Zero-jettiness soft and beam functions at NNLO to higher orders in epsilon

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Outline

1. Motivation
2. Slicing
3. Beam function
4. Soft function
Motivation

- **Fully differential** colour singlet production $pp \rightarrow V + \text{jets}$ currently known at $N^2\text{LO}$

- Few processes already known at $N^3\text{LO}$: Higgs rapidity distribution, fully inclusive cross section for $gg \rightarrow H$, $pp \rightarrow \gamma^*$ and $bb \rightarrow H$
  
[Dulat, Mistlberger, Peloni’18]

[Mistlberger’18][Duhr, Dulat, Mistlberger’20][Duhr, Dulat, Hirschi, Mistlberger’20]

- High experimental precision requires knowledge of $N^3\text{LO}$ corrections to interesting processes: Higgs boson production, Drell-Yan, . . .

[Dulat, Mistlberger, Peloni’18]
Problem:
- Soft, collinear and virtual singularities have to cancel each other → spread across phase-spaces with different jet multiplicity
- Systematic approach to compensate singularities (subtraction schemes) only known at $N^2$LO → not at $N^3$LO

Solution:
- Concentrate the most offending singularities in one phase-space region through smart choice of an observable

Example:
- Vector boson production $pp \rightarrow V + \text{jets}$ with $p_\perp$ as observable
Slicing

\[
\frac{d\sigma}{dp_\perp}
\]

Motivation

Zero-jettiness soft and beam functions

Beam function

Soft function

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Slicing

\[ \frac{d\sigma}{dp_\perp} \]

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Slicing

\[ N^3 \text{LO } pp \rightarrow V + 0 \text{ jets} \]

\[ \frac{d\sigma}{dp_{\perp}} \]

\[ p_0 \]

\[ p_{\perp} \]

\[ N^2 \text{LO } pp \rightarrow V + 1 \text{ jet} \]

soft or collinear

hard

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Slicing

\[ N^3 \text{LO } pp \rightarrow V + 0 \text{ jets } \frac{d\sigma}{dp_\perp} \]

- \( p_\perp > p_0 \): \( N^2 \text{LO} \rightarrow \text{known} \)
- \( p_\perp < p_0 \): \( N^3 \text{LO} \rightarrow \text{unknown, however soft and collinear limits lead to simplification} \Rightarrow \text{Slicing} \)

[Giele, Glover ’92]
Zero-jettiness

- zero-jettiness $\tau$ instead $p_\perp$ as slicing variable

$$\tau = \sum \min_{i \in 1, 2} \left[ \frac{2p_i \cdot k_j}{Q_i} \right]$$

- simplification through the factorization theorem derived in SCET

$$\lim_{\tau \to 0} d\sigma_{pp \to V}(\tau) = B_\tau \otimes B_\tau \otimes S_\tau \otimes H_\tau \otimes d\sigma^{LO}_V$$

[Stewart, Tackmann, Waalewijn ’10]

- beam function $B$, describes singularities due to collinear emission of incoming particles
- soft function $S$, describes singularities due to soft emission
- hard function $H$, describes corrections to the Born-like process
Zero-jettiness

- $B$ for $p_\perp$ known through N$^3$LO QCD  
  [Luo, Yang, H. Zhu, Y. Zhu ’19]

- $B$ and $S$ for zero-jettiness currently known through NNLO QCD  
  [Gaunt, Stahlhofen, Tackmann’14] [Gaunt, Stahlhofen, Tackmann’14]  
  [Boughezal, Petriello, Schubert, Xing ’17][Monni, Gehrmann, Luisoni ’11]  
  [Kelley, Schwartz, Schabinger, Zhu ’11]

- Since Yesterday: $B$ for zero-jettiness calculated through N$^3$LO QCD  
  [Ebert, Mistlberger, Vita ’20]

- All quantities needed to N$^3$LO → renormalization → NNLO through higher orders in $\epsilon$

- In case of $S$ develop tools for N$^3$LO calculation, test them on NNLO

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Zero-jettiness soft and beam functions
Calculation of $B_{ij}$

- $B_{ij}$ as an integral over the collinear QCD splitting function $P_{j \to i^* \{m\}}$

$$B_{ij} \sim \sum_{\{m\}} \int \text{dPS}^{(m)} P_{j \to i^* \{m\}}$$

[Ritzmann, Waalewijn'14]

