Towards charm- and bottom-quark masses with five-loop accuracy

Andreas Maier Peter Marquard York Schröder

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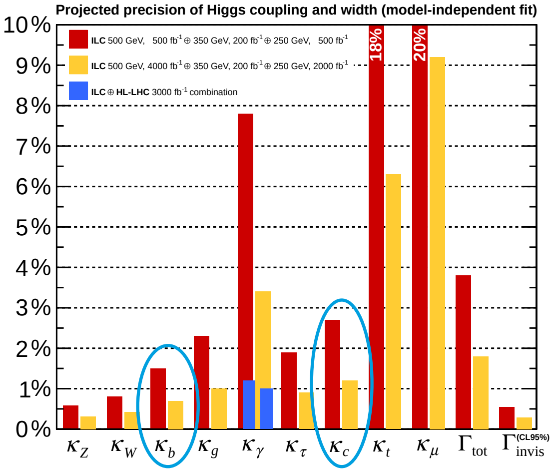
How much precision do we need?

[arXiv:1506.05992]



Coupling proportional to mass $\Rightarrow$ need $\Delta m_b < 0.5\%$, $\Delta m_c < 1\%$

How much precision do we have?

[Chetyrkin, Kühn, Maier, Maierhöfer, Marquard, Steinhauser, Sturm 2009 + 2017]

$$m_b(10\,\text{GeV}) = \big(3610 \pm 10(\text{exp}) \pm 12(\alpha_s) \pm 3(\mu)\big)\,\text{MeV}$$
$$m_b(m_b) = (4163 \pm 16)\,\text{MeV}$$
$$m_c(3\,\text{GeV}) = \big(993 \pm 7(\text{exp}) \pm 4(\alpha_s) \pm 2(\mu) \pm 1(\text{np})\big)\,\text{MeV}$$

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[Dehnadi, Hoang, Mateu, Zebarjad 2013; Dehnadi, Hoang, Mateu 2015]

$$m_b(m_b) = \big(4176 \pm 4(\text{stat}) \pm 19(\text{sys}) \pm 7(\alpha_s) \pm 10(\mu)\big)\,\text{MeV}$$
$$m_c(3\,\text{GeV}) = \big(994 \pm 6(\text{stat}) \pm 9(\text{sys}) \pm 10(\alpha_s) \pm 21(\mu) \pm 2(\text{np})\big)\,\text{MeV}$$

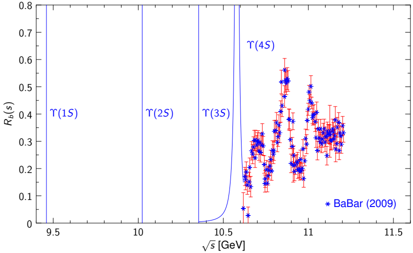
Goal: eliminate theory error

How do we determine heavy-quark masses?

Total inclusive cross section for $e^+e^- \to Q\bar{Q}$ near threshold

- Highly sensitive to heavy-quark mass m_Q
- Experimentally clean
- Perturbative predictions to high orders
- Non-perturbative effects
  - negligible for top quarks
  - highly suppressed in *sum rules* for charm and bottom quarks

Sum rules



Consider *moments* $\mathcal{M}_n$ of $R_Q(s) = \frac{\sigma(e^+e^- \to Q\bar{Q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$

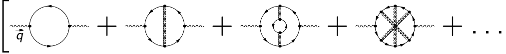
$$\mathcal{M}_n = \int_{s_0}^{\infty} ds\, \frac{R_Q(s)}{s^{n+1}} = \frac{12\pi^2}{n!} \underbrace{\left[\left(\frac{d}{dq^2}\right)^n \Pi_Q(q^2)\right]_{q^2=0}}_{\text{perturbative for } n \lesssim 10}$$

Extract quark mass from $\mathcal{M}_n^{\text{theory}}(m_Q) = \mathcal{M}_n^{\text{experiment or lattice}}$

Sum rules

Use fixed-order perturbation theory for $n \sim 1$:

$$\mathcal{M}_n^{\text{theory}} = \frac{12\pi^2}{n!} \left(\frac{d}{dq^2} \right)^n \Big[\ \cdots \ \Big]_{q^2=0}$$



- Exact three-loop results up to $n = 30$

[Chetyrkin, Kühn, Steinhauser 1995]

[Boughezal, Czakon, Schutzmeier 2006; Maier, Maierhöfer, Marquard 2007]

- Exact four-loop results up to $n = 4$

[Chetyrkin, Kühn, Sturm 2006; Boughezal, Czakon, Schutzmeier 2006]

[Maier, Maierhöfer, Marquard, Smirnov 2008–2009; Maier, Marquard 2017]

- Approximate four-loop results up to $n = 10$

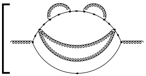
[Hoang, Mateu, Zebarjad 2008; Kiyo, Maier, Maierhöfer, Marquard 2009; Greynat, Masjuan, Peris 2011]

Next: $n = 1$ at five loops

The first moment at five loops

1. Generate diagrams (`qgraf` [Nogueira 1991])

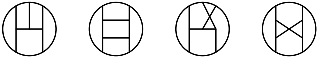
$$\mathcal{M}_1^{5\text{ loop}} = 12\pi^2 \frac{d}{dq^2} \Bigg[$$



$$+\ (18378\text{ others})\Bigg]_{q^2=0}$$

The first moment at five loops

1. Generate diagrams (`qgraf` [Nogueira 1991])
2. Identify vacuum topologies



34 combinations of massive and massless lines

The first moment at five loops

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3. Insert Feynman rules, Simplify expressions, . . .

