Karlsruhe Institute of Technology

On the kinetic mass of heavy quarks at three loops

SFB TRR 257 Mini workshop on quark masses

Matteo Fael | October 26, 2020

INSTITUTE FOR THEORETICAL PARTICLE PHYSICS - KIT KARLSRUHE

based on Fael, Schönwald, Steinhauser, PRL 125 (2020) 052003, JHEP 10 (2020) 087

- What is the kinetic mass?
- The kinetic mass up to three loops.
- Finite charm mass effects for m_b^{kin}
- $\overline{\mathrm{MS}}$–kinetic mass relation

What is the kinetic mass?

The total width for $B \to X_c \ell \bar{\nu}_\ell$ in the large β_0 limit:

$$\Gamma_{\rm sl} = \frac{G_F^2 |V_{cb}|^2 (m_b^{\rm OS})^5}{192\pi^3} f(0.25) \left[1 - 1.78\left(\frac{\alpha_s}{\pi}\right) - 15.8\left(\frac{\alpha_s}{\pi}\right)^2 - 230\left(\frac{\alpha_s}{\pi}\right)^3 + \ldots\right]$$

See: Bigi, Shifman, Uraltsev, Vainshtein PRD 50 (1994) 2234; Beneke, Braun, NPB 426 (1994) 301; Ball, Beneke, Braun, PRD 52 (1995) 3929; Melnikov, van Ritbergen, PLB 482 (2000) 99.

If we change the mass scheme $m_b^{\rm OS} \to \tilde{m}_b \left(1 + c\frac{\alpha_s}{\pi}\right)$

$$\Gamma_{\rm sl} \propto (\tilde{m}_b)^n \left[1 + (nc + a_1)\left(\frac{\alpha_s}{\pi}\right) + \left(\frac{n(n+1)}{2}c^2 + nc\, a_1 + a_2\right)\left(\frac{\alpha_s}{\pi}\right)^2 + \ldots\right]$$

Can we resum the n-enhanced $(n\alpha_s)^k$ terms (with $n = 5$)?

Bigi, Shifman, Uraltsev, Vainsthein, High Power n of m_b in beauty widths and $n = 5 \to \infty$ limit, PRD 56 (1997) 4017

The total width for $B \to X_c \ell \bar{\nu}_\ell$ in the large β_0 limit:

$$\Gamma_{\mathrm{sl}} = \frac{G_F^2 |V_{cb}|^2 (m_b^{\mathrm{OS}})^5}{192\pi^3} f(0.25) \left[1 - 1.78 \left(\frac{\alpha_s}{\pi}\right) - 15.8 \left(\frac{\alpha_s}{\pi}\right)^2 - 230 \left(\frac{\alpha_s}{\pi}\right)^3 + \dots \right]$$

See: Bigi, Shifman, Uraltsev, Vainshtein PRD 50 (1994) 2234; Beneke, Braun, NPB 426 (1994) 301; Ball, Beneke, Braun, PRD 52 (1995) 3929; Melnikov, van Ritbergen, PLB 482 (2000) 99.

If we change the mass scheme $m_b^{\mathrm{OS}} \to \tilde{m}_b \left(1 + c\frac{\alpha_s}{\pi}\right)$

$$\Gamma_{\mathrm{sl}} \propto (\tilde{m}_b)^n \left[1 + (nc + a_1) \left(\frac{\alpha_s}{\pi}\right) + \left(\frac{n(n+1)}{2} c^2 + nc\, a_1 + a_2 \right) \left(\frac{\alpha_s}{\pi}\right)^2 + \dots \right]$$

Can we resum the n-enhanced $(n\alpha_s)^k$ terms (with $n = 5$)?

Bigi, Shifman, Uraltsev, Vainsthein, High Power n of m_b in beauty widths and $n = 5 \to \infty$ limit, PRD 56 (1997) 4017

The total width for $B \to X_c \ell \bar{\nu}_\ell$ in the large β_0 limit:

$$\Gamma_{\rm sl} = \frac{G_F^2 |V_{cb}|^2 (m_b^{\rm OS})^5}{192\pi^3} f(0.25) \left[1 - 1.78 \left(\frac{\alpha_s}{\pi}\right) - 15.8 \left(\frac{\alpha_s}{\pi}\right)^2 - 230 \left(\frac{\alpha_s}{\pi}\right)^3 + \dots \right]$$

See: Bigi, Shifman, Uraltsev, Vainshtein PRD 50 (1994) 2234; Beneke, Braun, NPB 426 (1994) 301; Ball, Beneke, Braun, PRD 52 (1995) 3929; Melnikov, van Ritbergen, PLB 482 (2000) 99.

If we change the mass scheme $m_b^{\rm OS} \to \tilde{m}_b \left(1 + c\frac{\alpha_s}{\pi}\right)$

$$\Gamma_{\rm sl} \propto (\tilde{m}_b)^n \left[1 + (nc + a_1) \left(\frac{\alpha_s}{\pi}\right) + \left(\frac{n(n+1)}{2} c^2 + nc\, a_1 + a_2 \right) \left(\frac{\alpha_s}{\pi}\right)^2 + \dots \right]$$

Can we resum the n-enhanced $(n\alpha_s)^k$ terms (with $n = 5$)?

Bigi, Shifman, Uraltsev, Vainsthein, High Power n of m_b in beauty widths and $n = 5 \to \infty$ limit, PRD 56 (1997) 4017

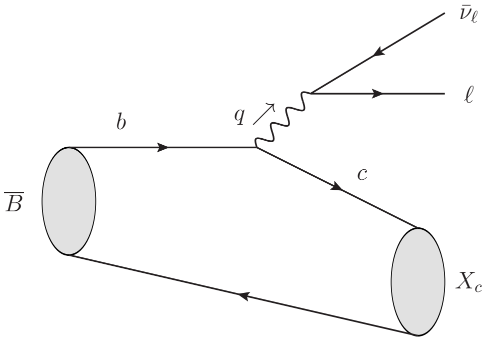
The relation between the masses of the heavy meson B and the heavy bottom b:

$$M_B = m_b + \overline{\Lambda} + \frac{\mu_\pi^2 + d_B \mu_G^2}{2m_b} + \ldots$$

- $\overline{\Lambda}$: the binding energy of the heavy hadron.
- $\boldsymbol{\mu_\pi}$: the kinetic energy induced by the residual motion of the heavy quark
- $\boldsymbol{\mu_G}$: the chromo-magnetic interaction between the heavy quark and the light degrees of freedom.

