Numerical methods and beam functions at NNLO and beyond

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1. At hadron colliders, there are many interesting final states to observe. Among these, the production of colour-neutral final states has gotten much attention.

2. There are many methods to compute cross sections at NLO and NNLO. However, at N\(^3\)LO the implementation of these methods is involved.

3. I shall present our methods to compute beam function which have already been computed at NLO [Becher, Neubert ‘11], NNLO [Gehrmann, Lübbert, Yang ’12, ’14] and recently N\(^3\)LO [Luo, Wang, Xu, Yang, Yang, Zhu ‘19, Ebert, Mistlberger, Vita, ‘20]

4. The beam function was the last ingredient needed to implement the q\(_T\) slicing method at N\(^3\)LO.
The $q_T$ slicing method

[Catani, Grazzini ‘07, ‘15]

\[ p + p \rightarrow F(q_T) + X \]

\[ \sigma_{N^{m\text{LO}}}^{F} = \int_{q_T,\text{cut}}^{\infty} dq_T \frac{d\sigma_{N^{m\text{LO}}}^{F}}{dq_T} + \int_{q_T,\text{cut}}^{\infty} dq_T \frac{d\sigma_{N^{m\text{LO}}}^{F}}{dq_T} \]

\[ = \int_{0}^{q_T,\text{cut}} dq_T \frac{d\sigma_{N^{m\text{LO}}}^{F}}{dq_T} + \int_{q_T,\text{cut}}^{\infty} dq_T \frac{d\sigma_{N^{m-1\text{LO}}}^{F+\text{jet}}}}{dq_T} \]

enough to know in small-$q_T$ approximation

known
Factorization

where $F = H, Z, W, ZZ, WW, tt, \ldots$.

If not colourless the final state must be at least massive

$$q^2 \sim q_T^2 \gg \Lambda_{QCD} \quad \text{collinear factorization}$$

$$\frac{d\sigma_F}{d\Phi} = \phi_1 \otimes \phi_2 \otimes C + O\left(\frac{1}{q^2}\right)$$

$$q^2 \gg q_T^2 > \Lambda_{QCD} \quad \text{small-}q_T \text{ factorization}$$

$$\frac{d\sigma_F}{d\Phi} = B_1 \otimes B_2 \otimes H \otimes S + O\left(\frac{q_T^2}{q^2}\right)$$
All those functions

To get the cross section at $N^m\text{LO}$, we need to know all those functions at $N^m\text{LO}$

$$\frac{d\sigma_{F}^{N^m\text{LO}}}{d\Phi} = B_1^{N^m\text{LO}} \otimes B_2^{N^m\text{LO}} \otimes H^{N^m\text{LO}} \otimes S^{N^m\text{LO}}$$

$B$ - beam function - radiation collinear to the beam, process-independent, known up to $N^3\text{LO}$

$H$ - hard function - virtual corrections, process-dependent

$S$ - soft function - soft, real radiation, process-dependent

Today, I will focus on $B$. 
Renormalization

\[
\frac{d\sigma_F}{d\Phi} = \mathcal{B}_1^{(\text{bare})} \otimes \mathcal{B}_2^{(\text{bare})} \otimes \text{Tr} \left[ \mathcal{H}^{(\text{bare})} \otimes \mathcal{S}^{(\text{bare})} \right]
\]

finite

\[
= Z_B \mathcal{B}_1^{(\text{bare})} \otimes Z_B \mathcal{B}_2^{(\text{bare})} \otimes \text{Tr} \left[ Z_H^{\dagger} \mathcal{H}^{(\text{bare})} Z_H \otimes Z_S^{\dagger} \mathcal{S}^{(\text{bare})} Z_S \right]
\]

\[
= \mathcal{B}_1(\mu) \otimes \mathcal{B}_2(\mu) \otimes \text{Tr} \left[ \mathcal{H}(\mu) \otimes \mathcal{S}(\mu) \right]
\]

\[
\frac{d}{d\mu} \frac{d\sigma_F}{d\Phi} = 0 \quad \rightarrow \quad \text{Renormalization Group Equations for } \mathcal{B}, \mathcal{H} \text{ and } \mathcal{S}
\]
Small-$q_T$ factorization in SCET