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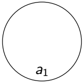
3. Insert Feynman rules, Simplify expressions, ...
4. Reduce to master integrals (`crusher` [Marquard, Seidel])

   Integration-by-parts (IBP) identities: [Chetyrkin, Tkachov 1981]

$$T(a_1) = $$



$$= \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - m_Q^2)^{a_1}}$$

$$\begin{aligned} 0 &= \int \frac{d^d l}{(2\pi)^d} \frac{\partial}{\partial l_\mu} l_\mu \frac{1}{(l^2 - m_Q^2)^{a_1}} \\ &= \int \frac{d^d l}{(2\pi)^d} \frac{d}{(l^2 - m_Q^2)^{a_1}} - a_1 \frac{2l^2}{(l^2 - m_Q^2)^{a_1+1}} \\ &= dT(a_1) - 2a_1[T(a_1) + m_Q^2 T(a_1 + 1)] \end{aligned}$$

- Insert numerical values for a_1
- Solve linear system of equations $\Rightarrow T(1)$ as master integral

[Laporta 2000]

The first moment at five loops

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5. Calculate & insert master integrals

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$$I_i = \int \frac{1}{D_{i,1}^{\alpha_{i,1}} \cdots D_{i,n}^{\alpha_{i,n}}}, \quad \alpha_{i,j} \in \{1, 2\}, \quad n \leq 12, \quad i \leq 183$$

Preferred method: difference equations (`SPADES` [Luthe, Schröder])

[Laporta 2000]

Alternative: sector decomposition (`FIESTA` [Smirnov et al. 2008–2015])

[Hepp 1966; Speer 1968; Speer 1977; Binoth, Heinrich 2000]

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5. Calculate & insert master integrals
6. Renormalise

Calculation of master integrals

Difference equations

- Raise one denominator to symbolic power x:

$$I_i(x) = \int \frac{1}{D_1^{x} \cdots D_{15}^{\alpha_P}}$$

- IBP reduction $\Rightarrow$ Coupled first-order difference equations

$$I_i(x \pm 1) = \sum_j p_{ij}^{\pm}(d, x) I_j(x)$$

- Recurrence relations from factorial series ansatz

$$I_i(x) = \sum_{s=0}^{\infty} a_{i,s} \frac{\Gamma(x+1)}{\Gamma(x+s-d/2+1)}$$

- Boundary conditions from $x \to \infty$

$\Rightarrow$ high-precision numerical solution

Calculation of master integrals

Sector decomposition

Goal: numerical integrations over hypercube

1. Introduce Feynman parameters $\left(N_\alpha = \sum_j \alpha_j\right)$

$$
\begin{aligned}
I(x) &= \int \frac{1}{D_1^{\alpha_1} \cdots D_N^{\alpha_N}} \\
&\propto \prod_{j=1}^{N} \left(\int_0^\infty dx_j \, x_j^{\alpha_j - 1} \right) \delta\left(1 - \sum_j x_j\right) \frac{\mathcal{U}^{N_\alpha - 3d}}{\mathcal{F}^{N_\alpha - 5d/2}}
\end{aligned}
$$

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2. Transform to sector integrals over hypercube
   - Split into *primary sectors* $s = 1, \ldots, N$ such that $x_s \geq x_j$
   - Rescale integration variables $x_j = t_j x_s$ with $t_j \in [0, 1]$

Calculation of master integrals

Sector decomposition

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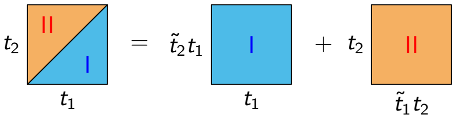
$$I(x) \propto \sum_{s=1}^{N} \prod_{j=1}^{N-1} \left(\int_0^1 dt_j \, t_j^{\alpha_j - 1} \right) \frac{\mathcal{U}_s^{N_\alpha - 3d}}{\mathcal{F}_s^{N_\alpha - 5d/2}}$$

3. Iteratively decompose until $\mathcal{F}_s, \mathcal{U}_s \neq 0$ for all $t_j = 0$

   Example: zero for $t_1 = t_2 = 0$



Calculation of master integrals

Sector decomposition

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4. Expand around $d = 4$

Calculation of master integrals

Sector decomposition

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3. Iteratively decompose until $\mathcal{F}_s, \mathcal{U}_s \neq 0$ for all $t_j = 0$
4. Expand around $d = 4$
5. Integrate $\Rightarrow$ low-precision numerical result

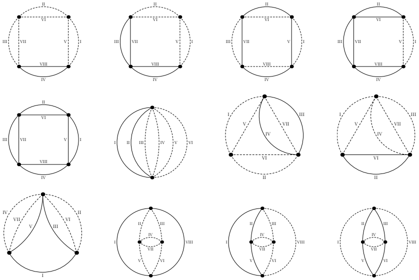
The first moment at five loops

First results

n_h: Closed heavy (massive) quark loops
n_l: Closed light (massless) quark loops

	n_l^0	n_l^1	n_l^2	n_l^3
n_h^1	✗	✗	✓	✓
n_h^2	✗	✗	✓	
n_h^3	✗	✓		
n_h^4	✓			

Master integrals:



The first moment at five loops

Preliminary results

$$\mathcal{M}_1^{5\text{ loop}} = \frac{3\pi^2}{m_Q^2}\left(\frac{\alpha_s}{\pi}\right)^4 n_h C_F \Big[0.6\,T_F^3 n_l^3 + 1.2\,T_F^3 n_l^2 n_h + 0.9\,T_F^3 n_l n_h^2 + 0.2\,T_F^3 n_h^3 + (C_F - 5C_A)T_F^2 n_l^2 + \ldots\Big]$$

Conclusions

- High-precision determinations of charm- and bottom-quark masses using sum rules
- Goal: eliminate theory uncertainty
- First results at five loops