The SV sum rules

- The small velocity (SV) sum rules are relations between the physical differential rate of $\overline{B} \to X_c \ell \bar{\nu}_\ell$ and the HQET parameters $\overline{\Lambda}$, μ_π, ...



$$I_n(\vec{q}^2) \equiv \int_{|\vec{q}|}^{q_0^{\max}} dq_0\, \omega^n \frac{d\Gamma}{dq_0\, d\vec{q}^2}$$

The excitation energy of the hadronic state $X_c = D, D^*, D\pi, D\pi\pi, \ldots$

$$\omega = q_0^{\max} - q_0 = M_B - \sqrt{M_D^2 + \vec{q}^{\,2}} - q_0$$

For further details: Bigi, Shifman, Uraltsev, Vainshtein, PRD 52 (1995) 196

The SV sum rules

Compare the experimental determination of I_n with the tree-level prediction within HQE,

$$I_n(\vec{q}^2) = \int_{|\vec{q}|}^{q_0^{\max}} dq_0\, \omega^n \frac{d\Gamma_{\text{tree}}}{dq_0\, d\vec{q}^2}$$

in the small velocity limit ($|\vec{q}| \ll m_c \sim m_b$ or $|\vec{v}| = |\vec{q}/m_c| \ll 1$):

$$I_0(\vec{q}^{\,2}) = |\vec{q}\,| \frac{G_F^2 |V_{cb}|^2}{8\pi^3} (m_b - m_c)^2 + O\left(|\vec{v}|^2, \frac{\Lambda_{\text{QCD}}}{m_b}\right)$$

$$I_1(\vec{q}^{\,2}) = I_0 \frac{\vec{v}^2}{2} \bar{\Lambda} + O\left(|\vec{v}|^3, \frac{\Lambda_{\text{QCD}}^2}{m_b^2}\right)$$

$$I_2(\vec{q}\,) = I_0 \frac{\vec{v}^2}{3} \mu_\pi^2 + O\left(|\vec{v}|^3, \frac{\Lambda_{\text{QCD}}^3}{m_b^3}\right)$$

The SV sum rules

- We obtain an operative definition (among others) of the parameters in HQET and m_b:

$$\overline{\Lambda} = \lim_{\vec{v}\to 0} \lim_{m_b \to 0} \frac{2}{\vec{v}^{\,2}} \frac{I_1(\vec{v}^{\,2})}{I_0(\vec{v}^{\,2})}$$

$$m_b = \overline{M}_B - \overline{\Lambda} - \frac{\mu_\pi^2}{2m_b} + \dots$$

$$\mu_\pi^2 = \lim_{\vec{v}\to 0} \lim_{m_b \to 0} \frac{3}{\vec{v}^{\,2}} \frac{I_2(\vec{v}^{\,2})}{I_0(\vec{v}^{\,2})}$$

with spin averaged $\overline{M}_B = \frac{M_B + 3M_{B^*}}{4}$

- The definition is independent on the current for $b \to c$ (heavy quark symmetries)

Let's include radiative corrections ...

Karlsruhe Institute of Technology

$$I_n(\vec{q}^2) = \int_{|\vec{q}|}^{q_0^{\max}} dq_0\, \omega^n \frac{d\Gamma_{\rm tree}}{dq_0\, d\vec{q}^2} + \int_{q_0^{\max}-\mu}^{q_0^{\max}} dq_0\, \omega^n \frac{d\Gamma_{\alpha_s}}{dq_0\, d\vec{q}^2} + \int_{|\vec{q}|}^{q_0^{\max}-\mu} dq_0\, \omega^n \frac{d\Gamma_{\alpha_s}}{dq_0\, d\vec{q}^2}$$

- We introduce a **Wilsonian cutoff μ**, with $\Lambda_{\rm QCD} \ll \mu \ll M_B$, to separate gluons with
  - $\omega < \mu$ that belong to the non-perturbative regime,
  - $\omega > \mu$ that can be described in pQCD.

Let's include radiative corrections . . .

$$I_1(\vec{q}^2) = \underbrace{\int_{|\vec{q}|}^{q_0^{\max}} dq_0\, \omega\, \frac{d\Gamma_{\rm tree}}{dq_0\, d\vec{q}^2} + \int_{q_0^{\max}-\mu}^{q_0^{\max}} dq_0\, \omega\, \frac{d\Gamma_{\alpha_s}}{dq_0\, d\vec{q}^2}}_{\text{use this to define in the SV limit } \overline{\Lambda}(\mu)} + \int_{|\vec{q}|}^{q_0^{\max}-\mu} dq_0\, \omega\, \frac{d\Gamma_{\alpha_s}}{dq_0\, d\vec{q}^2}$$

- We introduce a **Wilsonian cutoff** μ,with $\Lambda_{\rm QCD} \ll \mu \ll M_B$, to separate gluons with
  - $\omega < \mu$ that belong to the non-perturbative regime,
  - $\omega > \mu$ that can be described in pQCD.

Cancellation of the IR contributions in $\Gamma(\overline{B} \to X_c \ell \bar{\nu}_\ell)$

Karlsruhe Institute of Technology

- The relevant parameter in $\Gamma(\overline{B} \to X_c \ell \bar{\nu}_\ell)$ is $\boldsymbol{m_b^5}$ **not** $\boldsymbol{M_B^5}$:

$$\Gamma(\overline{B} \to X_c \ell \bar{\nu}_\ell) \simeq \frac{G_F^2 |V_{cb}|^5}{192\pi^3} (M_B - \overline{\Lambda})^5$$

Bigi, Shifman, Uraltsev, Vainshtein, PRD 56 (1997) 4017

- In pQCD, we can *peel off* the IR renormalon sensitivity from the on-shell mass identifying:

$$m_b(\mu) \to m_b^{\mathrm{kin}}(\mu), \qquad \overline{M}_B \to m_b^{\mathrm{OS}},$$

$$\overline{\Lambda}(\mu) \to [\overline{\Lambda}(\mu)]_{\mathrm{pert}}, \qquad [\mu_\pi^2(\mu)] \to [\mu_\pi^2(\mu)]_{\mathrm{pert}}.$$

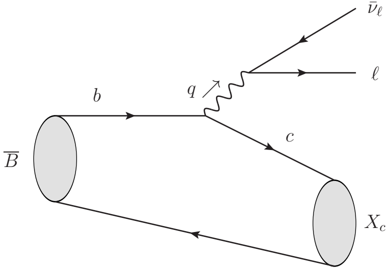
- Moreover, we do not generate large $(n\alpha_s)^k$ corrections.