Gluons’ momenta in light-cone coordinates

$$k_i^\mu = (k_i^+, k_i^-, k_i^\perp) \quad \text{where} \quad k^\pm = k^0 \pm k^3$$

Expansion parameter

$$\lambda = \sqrt{\frac{q_T^2}{q^2}} \ll 1$$

Phase space regions

- **collinear**
  $$k_i^\mu \sim (1, \lambda^2, \lambda) Q^2 \quad \mathcal{B}_1$$

- **anti-collinear**
  $$k_i^\mu \sim (\lambda^2, 1, \lambda) Q^2 \quad \mathcal{B}_2$$

- **hard**
  $$k_i^\mu \sim (1, 1, 1) Q^2 \quad \mathcal{H}$$

- **soft**
  $$k_i^\mu \sim (\lambda, \lambda, \lambda) Q^2 \quad \mathcal{S}$$
Soft Collinear Effective Theory (SCET)

\[ \text{SCET} \simeq \text{QCD} \bigg|_{\text{IR limit}} \]

Hard degrees of freedom are integrated out into Wilson coefficients, which are then used to adjust new couplings of the (effective) theory.

QCD fields written as sums of collinear, anti-collinear and soft components:

\[ \phi(x) = \phi_c(x) + \phi_{\bar{c}}(x) + \phi_s(x) \]

The new fields decouple in the Lagrangian

\[ \mathcal{L}_{\text{SCET}} = \mathcal{L}_c + \mathcal{L}_{\bar{c}} + \mathcal{L}_s \]

At leading power, the decoupled Lagrangians are copies of the QCD lagrangian.

The separation of fields in the Lagrangian into collinear, anti-collinear and soft sectors, facilitates proofs of factorization theorems.
Rapidity divergences and analytic regulator

\[
\int \frac{d^d k}{2\pi^d} \delta^+(k^2) \rightarrow \int d^d k \left( \frac{\nu}{k^+} \right)^\alpha \delta^+(k^2)
\]

The regulator is necessary at intermediate steps of the calculation.

Modification of the measure [Becher, Bell ‘12]
Rapidity divergences and analytic regulator

Rapidity divergences do not appear in QCD, hence, the complete SCET result has to stay finite in the limit $\alpha \to 0$.

They appear in real emission diagrams due to the expansion in different momentum regions (collinear, soft, ..).

$$s_{i_1i_2..i_k} = (p_{i_1} + p_{i_2} + .. + p_{i_k})^2$$

If we look at a phase space region in which $p_{i_1}$ and $p_{i_2}$ are collinear while all other momenta are anti-collinear, one expands

$$s_{i_1i_2..i_k} = (p_{i_1}^- + p_{i_2}^-)(p_{i_3}^+ + .. + p_{i_k}^+) + O(\lambda^2)$$

It can be shown that divergences arising from this expansion, can be regulated by analytic regularization [Becher, Bell ‘11].
NNLO and $N^3$LO beam function
The beam function

Represents corrections coming from emissions of real, collinear gluons, whose transverse momenta sum up to a fixed value $q_T$ and whose longitudinal component along $p$ sums up to $1 - z$

$$B_{\text{bare}}(q_T, z) \propto \sum \left( q_T - |\sum_i k_i\perp| \right) \prod_i \delta^+(k_i^2) \delta (\vec{n} \cdot \sum k_i - (1 - z) \vec{n} \cdot p)$$

$$p = \frac{\vec{n} \cdot p}{2} = \frac{p_\perp}{2} n$$

$$n^2 = \vec{n}^2 = 0$$

$$n \cdot \vec{n} = 2$$
N³LO propagators

Possible denominators that may cause divergencies.