- Obtain splitting function through collinear projection operator $\mathcal{P}$

[Catani, Grazzini '00]

$$P_{j \to i^* \{m\}} \sim \mathcal{P}|M_{j \to i^* \{m\}}|^2$$

- Since QCD is charge-conjugation invariant we only need to consider the sets $(i, j) \in \{(q_i, q_j), (q_i, g), (q_i, \bar{q}_j), (g, g), (g, q_j)\}$
Phase space measure

- Consider collinear emissions off $p_1$

\[ \tau = \sum \min_{i \in 1,2} \left[ \frac{2p_i \cdot k_m}{Q_i} \right] \Rightarrow \sum_{m} \frac{2p_1 \cdot k_m}{Q_1} \]

- $m$ particle phase space now defined as

\[
\int dPS^{(m)} = \left( \prod_m \int \frac{d^d k_m}{(2\pi)^{d-1}} \delta^+(k_m^2) \right) \\
\times \delta \left( 2 \sum_m k_m \cdot p - \frac{t}{z} \right) \delta \left( \frac{2 \sum_m k_m \cdot \bar{p}}{s} - (1 - z) \right)
\]
Reverse unitarity, IBPs

- Use reverse unitarity to rewrite phase-space delta functions
  \[ \delta (p^2 - m^2) = \frac{i}{2\pi} \left[ \frac{1}{p^2 - m^2 + i\epsilon} - \frac{1}{p^2 - m^2 - i\epsilon} \right], \]

- Use Integration-by-parts machinery to reduce beam function to sum of master integrals (MIs)

- Compute master integrals

Motivation
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  Slicing
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  Beam function
  ○○○○
  Soft function
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Real-real master integral

Example of a master integral:

\[
I = \int \frac{d^d k_1}{(2\pi)^{d-1}} \int \frac{d^d k_2}{(2\pi)^{d-1}} \delta^+(k_1^2) \delta^+(k_2^2) \delta (2k_{12} \cdot p - \frac{t}{z}) \\
\quad \times \delta (2k_{12} \cdot \bar{p} - (1 - z)) \\
\quad \times \frac{(p - k_1)^2 k_{12}^2 \bar{p} \cdot k_2}{(p - k_1)^2 k_{12}^2 \bar{p} \cdot k_2}.
\]

Its solution reads

\[
I = -\frac{(\Omega_d-2)^2}{4(2\pi)^{2d-2}} t^{-1-2\epsilon} \left(\frac{1 - z}{z}\right)^{-1-2\epsilon} \frac{\Gamma(1 - \epsilon)^2 \Gamma(-\epsilon)^2}{\Gamma(1 - 2\epsilon)^2} \\
\quad \times {}_3F_2 (1, 1, -\epsilon; 1 - 2\epsilon, 1 - \epsilon, 1)
\]
Beam functions at NNLO

- We find that all five beam functions $B_{q_i,q_j}$, $B_{q_i,g}$, $B_{q_i,qj}$, $B_{g,g}$, $B_{g,qj}$ can be expressed through 12 master integrals
- Integrals easy enough to be straightforwardly integrated
- Many integrals can be computed in closed form in $\epsilon$
- General structure of the result

$$I_{gg}^{(2)} = \sum_{k=0}^{5} \frac{1}{\mu^2} L_k \left( \frac{t}{\mu^2} \right) F_+^{(k)}(z) + \delta(t) F_\delta(z),$$

$$F_\delta(z) = C_{-1} \delta(1 - z) + \sum_{k=0}^{5} C_k L_k(1 - z) + F_{\delta,h}(z),$$

where we define the plus distribution

$$L_n(z) = \left[ \frac{\ln^n(z)}{z} \right]_+$$
Calculation of $S$

- Soft function $S$ as an integral over soft emission functions (eikonals)
  \[ S \sim \sum_{\{m\}} \int dP S^{(m)} \xi\{m\} \]