The kinetic mass

$$m_b^{\text{kin}}(\mu) = m_b^{\text{OS}} - [\overline{\Lambda}(\mu)]_{\text{pert}} - \frac{[\mu_\pi^2(\mu)]_{\text{pert}}}{2m_b^{\text{kin}}(\mu)} - \ldots .$$

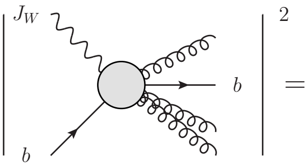
Bigi, Shifman, Uraltsev, Vainshtein, PRD 56 (1997) 4017.
see also: Czarnecki, Melnikov, Uraltsev, PRL 80 (1998) 3189;
Gambino, JHEP 09 (2011) 055;
Fael, Schönwald, Steinhauser, PRL 125 (2020) 052003.

The kinetic mass up to three loops



$$[\overline{\Lambda}(\mu)]_{\rm pert} = \lim_{\vec{v}\to 0} \lim_{m_b \to 0} \frac{2}{\vec{v}^{\,2}} \frac{\int_{q_0^{\rm max}-\mu}^{q_0^{\rm max}} dq_0 \, \omega \, \frac{d\Gamma}{dq_0 \, d\vec{q}^2}}{\int_{q_0^{\rm max}-\mu}^{q_0^{\rm max}} dq_0 \, \frac{d\Gamma}{dq_0 \, d\vec{q}^2}}$$

$$[\mu_\pi^2(\mu)]_{\rm pert} = \lim_{\vec{v}\to 0} \lim_{m_b \to 0} \frac{3}{\vec{v}^{\,2}} \frac{\int_{q_0^{\rm max}-\mu}^{q_0^{\rm max}} dq_0 \, \omega^2 \, \frac{d\Gamma}{dq_0 \, d\vec{q}^2}}{\int_{q_0^{\rm max}-\mu}^{q_0^{\rm max}} dq_0 \, \frac{d\Gamma}{dq_0 \, d\vec{q}^2}}$$



$$= W(\omega, \vec{v})$$

$$[\bar{\Lambda}(\mu)]_{\rm pert} = \lim_{\vec{v}\to 0} \lim_{m_b \to 0} \frac{2}{\vec{v}^{\,2}} \frac{\int_0^\mu d\omega\, \omega\, W(\omega, \vec{v})}{\int_0^\mu d\omega\, W(\omega, \vec{v})}$$

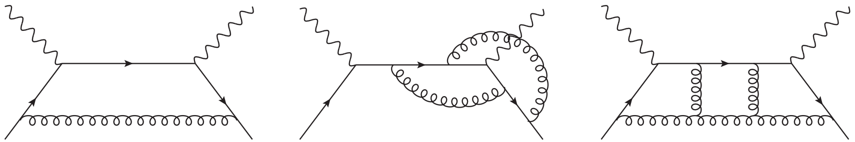
$$[\mu_\pi^2(\mu)]_{\rm pert} = \lim_{\vec{v}\to 0} \lim_{m_b \to 0} \frac{3}{\vec{v}^{\,2}} \frac{\int_0^\mu d\omega\, \omega^2\, W(\omega, \vec{v})}{\int_0^\mu d\omega\, W(\omega, \vec{v})}$$

$$[\overline{\Lambda}(\mu)]_{\mathrm{pert}} = \lim_{\vec{v}\to 0} \lim_{m_b \to 0} \frac{2}{\vec{v}^{\,2}} \frac{\int_0^\mu d\omega\, \omega\, W(\omega, \vec{v})}{\int_0^\mu d\omega\, W(\omega, \vec{v})}$$

$$[\mu_\pi^2(\mu)]_{\mathrm{pert}} = \lim_{\vec{v}\to 0} \lim_{m_b \to 0} \frac{3}{\vec{v}^{\,2}} \frac{\int_0^\mu d\omega\, \omega^2\, W(\omega, \vec{v})}{\int_0^\mu d\omega\, W\omega, \vec{v})}$$

m_b^{kin} at $O(\alpha_s^3)$: ingredients

- The structure function $W(\omega, \vec{v})$ up to $O(\alpha_s^3)$ (we chose scalar and vector current)
- $W(\omega, \vec{v})$ is given by the imaginary part of forward scattering amplitudes like these:



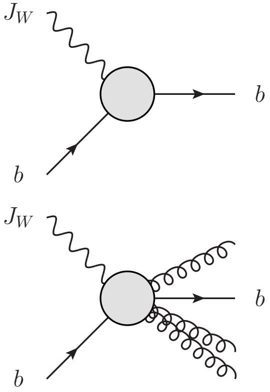
- We don't want to work with non-relativistic quantities.
- We need the leading term of $W(\omega, \vec{v})$ in an expansion in ω and $\vec{v}^2$.

Let's split the structure function in elastic and inelastic part:

$$W(\omega,\vec{v}) = W_{\rm el}(\vec{v}\,)\,\delta(\omega) + \frac{\vec{v}^2}{\omega} W_{\rm real}(\omega)\,\theta(\omega) + \mathcal{O}\left(v^4, \frac{\omega}{m_b}\right)$$

The expression for $\overline{\Lambda}$ reduces to:

$$[\overline{\Lambda}(\mu)]_{\rm pert} = \lim_{\vec{v}\to 0} \lim_{m_b\to\infty} \frac{2}{\vec{v}^{\,2}} \frac{\sum_{n=1} \alpha_s^n \int_0^\mu \omega\, \frac{\vec{v}^{\,2}}{\omega} W^{(n)}_{\rm real}\, {\rm d}\omega}{\sum_{n=0} \alpha_s^n W^{(n)}_{\rm el}}$$



- Virtual corrections don't contribute at numerator: $\int \omega\delta(\omega){\rm d}\omega = 0$.
- Real corrections don't contribute at denominator: they are killed by the $\vec{v} \to 0$ limit.

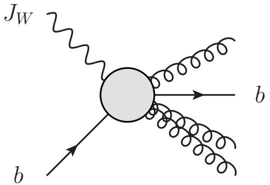
The kinetic mass as a threshold mass

- We compute the leading $\vec{v}^2/\omega$ term of $W(\omega, \vec{v}^2)$ via the *expansion by regions*.