<table>
<thead>
<tr>
<th>light-cone</th>
<th>internal only</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \cdot l_1$</td>
<td>$l_1 \cdot l_2$</td>
</tr>
<tr>
<td>$n \cdot l_2$</td>
<td>$l_1 \cdot l_3$</td>
</tr>
<tr>
<td>$n \cdot l_3$</td>
<td>$l_2 \cdot l_3$</td>
</tr>
<tr>
<td>$\bar{n} \cdot l_1$</td>
<td>$l_1 \cdot l_2 + l_1 \cdot l_3 + l_2 \cdot l_3$</td>
</tr>
<tr>
<td>$\bar{n} \cdot l_2$</td>
<td>internal + external</td>
</tr>
<tr>
<td>$\bar{n} \cdot l_3$</td>
<td>$p_\perp n \cdot l_1$</td>
</tr>
<tr>
<td>$n \cdot l_1 + n \cdot l_2$</td>
<td>$p_\perp n \cdot l_2$</td>
</tr>
<tr>
<td>$n \cdot l_1 + n \cdot l_3$</td>
<td>$p_\perp n \cdot l_3$</td>
</tr>
<tr>
<td>$n \cdot l_2 + n \cdot l_3$</td>
<td></td>
</tr>
<tr>
<td>$\bar{n} \cdot l_1 + \bar{n} \cdot l_2$</td>
<td>$l_1 \cdot l_2 - p_\perp n \cdot l_1 - p_\perp n \cdot l_2$</td>
</tr>
<tr>
<td>$\bar{n} \cdot l_1 + \bar{n} \cdot l_3$</td>
<td>$l_1 \cdot l_3 - p_\perp n \cdot l_1 - p_\perp n \cdot l_3$</td>
</tr>
<tr>
<td>$\bar{n} \cdot l_2 + \bar{n} \cdot l_3$</td>
<td>$l_2 \cdot l_3 - p_\perp n \cdot l_2 - p_\perp n \cdot l_3$</td>
</tr>
</tbody>
</table>
The way to go

The beam function

\[ B_{\text{bare}}(z, q_T) = \sum_i \mathcal{I}_i , \]

can be calculated if each integral is represented as

\[ \mathcal{I}_i = \sum_{j \in \text{sectors}} \int_0^1 \frac{dx_1}{x_1^{1+a_1\epsilon}} \frac{dx_2}{x_2^{1+a_2\epsilon}} \frac{dx_3}{x_3^{1+a_3\epsilon}} \frac{dx_4}{x_4^{1+a_4\epsilon}} \cdot \mathcal{W}_j(x_1, x_2, \ldots, x_9) . \]

\[ \mathcal{W}_j(x_1, x_2, \ldots, x_9) \] has to be finite if \( x_1, \ldots, x_4 \to 0. \)

Then we can use

\[
\frac{1}{x_i^{1+a_i\epsilon}} = - \frac{1}{a_i\epsilon} \delta(x_i) + \sum_{n=0}^{\infty} \frac{(-a_i\epsilon)^n}{n!} \left[ \log^n(x_i) \right]_+ .
\]

In order to further simplify the computation, we substitute the delta \( \delta(k_\perp) \) for \( e^{-k_\perp^2} \) and rescale the integral correspondingly.
N$^3$LO propagators

The first problem: It is impossible to parameterize the momenta such that all scalar products look simple simultaneously.

Example

\[ n = [1, 0, 0, 0, 1] \]
\[ \bar{n} = [1, 0, 0, 0, -1] \]
\[ l_1 = \left[ \frac{l_{1T}^2 + l_{1T}^2}{2l_1}, 0, 0, 0, \frac{l_{1T}^2 - l_{1T}^2}{2l_1} \right] \]
\[ l_3 = \left[ \frac{l_{3T}^2 + l_{3T}^2}{2l_3}, 0, l_3 \sin \chi_1, l_3 \cos \chi_1, \frac{l_{3T}^2 - l_{3T}^2}{2l_3} \right] \]
\[ l_2 = \left[ \frac{l_{2T}^2 + l_{2T}^2}{2l_2}, l_2 \sin \phi_1 \sin \phi_2, l_2 \cos \phi_2 \sin \phi_1, l_2 \cos \phi_1, \frac{l_{2T}^2 - l_{2T}^2}{2l_2} \right] \]
\[ \bar{n} \cdot l_1 = l_{1-} \]
\[ \bar{n} \cdot l_2 = l_{2-} \]
\[ \bar{n} \cdot l_3 = l_{3-} \]
\[ l_1 \cdot l_2 = \frac{l_{1T}l_{2-}}{2l_1} + \frac{l_{2T}l_{1-}}{2l_2} - l_{1T}l_{2T} \cos \phi_1 \Rightarrow \phi_1 = 0 \text{ and } \frac{l_{1T}}{l_{1-}} = \frac{l_{2T}}{l_{2-}} \]
\[ l_2 \cdot l_3 = \frac{l_{2T}l_{3-}}{2l_2} + \frac{l_{3T}l_{2-}}{2l_3} - l_{2T}l_{3T} \cos \chi_1 \cos \phi_1 - l_{2T}l_{3T} \cos \phi_2 \sin \chi_1 \sin \phi_1 \]
### Step 1: Selector Functions

