_{ss}}{m^2} \psi \dagger \psi \dagger \chi + \frac{d_{sv}}{m^2} \psi \dagger \psi \dagger \sigma_i \psi \dagger \sigma_i \chi$$

$$+ \frac{d_{vs}}{m^2} \psi \dagger T^a \psi \dagger T^a \chi + \frac{d_{vv}}{m^2} \psi \dagger T^a \sigma_i \psi \dagger T^a \sigma_i \chi$$

$$+ O \left( \frac{1}{m^3} \right).$$

With the Wilson coefficients $d_i$ contributing to different quark configurations (with historical naming convention):

<table>
<thead>
<tr>
<th>Color singlet</th>
<th>Spin singlet</th>
<th>Spin triplet</th>
</tr>
</thead>
<tbody>
<tr>
<td>Color octet</td>
<td>$d_{ss}$</td>
<td>$d_{sv}$</td>
</tr>
<tr>
<td></td>
<td>$d_{vs}$</td>
<td>$d_{vv}$</td>
</tr>
</tbody>
</table>

- $\mathcal{L}_G$ gives the gluon dynamics such as self-interaction of the chromo-magnetic and chromo-electric field
- $\mathcal{L}_{\text{light}}$ is the same as in full QCD with decoupled heavy quarks

Matching coefficients in nonrelativistic QCD to two-loop accuracy
COMPUTATION OF $d_{ss}$, $d_{sv}$, $d_{vs}$, $d_{vv}$

How to determine $d_{ss}$, $d_{sv}$, $d_{vs}$ and $d_{vv}$?
**Computation of** $d_{ss}, d_{sv}, d_{vs}, d_{vv}$

How to determine $d_{ss}, d_{sv}, d_{vs}$ and $d_{vv}$?

Using matching à la Wilson!
I.e. comparison of Green’s functions with the same particles in the full (QCD) and effective Theory (NRQCD).
Computation of \( d_{ss}, d_{sv}, d_{vs}, d_{vv} \)

How to determine \( d_{ss}, d_{sv}, d_{vs} \) and \( d_{vv} \)?

Using matching à la Wilson!
I.e. comparison of Green’s functions with the same particles in the full (QCD) and effective Theory (NRQCD).

\[
\langle \bar{Q}Q\bar{Q}Q \rangle_{\text{eff trunc}}^{\text{full trunc}} = \langle \bar{Q}Q\bar{Q}Q \rangle_{\text{full trunc}}^{\text{full trunc}}
\]

\[
\Leftrightarrow \tilde{Z}_{2}^{2} \langle \bar{Q}^{0}\bar{Q}^{0}Q^{0}\bar{Q}^{0} \rangle_{\text{eff trunc}}^{\text{eff trunc}} = Z_{2}^{2} \langle \bar{Q}^{0}\bar{Q}^{0}Q^{0}\bar{Q}^{0} \rangle_{\text{trunc}}^{\text{full trunc}}
\]

\[
\Rightarrow \tilde{Z}_{2}^{2} A^{\text{NRQCD}} = Z_{2}^{2} A^{\text{QCD}}
\]

\( Z_{2}, \tilde{Z}_{2} \): wave function renormalization constant in QCD and NRQCD respectively.
Computation of $d_{ss}, d_{sv}, d_{vs}, d_{vv}$

The amplitudes $\mathcal{A}$ can be split into two contributions:

- **hard**: Including momentum scaling around $\mathcal{O}(m)$. Hidden inside the Wilson coefficients of the EFT
- **soft**: Everything else, including soft, ultrasoft and potential. Represented by the EFT Feynman rules

⇒ That means for the full theory: Threshold expansion
Computation of $d_{ss}, d_{sv}, d_{vs}, d_{vv}$

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![Feynman diagram]

**hard**: $k_0 \sim m$, $k \sim m$

**soft**: $k_0 \sim \sqrt{y}$, $k \sim \sqrt{y}$

**potential**: $k_0 \sim \sqrt{y}$, $k \sim \frac{\sqrt{y}}{m}$

**ultrasoft**: $k_0 \sim \frac{\sqrt{y}}{m}$, $k \sim \frac{\sqrt{y}}{m}$

with $y = m^2 - p^2$

(same for $l$)
**Computation of** $d_{ss}, d_{sv}, d_{vs}, d_{vv}$

Wilson coefficients only includes *hard* contributions, leading to the following simplifications:

- Only the hard part of the loop momenta, $k, l, \ldots \sim m$
- Only leading order in heavy quark kinematics: static particles
  
  $$p = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- Suppress higher order terms in the EFT

**BUT:** There may appear uncanceled poles that only vanish in the complete sum *hard* + *soft* + *potential* + *ultrasoft*

The matching condition is then:

$$\Rightarrow \quad = \quad + \quad + \quad + \ldots$$
Computational Setup

- Automated diagram generation: qgraf
- Translation of analytic expressions in FORM
- Topology mapping: exp
- Evaluation of color structure: color
- Integral reduction to Master Integrals: FIRE
- Partial fractioning in Mathematica\(^2\)

⇒ Sum of 334 diagrams to compute at 2-loop level (“easy problem”)

\(^2\)Thanks to Florian Herren
Computational Setup: Projection

Further simplifications:

- Projection of tensor integrals onto scalar integrals, using a projection operator \( \hat{P} \)

\[
\mathcal{A} = \sum_i a_i B^{(1)}_i \otimes B^{(2)}_i
\]

\[
\Rightarrow a_i = \text{Tr} \left( \hat{P}_i A \right)
\]

- Simple ansatz:

\[
\hat{P}_i = \sum_j p_{ij} B^{(1)}_j \otimes B^{(2)}_j
\]

where \( A \otimes B \equiv (\bar{u} A u) (\bar{v} B v) \) or \( (\bar{u} A v) (\bar{v} B u) \)
Computational Setup: Projection

\[ B_1^{(1)} \otimes B_1^{(2)} = 1 \otimes 1, \quad B_9^{(1)} \otimes B_9^{(2)} = \gamma^\mu \gamma^\nu \otimes \gamma_\mu \gamma_\nu, \quad B_1^{(1)} \otimes B_1^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma, \]

\[ B_2^{(1)} \otimes B_2^{(2)} = \gamma \otimes 1, \quad B_{10}^{(1)} \otimes B_{10}^{(2)} = \gamma^\mu \gamma^\nu \gamma \otimes \gamma_\mu \gamma_\nu, \quad B_{18}^{(1)} \otimes B_{18}^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma, \]

\[ B_3^{(1)} \otimes B_3^{(2)} = 1 \otimes \gamma, \quad B_{11}^{(1)} \otimes B_{11}^{(2)} = \gamma^\mu \gamma^\nu \otimes \gamma_\mu \gamma_\nu \gamma, \quad B_{19}^{(1)} \otimes B_{19}^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma, \]

\[ B_4^{(1)} \otimes B_4^{(2)} = \gamma \otimes \gamma, \quad B_{12}^{(1)} \otimes B_{12}^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma, \quad B_{20}^{(1)} \otimes B_{20}^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\lambda \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma \gamma_\lambda, \]

\[ B_5^{(1)} \otimes B_5^{(2)} = \gamma^\mu \otimes \gamma_\mu, \quad B_{13}^{(1)} \otimes B_{13}^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \otimes \gamma_\mu \gamma_\nu \gamma_\rho, \quad B_{21}^{(1)} \otimes B_{21}^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\lambda \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma \gamma_\lambda, \]

\[ B_6^{(1)} \otimes B_6^{(2)} = \gamma^\mu \gamma \otimes \gamma_\mu \gamma_\mu, \quad B_{14}^{(1)} \otimes B_{14}^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \gamma \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\rho, \quad B_{22}^{(1)} \otimes B_{22}^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\lambda \gamma \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma \gamma_\lambda, \]

\[ B_7^{(1)} \otimes B_7^{(2)} = \gamma^\mu \otimes \gamma_\mu \gamma \gamma, \quad B_{15}^{(1)} \otimes B_{15}^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma \gamma, \quad B_{23}^{(1)} \otimes B_{23}^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\lambda \gamma \gamma \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma \gamma_\lambda, \]

\[ B_8^{(1)} \otimes B_8^{(2)} = \gamma^\mu \gamma \otimes \gamma_\mu \gamma \gamma, \quad B_{16}^{(1)} \otimes B_{16}^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \gamma \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\gamma, \quad B_{24}^{(1)} \otimes B_{24}^{(2)} = \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\lambda \gamma \gamma \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma \gamma_\lambda, \]

with \( \gamma = p / m \)