  Beneke, Smirnov, NPB 522 (1998) 321; Smirnov Springer Tracts Mod. Phys. 250 (2010)

- We trade the ω expansion with a **threshold expansion**:

$$y \equiv m_b^2 - s = -2\omega m_b + O(\omega^2, \vec{v}^2)$$



- We realize the $\vec{v} \to 0$ limit as an expansion in

$$q^2 = -m_b \vec{v}^2 (m_b - \omega) + O(\omega^2, \vec{v}^4)$$

- We have to consider terms up to $O(1/y)$ and $O(q^2)$.

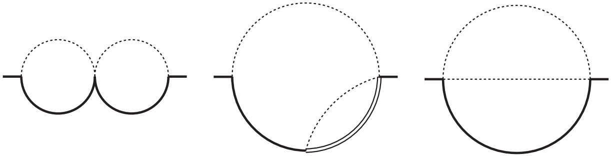
For the statistics ...

- 4, 66 and 1586 diagrams at one, two and three loops
- The scaling of the loop momenta cross checked with `asy`

  Pak, Smirnov, EPJC 71 (2011) 1626
- Partial fractioning and mapping between families implement in `FORM` thanks to private code `LIMIT`. Herren, PhD thesis, KIT, 2020
- `FIRE` and `LiteRed` reduction of integral families:

  Smirnov, Chuharev, hep-ph/1901.07808; Lee, hep-ph/1212.2685.
  - $\{2, 2\}$ in the $\{(uu), (uh)\}$ regions at two loops
  - $\{14, 4, 3\}$ in the $\{(uuu), (uuh), (uhh)\}$ regions at three loops
- New master integrals:
  - 3 in the (uu) region
  - 20 in the (uuu) region
- Heavy quark form factors up to $O(\alpha_s^2)$ (static limit)

  Lee, Smirnov, Smirnov, Steinhauser, JHEP 1805 (2018) 187; Blümlein, Marquard, Rana, PRD 99 (2019) 016013



I_1^{2l} I_2^{2l} I_3^{3l}

- The new master integrals contains linear-massive propagators:

$$I_2^{2l} = \int d^d k_1 d^d k_2 \frac{1}{k_1^2 (k_1 - k_2)^2 (2k_1 \cdot p - y)(2k_2 \cdot p)}$$

- **Mellin-Barnes method**

  `MB` package
  Czakon, Comput. Phys. Commun. 175 (2006) 559;
  Smirnov[2], EPJC 62 (2009) 445.

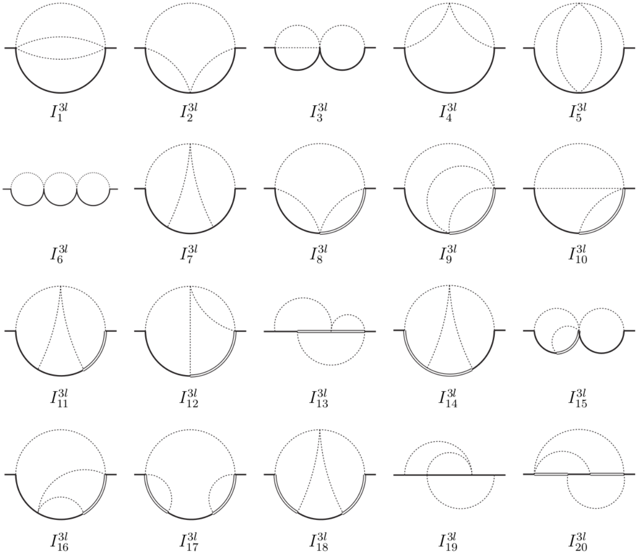
- **PSLQ**

- **Analytic summation of residues**

  HarmincSums
  www3.risc.jku.at/research/combinat/software/HarmonicSums/

- **Differential equations in auxiliary variable**

  Kotikov, PLB 254 (1991), 158
  Gehrmann, Remiddi, NPB 580 (2000) 485
  Henn, PRL 110 (2013), 251601.



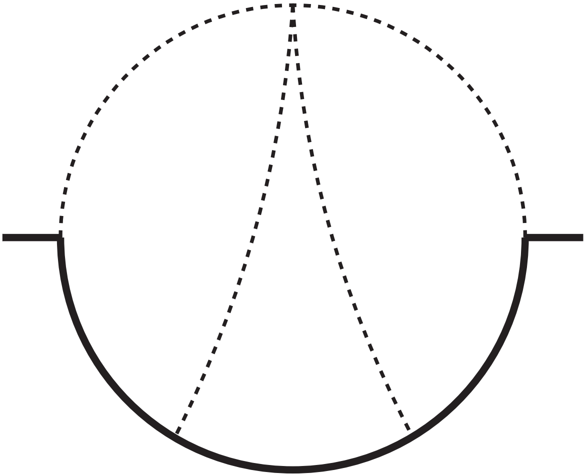
- Mellin-Barnes method

  MB package
  Czakon, Comput. Phys. Commun. 175 (2006) 559;
  Smirnov[2], EPJC 62 (2009) 445.

- PSLQ
- Analytic summation of residues

  HarmincSums
  www3.risc.jku.at/research/combinat/software/HarmonicSums/

- Differential equations in auxiliary variable
  Kotikov, PLB 254 (1991), 158
  Gehrmann, Remiddi, NPB 580 (2000) 485
  Henn, PRL 110 (2013), 251601.