<table>
<thead>
<tr>
<th>7 triple collinear</th>
<th>12 double collinear</th>
</tr>
</thead>
<tbody>
<tr>
<td>((l_1 \cdot l_2) (n \cdot l_1) (n \cdot l_2))</td>
<td>((n \cdot l_1) (\bar{n} \cdot l_2))</td>
</tr>
<tr>
<td>((l_1 \cdot l_3) (n \cdot l_1) (n \cdot l_3))</td>
<td>((l_1 \cdot l_3) (n \cdot l_2))</td>
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</tr>
<tr>
<td>((l_1 \cdot l_2) (l_1 \cdot l_3) (l_2 \cdot l_3))</td>
<td>((l_1 \cdot l_2) (\bar{n} \cdot l_3))</td>
</tr>
</tbody>
</table>

\[ S_{1,2;2} = \frac{1}{d_{1,2;1} \mathcal{D}}, \]
\[ d_{1,2;1} = (l_1 \cdot l_2) (\bar{n} \cdot l_1) (\bar{n} \cdot l_2), \]
\[ \mathcal{D} = \sum_{i,j,k} \frac{1}{d_{i,j;k}} + \sum_{i,j,k,l} \frac{1}{d_{i,j;k,l}}, \]
\[ S_{1,2;2} = \frac{1}{1 + \frac{(l_1 \cdot l_2) (\bar{n} \cdot l_2)}{(l_1 \cdot l_3) (\bar{n} \cdot l_3)} + \frac{(l_1 \cdot l_2) (\bar{n} \cdot l_1)}{(l_1 \cdot l_3)} + \cdots} \]
Step 2: nonlinear transformations

Let’s focus on the sector \((l_1 \cdot l_2)(\vec{n} \cdot l_1)(\vec{n} \cdot l_2)\). All other singularities are suppressed by the corresponding selector functions.

In this sector, divergencies can be generated by the following propagators:

\[
\begin{align*}
\vec{n} \cdot l_1 & \quad \rightarrow \quad l_{1-} \\
\vec{n} \cdot l_2 & \quad \rightarrow \quad l_{2-} \\
n \cdot l_1 & \\
n \cdot l_2 & \\
l_1 \cdot l_2 & \quad \rightarrow \quad \frac{l_{1T}l_{2-}}{2l_{1-}} + \frac{l_{2T}l_{1-}}{2l_{2-}} - l_{1T}l_{2T} \cos \phi_1 \\
n \cdot l_1 + n \cdot l_2 & \\
\vec{n} \cdot l_1 + \vec{n} \cdot l_2 & \quad \rightarrow \quad l_{1-} + l_{2-} \\
l_1 \cdot l_2 + l_1 \cdot l_3 + l_2 \cdot l_3 &
\end{align*}
\]
Step 2: nonlinear transformations

The nonlinear transformation

\[ \zeta = \frac{1}{2} \frac{(l_1^T l_2^- - l_1^- l_2^T)^2 (1 + \cos \phi_1)}{l_1^2 + l_2^2 - 2 l_1^- l_2^- l_1^T l_2^T \cos \phi_1} \]

turns

\[ l_1 \cdot l_2 = \frac{l_2^T l_2^-}{2 l_1^-} + \frac{l_1^2 l_1^-}{2 l_2^-} - l_1^T l_2^T \cos \phi_1 \]

into

\[ l_1 \cdot l_2 = \frac{(l_1^2 l_2^- - l_1^- l_2^T)^2}{2 l_1^- l_2^- (l_2^T l_2^- + l_1^2 l_2^T - 2 l_1^- l_2^- l_1^T l_2^T (1 - 2 \zeta))} \]
NNLO beam function

Known analytically [Gehrmann, Lübbert, Yang ’12, ’14].

We checked that our method reproduces that result.

Furthermore, we are able to get higher orders in the expansion parameters at the cost of more computing time.