- RV contribution simple, focus on RR
Phase space measure

- No simplification of $\tau$ due to soft limit

$$\tau = \sum_m \min_{i \in 1,2} \left[ \frac{2p_i \cdot k_m}{Q_i} \right]$$

- The $m$ particle phase space reads

$$\int dPS^{(m)}(m) = \prod_n \frac{d^d k_n}{(2\pi)^{d-1}} \delta^+(k_n^2) M_m$$

- $m$ particle measurement function $M_m$ splits integrand into sectors according to the emission of partons into different hemispheres
Phase space measure at NNLO

\[ M_2 = \delta (\tau - 2p \cdot k_1 - 2p \cdot k_2) \theta (2\bar{p} \cdot k_1 - 2p \cdot k_1) \theta (2\bar{p} \cdot k_2 - 2p \cdot k_2) + (p^\mu \leftrightarrow \bar{p}^\mu) \]
\[ + \; \delta (\tau - 2\bar{p} \cdot k_1 - 2p \cdot k_2) \theta (2p \cdot k_1 - 2\bar{p} \cdot k_1) \theta (2\bar{p} \cdot k_2 - 2p \cdot k_2) + (p^\mu \leftrightarrow \bar{p}^\mu) \]

- Integrand symmetric under exchange of \( p \) and \( \bar{p} \) \( \rightarrow \) only need to consider two configurations \( A \) and \( B \)
- Partons emitted into the same hemisphere
  \[ M_A(k_1, k_2) = \delta (\tau - 2p \cdot k_1 - 2p \cdot k_2) \theta (2\bar{p} \cdot k_1 - 2p \cdot k_1) \theta (2\bar{p} \cdot k_2 - 2p \cdot k_2) \]
- Partons emitted into different hemispheres
  \[ M_B(k_1, k_2) = \delta (\tau - 2\bar{p} \cdot k_1 - 2p \cdot k_2) \theta (2p \cdot k_1 - 2\bar{p} \cdot k_1) \theta (2\bar{p} \cdot k_2 - 2p \cdot k_2) \]
Treatment of $\theta$ functions

- We would like to use IBP through reversed unitarity. But what do with $\theta$ functions?

- Use the identity

$$\theta(b - a) = \int_0^1 dz \delta(zb - a) b$$

which holds for $a, b \in [0, \infty)$.

- $M_{A,B}$ now functions of auxiliary parameters $z_1$ and $z_2$

$$M_A = \int_0^1 dz_1 \int_0^1 dz_2 \delta(\tau - 2p \cdot k_1 - 2p \cdot k_2) \delta(2z_1 \bar{p} \cdot k_1 - 2p \cdot k_1) 2\bar{p} \cdot k_1$$

$$\times \delta(2z_2 \bar{p} \cdot k_2 - 2p \cdot k_2) 2\bar{p} \cdot k_2$$

- Commute $z_i$ and $k_i$ integration, use reverse unitarity and IBPs to obtain master integrals
Master integrals

- Only 9 MIs that can be solved in closed form in $\epsilon$
- For example

$$I = \left( \prod_{n}^{2} \int \frac{d^d k_n}{(2\pi)^{d-1}} \delta^+ (k_n^2) \right) \delta (\tau - 2p \cdot k_1 - 2p \cdot k_2) \delta (2p \cdot k_1 - z_1 2\bar{p} \cdot k_1)$$

$$\times \delta (2p \cdot k_2 - z_2 2\bar{p} \cdot k_2) \left( p \cdot k_1 - \frac{\tau z_1}{2(z_1 - z_2)} \right)^{-1} (k_1 \cdot k_2)^{-1}$$

$$= -\frac{\tau^{-2-4\epsilon}}{(z_1 z_2)^{1-\epsilon}} \frac{\Gamma^2 (-2\epsilon)}{\Gamma (-4\epsilon)} \frac{z_2}{z_1} \frac{z_1 - z_2}{z_1} \, _2F_1 \left( 1, 1 + \epsilon, 1 - \epsilon, \frac{z_2}{z_1} \right)$$

$$\times \, _2F_1 \left( 1, -2\epsilon, -4\epsilon, \frac{z_1 - z_2}{z_1} \right) .$$
Integration over auxiliary parameters

Integrate over $z_1$ and $z_2$

\[ S_{qq,A}^{(2)} = 2 \int_0^1 \, dz_1 \int_0^{z_1} \, dz_2 \left[ \frac{32\epsilon(2\epsilon - 1)z_1z_2}{\tau^2(z_1 - z_2)^2} I_{q\bar{q},3}^{q\bar{q},3} + \frac{8\epsilon(2\epsilon - 1)z_1z_2(z_1 + z_2)}{\tau(z_1 - z_2)^3} I_{00}^{q\bar{q},3} \right. \\
+ \frac{8(z_1 + z_2)}{(4\epsilon - 1)(z_1 - z_2)^4} \left( 16\epsilon^3 z_1 z_2 - \epsilon^2(z_1 + z_2)^2 + \epsilon \left( z_1^2 - 6z_1 z_2 + z_2^2 \right) + z_1 z_2 \right) \right] \\
+ \frac{8\tau z_1 z_2}{(z_1 - z_2)^5} \left( \epsilon^2(z_1 + z_2)^2 - z_1 z_2 \right) I_{11}^{q\bar{q},3}\]