The kinetic mass

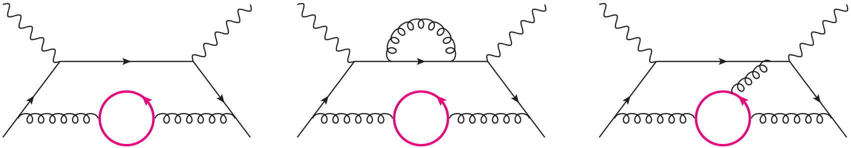
$$\frac{m^{\rm kin}}{m^{\rm OS}} = 1 - \frac{\alpha_s^{(n_l)}}{\pi} C_F \left(\frac{4}{3}\frac{\mu}{m^{\rm OS}} + \frac{1}{2}\frac{\mu^2}{(m^{\rm OS})^2}\right) + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^2 C_F \left\{ \frac{\mu}{m^{\rm OS}} \left[C_A \left(-\frac{215}{27} + \frac{2\pi^2}{9} + \frac{22}{9} l_\mu \right) + n_l T_F \left(\frac{64}{27} - \frac{8}{9} l_\mu \right) \right] \right.$$
$$\left. + \frac{\mu^2}{(m^{\rm OS})^2} \left[C_A \left(-\frac{91}{36} + \frac{\pi^2}{12} + \frac{11}{12} l_\mu \right) + n_l T_F \left(\frac{13}{18} - \frac{1}{3} l_\mu \right) \right] \right\} + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^3 C_F \left\{ \frac{\mu}{m^{\rm OS}} \left[C_A^2 \left(-\frac{130867}{1944} \right.\right.\right.$$
$$+ \frac{511\pi^2}{162} + \frac{19\zeta_3}{2} - \frac{\pi^4}{18} + \left(\frac{2518}{81} - \frac{22\pi^2}{27} \right) l_\mu - \frac{121}{27} l_\mu^2 \Big) + C_A n_l T_F \left(\frac{19453}{486} - \frac{104\pi^2}{81} - 2\zeta_3 \right.$$
$$\left. + \left(-\frac{1654}{81} + \frac{8\pi^2}{27} \right) l_\mu + \frac{88}{27} l_\mu^2 \right) + C_F n_l T_F \left(\frac{11}{4} - \frac{4\zeta_3}{3} - \frac{2}{3} l_\mu \right) + n_l^2 T_F^2 \left(-\frac{1292}{243} + \frac{8\pi^2}{81} + \frac{256}{81} l_\mu - \frac{16}{27} l_\mu^2 \right) \Big]$$
$$+ \frac{\mu^2}{(m^{\rm OS})^2} \left[C_A^2 \left(-\frac{96295}{5184} + \frac{445\pi^2}{432} + \frac{57\zeta_3}{16} - \frac{\pi^4}{48} + \left(\frac{2155}{216} - \frac{11\pi^2}{36} \right) l_\mu - \frac{121}{72} l_\mu^2 \right) + C_A n_l T_F \left(\frac{13699}{1296} - \frac{23\pi^2}{54} \right.\right.$$
$$\left.\left.\left. - \frac{3\zeta_3}{4} + \left(-\frac{695}{108} + \frac{\pi^2}{9} \right) l_\mu + \frac{11}{9} l_\mu^2 \right) + C_F n_l T_F \left(\frac{29}{32} - \frac{\zeta_3}{2} - \frac{1}{4} l_\mu \right) + n_l^2 T_F^2 \left(-\frac{209}{162} + \frac{\pi^2}{27} + \frac{26}{27} l_\mu - \frac{2}{9} l_\mu^2 \right) \right] \right\}, \quad (4)$$

Fael, Schönwald, Steinhauser, PRL 125 (2020) 052003

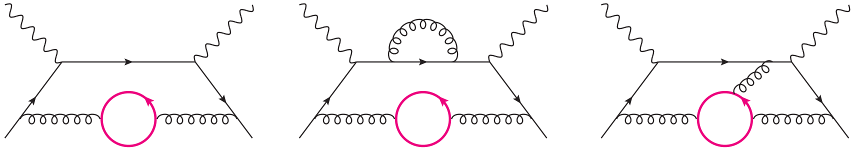
- The mass relation is written in terms of $\alpha_s^{(n_l)}$.
- n_l = number of massless quark, $l_\mu = \log(2\mu/\mu_s)$.
- The α_s^2 and $\alpha_s^3 \beta_0^2$ terms agree with Czarnecki, Melnikov, Uraltsev, PRL 80 (1998) 3189

Finite charm mass effects in m_b^{kin}

Charm mass effects: $0 < m_c$, $m_c \neq m_b$



Charm mass effects: $0 < m_c, m_c \neq m_b$



- Charm mass effects to the $\overline{\mathrm{MS}}$-on-shell mass relation known up to $O(\alpha_s^3)$.
  Davydychev, Grozin, PRD 59 (1999) 054024; Broadhurst, Gray, Schlicher, Z. Phys. C 52 (1991) 111;
  Bekavac, Grozin, Seidel, Steinhauser, JHEP 10 (2007) 006; Fael, Schönwald, Steinhauser, JHEP 10 (2020) 087.
- No charm mass effects known for the kinetic-on-shell mass relation.
- For their evaluation we demand $|y| \ll m_c^2, m_b^2$, i.e. no cut through a charm loop

Charm mass effects: $0 < m_c$, $m_c \neq m_b$

- Charm mass effects introduce additional families and master integrals in the mixed ultrasoft-hard regions.

  Fael, Schönwald, Steinhauser, JHEP 10 (2020) 087.

- For the bare three-loop diagrams we obtain a non trivial m_c/m_b dependence.
- However after on-shell renormalization, only $\log(\mu_s/m_c)$ remains.

The decoupling miracle

Karlsruhe Institute of Technology

Let's stare at $[\overline{\Lambda}(\mu)]_{\rm pert}$ with 5 active quarks for α_s:

$$[\overline{\Lambda}(\mu)]_{\rm pert} = \frac{\alpha_s^{(5)}(\mu_s)}{\pi} C_F\, \mu \left\{ \frac{4}{3} + \frac{\alpha_s^{(5)}(\mu_s)}{\pi} \left[C_A \left(\frac{215}{27} - \frac{2}{9}\pi^2 - \frac{22}{9} l_\mu \right) - n_l T_F \left(\frac{64}{27} - \frac{8}{9} l_\mu \right) - \frac{4}{9} n_c \log\left(\frac{\mu_s}{m_c^{\rm OS}} \right) - \frac{4}{9} n_h \log\left(\frac{\mu_s}{m_b^{\rm OS}} \right) \right] \right\}$$

$$\alpha_s^{(5)}(\mu_s) \to \alpha_s^{(4)}(\mu_s)$$

The decoupling miracle

Karlsruhe Institute of Technology

Let's stare at $[\overline{\Lambda}(\mu)]_{\rm pert}$ with 5 active quarks for α_s:

$$[\overline{\Lambda}(\mu)]_{\rm pert} = \frac{\alpha_s^{(5)}(\mu_s)}{\pi} C_F\, \mu \Bigg\{ \frac{4}{3} + \frac{\alpha_s^{(5)}(\mu_s)}{\pi} \Bigg[C_A \left(\frac{215}{27} - \frac{2}{9}\pi^2 - \frac{22}{9} l_\mu \right) - n_l T_F \left(\frac{64}{27} - \frac{8}{9} l_\mu \right) - \frac{4}{9} n_c \log\left(\frac{\mu_s}{m_c^{\rm OS}} \right) - \frac{4}{9} n_h \log\left(\frac{\mu_s}{m_b^{\rm OS}} \right) \Bigg] \Bigg\}$$