- For technical reasons, applying Dirac's equation is only possible after the projection
- No simplification since Fierz identities are only defined in \( d = 4 \) dimensions
Computational Setup: Operator basis

Additionally: Technical operator basis

\[ O_s^{(0)} = \psi^\dagger \psi^\dagger \chi , \]
\[ O_s^{(1)} = - \frac{1}{8} \psi^\dagger [\sigma^i, \sigma^j] \psi^\dagger [\sigma^i, \sigma^j] \chi , \]
\[ O_s^{(2)} = \frac{1}{64} \psi^\dagger [\sigma^i, \sigma^j] [\sigma^k, \sigma^l] \psi^\dagger [\sigma^i, \sigma^j] [\sigma^k, \sigma^l] \chi , \]
\[ \ldots \]
\[ O_o^{(0)} = \psi^\dagger T^a \psi^\dagger T^a \chi , \]
\[ O_o^{(1)} = - \frac{1}{8} \psi^\dagger T^a [\sigma^i, \sigma^j] \psi^\dagger T^a [\sigma^i, \sigma^j] \chi , \]
\[ O_o^{(2)} = \frac{1}{64} \psi^\dagger T^a [\sigma^i, \sigma^j] [\sigma^k, \sigma^l] \psi^\dagger T^a [\sigma^i, \sigma^j] [\sigma^k, \sigma^l] \chi , \]
\[ \ldots \]

- Prevent undefined Levi-Civita-symbol \( \epsilon_{ijk} \) in \( d \) dimensions
- Straightforward translation in regular \( d = 4 \) basis
Classification of diagrams:

“Scattering” Diagrams:
- Only contributions when two different flavors with different masses are regarded
- For simplification, we computed only the case with a single mass

“Annihilation” Diagrams:
- Remaining same flavor contribution

⇒ Providing cross checks for future computations
Results

Only scattering diagrams:

\[ \mathcal{L}_{\psi\chi} = \frac{d_{ss}}{m^2} \psi^\dagger \psi \chi^\dagger \chi + \frac{d_{sv}}{m^2} \psi^\dagger \sigma^i \psi \chi^\dagger \sigma^i \chi + \frac{d_{sv}}{m^2} \psi^\dagger T^a \psi \chi^\dagger T^a \chi + \frac{d_{sv}}{m^2} \psi^\dagger \sigma^i \psi \chi^\dagger \sigma^i \chi \]

\[ d_{ss}^{(2)} = \frac{C_F}{N_C} \left[ \frac{3 \zeta_3}{2} - \frac{57}{4} + \frac{55 \pi^2}{24} \right] + \frac{C_F n_f T_F}{N_C} \left[ \frac{1}{6 \epsilon^2} - \frac{7}{18 \epsilon} - \frac{\ln \left( \frac{\mu^2}{m^2} \right)}{6} - \frac{\pi^2}{9} - \frac{19}{9} \right] + \frac{C_F n_f T_F}{N_C} \left[ \frac{\pi^2}{9} - \frac{20}{27} \right]

\[ + \frac{C_A C_F}{N_C} \left[ - \frac{11}{24 \epsilon^2} - \frac{8}{9} - \frac{7 \pi^2}{24} + \frac{11 \ln \left( \frac{\mu^2}{m^2} \right)}{24} - \frac{119 \zeta_3}{16} + \frac{685 \pi^2}{288} + \frac{791}{24} + \frac{5}{8} \pi^2 \log 2 \right] + \mathcal{O}(\epsilon), \]