- Rescale $z_2 = tz_1 \rightarrow z_1$ completely factors out
- Remaining integral singular in $t = 1 \rightarrow$ endpoint subtraction

Motivation

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Zero-jettiness soft and beam functions
Integration over auxiliary parameters

\[ S_{q\bar{q},B}^{(2)} = 2 \int_0^1 dz_1 \int_0^{z_1} dz_2 \left[ \frac{32 \epsilon (4 \epsilon - 1) z_1 z_2}{\tau^2 (z_1 z_2 - 1)^2} I_{q\bar{q},2}^{00} \right. \]

\[ - 8 \tau z_2 \left( \epsilon^2 (z_2 + 1) \left( z_1^2 z_2^2 - 1 \right) + \epsilon (z_1 - 1) z_2 (z_1 z_2 + 1) - z_1 (z_2 - 1) z_2 \right) \frac{I_{q\bar{q},4}^{11}}{(z_2 - 1)^3 (z_1 z_2 - 1)^3} \]

\[ + \left( \frac{16 z_1 z_2 (z_1 (-z_2) + z_1 + z_2 - 1)}{(z_1 - 1)^2 (z_2 - 1)^2 (z_1 z_2 - 1)^2} \right. \]

\[ + \epsilon \frac{8 \left( z_1^3 (- (z_2 - 3)) z_2^2 + z_1^2 z_2 \left( 3 z_2^2 - 11 z_2 + 6 \right) + z_1 \left( 6 z_2^2 - 11 z_2 + 3 \right) + 3 z_2 - 1 \right)}{(z_1 - 1)^2 (z_2 - 1)^2 (z_1 z_2 - 1)^2} \]

\[ + \epsilon^2 \frac{32 \left( z_1^2 z_2^2 + z_1^2 z_2 \left( z_2^2 - 4 z_2 + 2 \right) + z_1 \left( 2 z_2^2 - 4 z_2 + 1 \right) + z_2 \right)}{(z_1 - 1)^2 (z_2 - 1)^2 (z_1 z_2 - 1)^2} \left. \right] \]

Emission into different hemispheres \( \rightarrow \) "finite" \( \rightarrow \) expand in \( \epsilon \)
Results for the soft function

- The final result has the following form

\[
S^{(2)} = \tau^{-1-4\epsilon} \left( C_a^2 S_A^{(2)} + C_a T_F n_f S_B^{(2)} + C_A C_a S_C^{(2)} \right)
\]

- For example

\[
S_A^{(2)} = -\frac{8}{\epsilon^3} + \frac{16\pi^2}{3\epsilon} + 128\zeta(3) + \epsilon \frac{16\pi^4}{5} + \epsilon^2
\]

\[
\left( 1536\zeta(5) - \frac{256\pi^2\zeta(3)}{3} \right) + \epsilon^3 \left( \frac{2528\pi^6}{945} - 1024\zeta(3)^2 \right)
\]
Conclusion and Outlook

- Simplification of already existing NNLO calculations due to advanced tools
- Beam functions can be calculated as phase space integrals over matrix elements squared in the collinear limit
- Soft function can be calculated as phase space integrals over eikonal functions
- The identity \( \theta(b - a) = \int_0^1 dz \delta(z b - a) b \) allows for the use of multi-loop technology
- At NNLO all five partonic beam functions \( B_{q_i,q_j}, B_{q_i,g}, B_{q_i,\bar{q}_j}, B_{g,g}, B_{g,q_j} \) in terms of 12 MIs, soft function in terms of 9 MIs
- All quantities successfully calculated to order \( \epsilon^2 \), and \( \mathcal{O}(\epsilon^0) \) results checked against literature [DB’20]
- Outlook: N3LO beam function calculations [Ebert et al.’20][Behring et al.’19] N3LO soft function calculation ongoing...