$$\alpha_s^{(5)}(\mu_s) \to \alpha_s^{(4)}(\mu_s)$$

The decoupling miracle

Let's stare at $[\overline{\Lambda}(\mu)]_{\rm pert}$ with 4 active quarks for α_s:

$$[\overline{\Lambda}(\mu)]_{\rm pert} = \frac{\alpha_s^{(4)}(\mu_s)}{\pi} C_F\, \mu \Bigg\{ \frac{4}{3} + \frac{\alpha_s^{(4)}(\mu_s)}{\pi} \Bigg[C_A \left(\frac{215}{27} - \frac{2}{9}\pi^2 - \frac{22}{9} l_\mu \right) - n_l T_F \left(\frac{64}{27} - \frac{8}{9} l_\mu \right) - \frac{4}{9} n_c \log\left(\frac{\mu_s}{m_c^{\rm OS}} \right) \Bigg] \Bigg\}$$

$$\alpha_s^{(4)}(\mu_s) \to \alpha_s^{(3)}(\mu_s)$$

The decoupling miracle

Let's stare at $[\overline{\Lambda}(\mu)]_{\text{pert}}$ with 4 active quarks for α_s:

$$[\overline{\Lambda}(\mu)]_{\text{pert}} = \frac{\alpha_s^{(4)}(\mu_s)}{\pi} C_F\, \mu \Bigg\{ \frac{4}{3} + \frac{\alpha_s^{(4)}(\mu_s)}{\pi} \Bigg[C_A \left(\frac{215}{27} - \frac{2}{9}\pi^2 - \frac{22}{9} l_\mu \right) - n_l T_F \left(\frac{64}{27} - \frac{8}{9} l_\mu \right) - \frac{4}{9} n_c \log\left(\frac{\mu_s}{m_c^{\text{OS}}} \right) \Bigg] \Bigg\}$$

$$\alpha_s^{(4)}(\mu_s) \to \alpha_s^{(3)}(\mu_s)$$

The decoupling miracle

Karlsruhe Institute of Technology

Let's stare at $[\overline{\Lambda}(\mu)]_{\rm pert}$ with 3 active quarks for α_s:

$$[\overline{\Lambda}(\mu)]_{\rm pert} = \frac{\alpha_s^{(3)}(\mu_s)}{\pi} C_F\, \mu \left\{ \frac{4}{3} + \frac{\alpha_s^{(3)}(\mu_s)}{\pi} \left[C_A \left(\frac{215}{27} - \frac{2}{9}\pi^2 - \frac{22}{9} l_\mu \right) - n_l T_F \left(\frac{64}{27} - \frac{8}{9} l_\mu \right) \right] \right\}$$

The decoupling miracle

Let's stare at $[\overline{\Lambda}(\mu)]_{\rm pert}$ with 3 active quarks for α_s:

$$[\overline{\Lambda}(\mu)]_{\rm pert} = \frac{\alpha_s^{(3)}(\mu_s)}{\pi} C_F\, \mu \left\{ \frac{4}{3} + \frac{\alpha_s^{(3)}(\mu_s)}{\pi} \left[C_A \left(\frac{215}{27} - \frac{2}{9}\pi^2 - \frac{22}{9} l_\mu \right) - n_l T_F \left(\frac{64}{27} - \frac{8}{9} l_\mu \right) \right] \right\}$$

The kinetic mass

$$
\begin{aligned}
\frac{m^{\mathrm{kin}}}{m^{\mathrm{OS}}} = 1 &- \frac{\alpha_s^{(n_l)}}{\pi} C_F \left(\frac{4}{3}\frac{\mu}{m^{\mathrm{OS}}} + \frac{1}{2}\frac{\mu^2}{(m^{\mathrm{OS}})^2} \right) + \left(\frac{\alpha_s^{(n_l)}}{\pi} \right)^2 C_F \Bigg\{ \frac{\mu}{m^{\mathrm{OS}}} \Bigg[C_A \left(-\frac{215}{27} + \frac{2\pi^2}{9} + \frac{22}{9} l_\mu \right) + n_l T_F \left(\frac{64}{27} - \frac{8}{9} l_\mu \right) \Bigg] \\
&+ \frac{\mu^2}{(m^{\mathrm{OS}})^2} \Bigg[C_A \left(-\frac{91}{36} + \frac{\pi^2}{12} + \frac{11}{12} l_\mu \right) + n_l T_F \left(\frac{13}{18} - \frac{1}{3} l_\mu \right) \Bigg] \Bigg\} + \left(\frac{\alpha_s^{(n_l)}}{\pi} \right)^3 C_F \Bigg\{ \frac{\mu}{m^{\mathrm{OS}}} \Bigg[C_A^2 \Bigg(-\frac{130867}{1944} \\
&+ \frac{511\pi^2}{162} + \frac{19\zeta_3}{2} - \frac{\pi^4}{18} + \left(\frac{2518}{81} - \frac{22\pi^2}{27} \right) l_\mu - \frac{121}{27} l_\mu^2 \Bigg) + C_A n_l T_F \Bigg(\frac{19453}{486} - \frac{104\pi^2}{81} - 2\zeta_3 \\
&+ \left(-\frac{1654}{81} + \frac{8\pi^2}{27} \right) l_\mu + \frac{88}{27} l_\mu^2 \Bigg) + C_F n_l T_F \left(\frac{11}{4} - \frac{4\zeta_3}{3} - \frac{2}{3} l_\mu \right) + n_l^2 T_F^2 \left(-\frac{1292}{243} + \frac{8\pi^2}{81} + \frac{256}{81} l_\mu - \frac{16}{27} l_\mu^2 \right) \Bigg] \\
&+ \frac{\mu^2}{(m^{\mathrm{OS}})^2} \Bigg[C_A^2 \Bigg(-\frac{96295}{5184} + \frac{445\pi^2}{432} + \frac{57\zeta_3}{16} - \frac{\pi^4}{48} + \left(\frac{2155}{216} - \frac{11\pi^2}{36} \right) l_\mu - \frac{121}{72} l_\mu^2 \Bigg) + C_A n_l T_F \Bigg(\frac{13699}{1296} - \frac{23\pi^2}{54} \\
&- \frac{3\zeta_3}{4} + \left(-\frac{695}{108} + \frac{\pi^2}{9} \right) l_\mu + \frac{11}{9} l_\mu^2 \Bigg) + C_F n_l T_F \left(\frac{29}{32} - \frac{\zeta_3}{2} - \frac{1}{4} l_\mu \right) + n_l^2 T_F^2 \left(-\frac{209}{162} + \frac{\pi^2}{27} + \frac{26}{27} l_\mu - \frac{2}{9} l_\mu^2 \right) \Bigg] \Bigg\}, \quad (4)
\end{aligned}
$$

Fael, Schönwald, Steinhauser, PRL 125 (2020) 052003

- The mass relation is written in terms of $\alpha_s^{(n_l)}$.
- n_l = number of massless quark, $l_\mu = \log(2\mu/\mu_s)$.