\[ d_{sv}^{(2)} = \frac{C_F}{N_C} \left[ \frac{\pi^2}{12 \epsilon} + \frac{15 \zeta_3}{4} + \frac{77 \pi^2}{48} - \frac{17}{12} - \frac{5 \pi^2 \log 2}{6} \right] - \frac{5 C_F n_f T_F}{9 N_C} + \frac{C_A C_F}{N_C} \left[ - \frac{\pi^2}{48 \epsilon} + \frac{7 \pi^2}{288} + \frac{5 \zeta_3}{2} + \frac{7 \pi^2}{288} + \frac{1}{36} + \frac{1}{3} \pi^2 \log 2 \right] + \frac{4 C_F n_f T_F}{9 N_C} + \mathcal{O}(\epsilon), \]

\[ d_{sv}^{(2)} = \frac{C_F}{N_C} \left[ \frac{9 \zeta_3}{2} - \frac{103}{2} + \frac{227 \pi^2}{24} \right] + \frac{C_A n_f T_F}{N_C} \left[ - \frac{5}{12 \epsilon^2} + \frac{35}{36 \epsilon} - \frac{5 \ln \left( \frac{\mu^2}{m^2} \right)}{12} + \frac{5 \pi^2}{18} + \frac{77}{18} \right] + \frac{C_A C_F}{N_C} \left[ - \frac{7}{3 \epsilon^2} + \frac{113}{36} - \frac{5 \pi^2}{6} + \frac{11 \ln \left( \frac{\mu^2}{m^2} \right)}{6} - \frac{65 \zeta_3}{144} + \frac{575 \pi^2}{36} - \frac{\pi^2 \log 2}{2} \right]

\[ + \frac{C_A n_f T_F}{N_C} \left[ - \frac{1}{5 \epsilon} - \frac{5 \pi^2}{18} + \frac{1289}{675} \right] + \frac{C_A n_f T_F}{N_C} \left[ - \frac{1}{5 \epsilon} - \frac{5 \pi^2}{18} + \frac{1289}{675} \right] + \xi \left[ \frac{3 C_A^2}{32} + 3 \left( \frac{13}{150} - \frac{1}{20 \epsilon} \right) \right] n_f T_F + \mathcal{O}(\epsilon), \]

\[ d_{sv}^{(2)} = \frac{C_F}{N_C} \left[ \frac{\pi^2}{4 \epsilon} + \frac{43 \zeta_3}{4} + \frac{155 \pi^2}{24} - \frac{29}{6} + \frac{23 \pi^2 \log 2}{6} \right] - \frac{20}{9} C_F n_f T_F + \frac{16 C_F n_f T_F}{9} + \frac{C_A C_F}{N_C} \left[ - \frac{3 \pi^2}{4 \epsilon} - \frac{1}{12 \epsilon} + \frac{13 \zeta_3}{8} + \frac{137 \pi^2}{36} + \frac{121}{36} + \frac{59 \pi^2 \log 2}{12} \right]

\[ + \frac{C_A n_f T_F}{N_C} \left[ 35 \pi^2 + \frac{1}{18 \epsilon} + \frac{1}{12 \epsilon} + \frac{1}{18 \epsilon} + \frac{1}{18 \epsilon} + \frac{\ln \left( \frac{\mu^2}{m^2} \right)}{12} + \frac{\pi^2}{18} - \frac{54}{31} \right] + \mathcal{O}(\epsilon). \]

[hep-ph/1907.08227]

⇒ Not fully renormalized, but sufficient for pNRQCD building blocks

⇒ Found agreement with 1-loop results from [hep-ph/9802365]
Conclusion

- Calculation of 2-loop matching coefficients in NRQCD with heavy quarks and antiquarks of same mass
- Usage of asymptotic threshold expansion lead to kinematically simple diagrams
- Splitting in “scattering” and “annihilation” diagrams to enable building blocks and cross checks for different processes
- Definition of a DR-compatible operator basis and thus avoiding evanescent operators
- Result is analytically exact: Translation to NRQED easily obtained from $C_A = 0$, $C_F = 1$, $T_F = 1$
- Relevant for $N^3LL$ (and $N^4LO$) threshold predictions of $t\bar{t}$ production
- Also other applications to describe heavy bound systems, like $\Upsilon$