The kinetic-$\overline{\mathrm{MS}}$ mass relation

The kinetic-$\overline{\mathrm{MS}}$ mass relation

- Quark mass relations:
  - kinetic-on-shell

    Fael, Schönwald, Steinhauser, PRL 125 (2020) 052003
  - $\overline{\mathrm{MS}}$-on-shell with charm mass effects

    Broadhurst, Gray, Schlicher, Z. Phys. C 52 (1991) 111;
    Bekavac, Grozin, Seidel, Steinhauser, JHEP 10 (2007) 006; Fael, Schönwald, Steinhauser, JHEP 10 (2020) 087.
- Input values:
  - $\alpha_s^{(5)}(M_Z) = 0.1179$
  - $\overline{m}_c(3\text{ GeV}) = 0.993\text{ GeV}$
  - $\overline{m}_b(\overline{m}_b) = 4.163\text{ GeV}$

$m_b^{\rm kin}(1\text{ GeV})$ from $\overline{m}_b(\overline{m}_b)$

scheme	$\alpha_s^{(n_r)}$	m_c in $\overline{\rm MS}$-OS	m_c in kin-OS	in MeV
(A)	3	✓	–	$4163 + 248 + (81 + 7_{\Delta_{m_c}} + 12_{dec} - 20_{n_c}) + (30 + 14_{\Delta_{m_c}} + 16_{dec} - 30_{n_c} - 1_{n_c\times dec} + 0.4_{\Delta_{m_c}\times dec}) = 4163 + 248 + 80 + 30 = 4520$
(B)	4	✓	✓	$4163 + 259 + (88 + 7_{\Delta_{m_c}} + 5_{\Delta^{\rm kin}_{m_c}} - 22_{n_c}) + (34 + 16_{\Delta_{m_c}} + 10_{\Delta^{\rm kin}_{m_c}} - 34_{n_c}) = 4163 + 259 + 78 + 26 = 4526$

Fael, Schönwald, Steinhauser, hep-ph/2011.XXXX

m_b^{kin}(1 GeV) from $\overline{m}_b(\overline{m}_b)$

Karlsruhe Institute of Technology

in MeV

	$\alpha_s^{(n_r)}$	m_c in $\overline{\mathrm{MS}}$-OS	m_c in kin-OS	
(C)	4	✓	✗	$4163 + 259 + (99 + 7_{\Delta_{m_c}} - 22_{n_c}) + (59 + 16_{\Delta_{m_c}} - 34_{n_c}) = 4163 + 259 + 84 + 41 = 4547$
(D)	3	✗	✗	$4163 + 248 + 81 + 30 = 4521$

Fael, Schönwald, Steinhauser, hep-ph/2011.XXXX

Scheme conversion uncertainty

- Half of the 3 loop correction

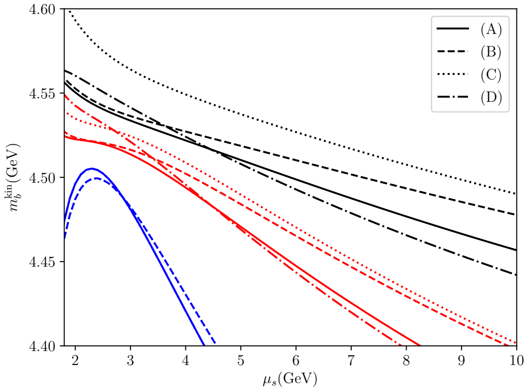
$$\delta m_b^{\text{kin}} \simeq 15 \text{ MeV}$$

- Scale variation $3 < \mu_s < 9$ GeV

$$\delta m_b^{\text{kin}} \simeq 17 \text{ MeV}$$

- Four-loop large-β_0 approximation

$$\delta m_b^{\text{kin}} \simeq 8 \text{ MeV}$$



Fael, Schönwald, Steinhauser, hep-ph/2011.XXXX

To be compared with:

- scheme conversion uncertainty at two-loops: $\delta m_b^{\text{kin}} = 30$ MeV Gambino, JHEP 09 (2011) 055
- $m_b^{\text{kin}}(1\ \text{GeV}) = 4554 \pm 18$ MeV from $B \to X_c \ell \bar{\nu}_\ell$ global fit HFLAV 2019

Conclusions

- We computed the relation between the kinetic mass and the on-shell mass up to $O(\alpha_s^3)$.
- We studied finite charm mass effects in m_b^{kin}: the charm *wants* to be heavy!
- We obtain precise predictions for the m_b^{kin}–$\overline{\mathrm{MS}}$ relation up to $O(\alpha_s^3)$.
- Now implemented in `(C)RunDec`.
- Mass scheme conversion uncertainty is reduced by about a factor of two compared to two-loop.
- Our results are relevant for future extractions of $|V_{cb}|$ inc. at Belle II.

Open Questions

- Higher terms of order μ^3 in the kinetic mass

$$M_B = m_b + \overline{\Lambda} + \frac{\mu_\pi^2 + d_B \mu_G^2}{2m_b} + \frac{\rho_D^3 - \rho_{\text{non local}}^3}{4m_b^2} + \ldots$$

- SV sum rules determine ρ_D, however there are non-local contributions:

$$\rho_{\text{non local}}^3 \simeq \frac{1}{2M_B} \int d^4x \, \langle B| \, \bar{b}_v(x) \pi^2 b_v(x) \, \bar{b}_v(0) \pi^2 b_v(0) \, |B\rangle$$

See also: Bigi, Shifman, Uraltsev, Vainsthein, PRD 52 (1995) 196;
Mannel, PRD 50 (1994) 428;
Gremm, Kapustin, PRD 55 (1997) 6924.

Spare

Charm quark mass

- $m_c^{\text{kin}}(0.5\text{ GeV})$:

$$\begin{aligned}
m_c^{\text{kin}}(0.5\text{ GeV}) &= 993 + 191 + 100 + 52\text{ MeV} = 1336\text{ MeV}\,,\\
m_c^{\text{kin}}(0.5\text{ GeV}) &= 1099 + 163 + 76 + 34\text{ MeV} = 1372\text{ MeV}\,,\\
m_c^{\text{kin}}(0.5\text{ GeV}) &= 1279 + 84 + 30 + 11\text{ MeV} = 1404\text{ MeV}\,.
\end{aligned}$$

- $m_c^{\text{kin}}(1\text{ GeV})$:

$$\begin{aligned}
m_c^{\text{kin}}(1\text{ GeV}) &= 993 + 83 + 35 + 14\text{ MeV} = 1125\text{ MeV}\,,\\
m_c^{\text{kin}}(1\text{ GeV}) &= 1099 + 37 + 2 - 3\text{ MeV} = 1135\text{ MeV}\,,\\
m_c^{\text{kin}}(1\text{ GeV}) &= 1279 - 73 - 61 - 17\text{ MeV} = 1128\text{ MeV}\,,
\end{aligned}$$

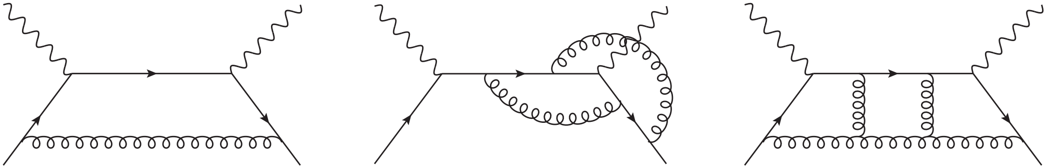
where from top to bottom $\mu_s = 3$ GeV, 2 GeV and $\overline{m}_c$ for $\overline{m}_c(\mu_s)$ and $\alpha_s^{(3)}(\mu_s)$.

Expansion by regions

- For one heavy particle threshold, there are two regions:

  see also: Smirnov Springer Tracts Mod. Phys. 250 (2010)

$$\text{hard (h): } k_i \sim m_b$$
$$\text{ultra-soft (us): } k_i \sim y/m_b$$



(u) (h)   (uu) (uh) (hh)   (uuu) (uuh) (uhh) (hhh)

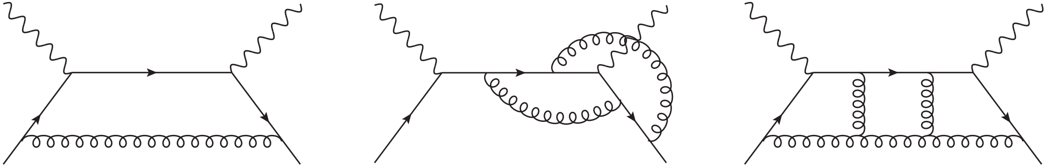
Expansion by regions

- For one heavy particle threshold, there are two regions:

see also: Smirnov Springer Tracts Mod. Phys. 250 (2010)

hard (h): $k_i \sim m_b$

ultra-soft (us): $k_i \sim y / m_b$



(u) ~~(h)~~ (uu) (uh) ~~(hh)~~ (uuu) (uuh) (uhh) ~~(hhh)~~

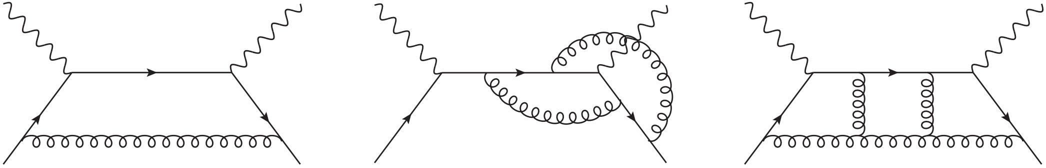
- **All hard** regions don't contribute: no imaginary part.
- After renormalization and decoupling $\alpha_s^{(n_l+n_h)} \to \alpha_s^{(n_l)}$,
  only all ultra-soft part remains

Expansion by regions

- For one heavy particle threshold, there are two regions:

  see also: Smirnov Springer Tracts Mod. Phys. 250 (2010)

$$\text{hard (h)}: k_i \sim m_b$$
$$\text{ultra-soft (us)}: k_i \sim y/m_b$$



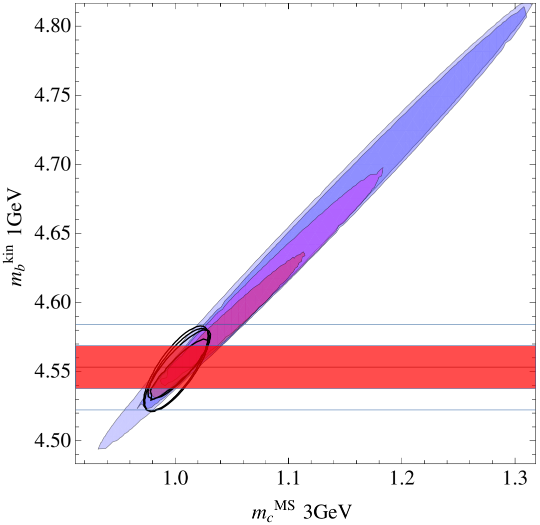
(u) ~~(h)~~ (uu) ~~(uh)~~ ~~(hh)~~ (uuu) ~~(uuh)~~ ~~(uhh)~~ ~~(hhh)~~

- **All hard** regions don't contribute: no imaginary part.
- After renormalization and decoupling $\alpha_s^{(n_l+n_h)} \to \alpha_s^{(n_l)}$, **only all ultra-soft part remains**

The total width for $B \to X_c \ell \bar{\nu}_\ell$ in the large β_0 limit:

$$\Gamma_{\rm sl} = \frac{G_F^2 |V_{cb}|^2 (m_b^{\rm kin})^5}{192\pi^3} f(0.25) \left[1 - 0.94 \left(\frac{\alpha_s}{\pi}\right) - 5.1 \left(\frac{\alpha_s}{\pi}\right)^2 - 17 \left(\frac{\alpha_s}{\pi}\right)^3 + \dots\right]$$

Benson, Bigi, Mannel, Uraltsev, NPB 665 (2003) 367



Original plot from

Gambino, Schwanda, PRD 89 (2014) 014022

Horizontal error bands superimposed by MF